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BOOK REVIEW

JAGERS, P., *Branching Processes with Biological Applications.* John Wiley and Sons, 1975, 268 pp. $28.50.

Review by CHARLES J. MODE

Drexel University

This book is highly recommended to those interested in stochastic models of populations. The first of the nine chapters contains a historical sketch, some generalities about populations, and a survey of results. Particularly impressive from the historical point of view is the list of luminaries whose widely varied careers at one time or another touched on problems of branching processes, among them Euler in mathematics, Lotka in demography, Fisher in statistics and mathematical genetics, and Kolmogorov in probability.

Chapters 2 and 3, preparatory to the study of the general process, are devoted to the classical Galton–Watson process and its neighbors. They cover such standard topics as the extinction probability for the Galton–Watson process and the usual trichotomy of critical, subcritical, and supercritical processes. In addition there are sections dealing with the total progeny of a branching process and such statistical topics as estimating the offspring distribution by maximum likelihood and Bayesian methods. Neighbors of the Galton–Watson process include a branching process with immigration, limit theorems for increasing numbers of ancestors, and a diffusion approximation first outlined by Feller. Also treated is the Galton–Watson process in varying environments (varying offspring distributions), which is of interest in its own right and also leads naturally to a brief discussion of branching processes in random environments. Throughout Chapters 2 and 3 theorems are stated and proved in terms of minimal conditions currently in vogue in the analysis of branching processes. Because the writing style is clear and terse, some readers may find Jager's treatment of the Galton–Watson process and its neighbors easier to read than previous ones. Aside from a brief account of multitype Galton–Watson process in Chapter 4 and some discussion of a special kind of two-type general branching process in Chapter 9, the theory of multitype branching processes is ignored.

Chapter 5 contains material on martingales, renewal theory, and point processes useful in the construction and analysis of branching processes. The chapter starts out with nice proofs of the martingale theorem, the zero-one laws of Kolmogorov and Hewitt–Savage, and the ballot theorem via martingale theory. There follows a section on aspects of renewal theory needed in the analysis of the general branching process. The chapter closes with a very brief discussion of point processes later used in the definition of the general branching process.

It is Chapters 6 through 9 that contribute most to distinguishing the book from others on branching processes. Chapter 6 contains a thorough development of the general branching process. In a general branching process reproduction may occur at any time during the lifetime of a particle, in contrast to the classical Bellman–Harris process where reproduction and death coincide. After the general branching process has been defined and related to special cases introduced by Bellman–Harris and Sevast'yanov, conditions for almost sure finiteness of the process are derived. A treatment of generating functions, moments, and the extinction probability then follows. A highlight of the section on the critical case for the general process is a proof of the exponential limit law by a method due to Holte, a method avoiding the heavy moment assumptions used by Weiner and his students in their investigations of the problem. The subcritical case is taken up next, and here the author performs a valuable service by including results of Ryan available previously only in thesis form. The next sections concern the supercritical process, populations enumerated according to a random characteristic, and almost sure convergence in the supercritical case. The proof of this last result still hinges on a lemma due to Harris; a proof recently developed by Athreya and Kaplan for the Bellman–Harris process, using a $E[\xi \log \xi]$-condition, may also work in the general case.

The stable age distribution, which is a basic concept in demography, is next taken up. Here a significant original contribution of the author is a rigorous proof that this distribution is indeed an eigenfunction of a certain mean kernel. This eigenfunction property, which was conjectured by Lotka, warrants attaching the adjective *stable* to the limiting age distribution. The concluding sections of Chapter 6 are devoted to the total progeny in a general process, integrals of branching processes, and the maximum-likelihood estimation of the reproduction mean. Apart from the problem of incorporating random environments into the framework of general branching processes, the development in Chapter 7 follows very closely that in Chapter 3.

In Chapters 7 and 8, important steps are taken toward making the general process useful and interesting to scientists outside a rather small circle of workers in stochastic processes. Chapter 8 is devoted to branching processes in demography and Chapter 9 to the study of branching processes in cell kinetics. In the chapter on demography an attempt is made to relate age-dependent birth and death processes to classical mathematical demography in the continuous case. Also contained in Chapter 8 are sections dealing with Lotka's renewal equation, the Fisherian reproductive value, the age of child-bearing, and the demographic notion of length of generation. In the two chapters on applications the treatment of cell kinetics doubtless required the greatest research effort—69 references are cited. The topics here—experimental techniques and mathematical models that have been proposed—are readily comprehended only by research workers in cell kinetics. The author derives many formulas, within the context of reproduction by binary splitting, that link branching processes to parameters and functions used by experimenters. Many of the results appear to be new, and the chapter linking branching processes to cell kinetics is probably the most extensive examination of the intersection of these two fields yet to appear.

Of all the chapters in the book, the one on branching processes in demography should be read most critically. In attempts to apply general branching processes to human populations, the reproductive process becomes a stochastic process describing variations in birth patterns experienced by a cohort of women during the childbearing ages. The point of view is longitudinal in the sense that the process is concerned with a cohort of women over the entire time span of the childbearing ages. In the one-six theory, only daughters are counted, so the final mean of the reproductive process times the probability a baby is female becomes the mean number of live births (MNLB) experienced by a woman during her entire reproductive career. However, demographers usually do not observe the MNLB. What is frequently observed is a cross-sectional sample of women belonging to the childbearing ages in a particular year. From these data age-specific birth rates may be estimated by partitioning the sample according to some selected age scale. The sum of the age-specific birth rates yields what demographers call the gross reproductive rate (GRR). If one multiplies each age-specific birth rate by a lifetable estimate of the probability of surviving to that age and then sums over all ages, one obtains a quantity demographers call the net reproductive rate (NRR). Despite the fact that a function and estimates of it are rarely distinguished in empirical demography, in attempting to make connections between demography and generalized branching processes it does not seem wise to follow the pseudo-probabilistic traditions of demography and identify the NRR with the MNLB as suggested by the author. For clearly the two quantities can differ. It would seem desirable, therefore, to reserve the symbol NRR for the empirically determined quantity. Indeed, for a particular sample, it is not clear in what sense NRR is an estimate of the MNLB, even if the laws of evolution of the process are stationary.

Turning now to suggestions for further research, an obvious problem is to investigate connections between age-specific birth rates and the reproductive process in a general branching model. One approach may be roughly described as follows: consider a large population with $N(x)$ individuals in age group x out of a total population of size $N = \sum_x N(x)$. For each age group x, consider $N(x)$ independent general branching processes conditioned on the initial individual having survived to age x. Suppose the evolution of the population described by a collection of N such independent processes conditioned on the ages of the initial individuals at $t = 0$. If the problem of sampling such a collective process is ignored, then the age-specific birth rates may be expressed in terms of the collective process in a straightforward way for any $t > 0$. At least two types of problems arise. First, for a fixed $t > 0$, search for good approximations to the age-specific birth rates as $N(x) \to \infty$ for all x. One would expect these approximations to involve the mean functions of the processes making up the collective process as well as the initial numbers $N(x)$. The presence of initial numbers in these approximations would be important in computer studies of the effects of deviations from a stable age distribution. Second, one could derive formulas for age-specific birth rates in a stable population by searching for limits as $t \to \infty$. Such investigations should enhance our understanding of the relationship between the NRR and the MNLB. A more ambitious investigator might try this with the NRR computed from a sample of the collective process.

Another class of problems falling within the scope of the general process and needing further investigation concerns the reproductive process, particularly as it applies to humans. Stochastic models of human reproduction are of obvious interest, but they have only an incidental connection with the general branching process. These models are most useful when they aid in the interpretation of large data systems generated in the study of human reproduction. In dealing with stochastic models in relation to data, a constructive-algorithmic approach to modeling with a view toward computerization seems to be more useful than the usual hard analysis associated with stochastic processes; see the review article [1]. For additional hard analysis not appearing in Jager's book, see [2], where further attempts are made to connect classical mathematical demography with the general branching process.

Although human reproduction is bisexual, the general branching process, and even the collective process described above, is essentially asexual, which limits its applicability to human populations. There is thus the eternal problem of constructing and analyzing genuinely bisexual processes, and here two problems arise. First, although bisexual processes fall naturally in the multitype class, it is difficult to get agreement on what the structure of these processes should be; and second, once a structure is decided upon formidable analytic difficulties usually arise. For references on some bisexual models proposed thus far, particularly as they relate to demography, see [1].

It appears that a constructive-algorithmic approach will be fruitful also in formulating bisexual processes, for computer experimentation should facilitate objective judgements of what structures are reasonable and feasible. In the absence of a capability for experimentation, the proper structure for these processes is elusive and the promise of particular choices sometimes proves illusory. A constructive-algorithmic approach may for some be an attractive research alternative to the field of branching processes, a field presently burdened with over-simplified models and limit theorems. Yet these limit theorems will in all likelihood prove useful in the quest for stochastic models of populations. Jager's book helps make clear which results on branching processes may be important in the search.

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DREXEL UNIVERSITY
PHILADELPHIA, PA 19104