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A STRONG INVARIANCE THEOREM FOR THE STRONG LAW OF LARGE NUMBERS[1]

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Let $X_1, X_2, \cdots$ be i.i.d. random variables with mean 0 and variance 1. Let $S_n = X_1 + \cdots + X_n$, and let $\{H_n\}$ be the standard partial sum processes on $[0, \infty)$ defined in terms of the S_n's and normalized as in Strassen. Each function of the "tail" behavior of the process H_n is the dual of a function of the "initial" behavior of the process H_n, the duality being induced by the time inversion map R. The dual role of "initial" and "tail" functions is used to exploit an extension of Strassen's invariance theorem for the law of the iterated logarithm due to Wichura, and thereby obtain limit theorems for a variety of functions of the "tail" behavior of the sums S_n. For example, with probability one,

$$\limsup_{n\to\infty} (n/2 \log\log n)^{\frac{1}{2}} \max_{n\le k<\infty} (k^{-1}S_k) = 1$$

and

$$\limsup_{n\to\infty} n^{-1} \max \{k \geqq 1 : k^{-1}S_k \geqq \theta(2 \log\log n/n)^{\frac{1}{2}}\} = \theta^{-2} .$$

1. Introduction. Let $X_1, X_2 \cdots$ be i.i.d. rv's with mean 0 defined on a common probability space $(\Omega, \mathscr{F}, P)$. Let $S_n = X_1 + \cdots + X_n$; the well-known strong law of large numbers then asserts that with probability one (w.p. 1)

$$\lim_{n\to\infty} n^{-1}S_n = 0 . \tag{1}$$

If the X_i's have finite variance (1 without loss of generality), the law of the iterated logarithm (Hartman and Wintner (1941), Strassen (1964)) yields a rate for the convergence in (1): w.p. 1 $n^{-1}S_n = O((n^{-1} \log\log n)^{\frac{1}{2}}$, or, more precisely,

$$\limsup_{n\to\infty} (n/2 \log\log n)^{\frac{1}{2}}(n^{-1}S_n) = 1 \quad \text{w.p. } 1 . \tag{2}$$

But the strong law (1) also has a variety other consequences; for example

$$\lim_{n\to\infty} \max_{n\le k<\infty} (k^{-1}S_k) = 0 \quad \text{w.p. } 1 , \tag{3}$$

and, for any $\varepsilon > 0$

$$\max \{k \geqq 1 : k^{-1}S_k \geqq \varepsilon\} < \infty \quad \text{w.p. } 1 . \tag{4}$$

Our object in this note is to show that when the X_i's have variance 1 then (3) and (4) can be strengthened in much the same way that (2) strengthens (1). To do this we make use of an extension of Strassen's (1964) invariance principle for the law of the iterated logarithm which is due to Wichura (1974). We then exploit this theorem by way of a duality relationship induced by time inversion to obtain iterated logarithm type limit theorems for a variety of "tail" functions

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of the S_n's. Strengthened versions of (3) and (4) emerge as special cases of this general duality in Examples 2 and 3 in Section 3.

2. The main results. Define functions H_n, $n \geqq 3$, on $[0, \infty)$ in terms of the sums S_n by

$$H_n(t) = (S_{[nt]} + (nt - [nt])X_{[nt]+1})/(2n \log\log n)^{\frac{1}{2}}\,, \qquad t \geqq 0\,.$$

H_n takes values in $C[0, \infty)$, the space of continuous functions on $[0, \infty)$. Let $p_0(t) = (t \vee 1)$, $p_1(t) = ((t \vee 3) \log\log (t \vee 3))^{\frac{1}{2}}$ for $t \geqq 0$, and set

$$B_i = \{x \in C[0, \infty) : \sup_{0 \leqq t < \infty} |x(t)|/p_i(t) < \infty\}$$

$i = 0, 1$. Define the metric ρ_i for functions $x, y \in B_i$ by

$$\rho_i(x, y) = \sup_{0 \leqq t < \infty} |x(t) - y(t)|/p_i(t) \qquad i = 0, 1\,.$$

Let

$$\mathbb{K} \equiv \{x \in C[0, \infty) : x(0) = 0,\ \textstyle\int_0^t \dot{x}(s)\,ds,\ \int_0^\infty \dot{x}(s)^2\,ds \leqq 1\}\,.$$

THEOREM 1 (Strassen–Wichura). *With probability one the sequence $\{H_n\}_{n \geqq 1}$ is relatively compact in the topology induced by the metric ρ_1 on $C[0, \infty)$ and has limit set $\mathbb{K}$.*

For the (easy) proof of Theorem 1 as a consequence of Strassen's (1964) invariance theorem see Wichura (1974). Our primary interest here is in the metric ρ_0; since $p_0 \geqq p_1$ (for all $t \geqq 0$) and hence $\rho_0(x, y) \leqq \rho_1(x, y)$ for all $x, y \in B_1$, Theorem 1 implies that we also have convergence with respect to ρ_0:

COROLLARY 1. *With probability one the sequence $\{H_n\}_{n \geqq 1}$ is relatively compact in the topology induced by the metric ρ_0 on $C[0, \infty)$ and has limit set $\mathbb{K}$.*

Before proceeding to the consequences of Theorem 1 and Corollary 1 we should remark that similar theorems hold for a wide variety of processes in addition to the "partial sum" processes of the preceding discussion. Many authors have considered similar processes on $[0, 1]$ under a wide range of probabilistic assumptions and have proved analogues of Strassen's (1964) original theorem for partial sums: i.e., when properly normalized the processes are, w.p. 1, relatively compact in the topology of uniform convergence on $C[0, 1]$ and have limit set $\mathbb{K}$ (restricted to $[0, 1]$). For example, results of this type have been established for martingales and processes with stationary increments by Heyde and Scott (1973), and for sums of weakly dependent variables by Phillip and Stout (1975). But now note that Wichura's extension of Strassen's theorem proceeds by an argument which is independent of the probabilistic assumptions imposed on the summands, and hence convergence in the topology on $C[0, \infty)$ induced by ρ_1 follows for any processes of this type which are relatively compact in the uniform topology on $C[0, 1]$.

A different approach to the convergence question (with respect to ρ_0 on $[0,\infty)$) is by way of imbedding; e.g., (1.2) of page 2 of Phillip and Stout (1975) or (3.5) page 123 or (4.5) page 127 of Jain, Jogdeo, and Stout (1975). The point is that it is no more difficult to establish convergence with respect to ρ_0 on $[0, \infty)$ than to establish uniform convergence on finite intervals. As a consequence, the following considerations apply to a wide range of processes "like" partial sum processes.

Now let R be the standard time inversion map defined by

$$\begin{aligned}(Rx)(t) &= tx(1/t) \qquad \text{for} \quad 0 < t < \infty \\ &= 0 \qquad\qquad \text{for} \quad t = 0 .\end{aligned}$$

Let $C_0 = \{x \in C[0, \infty) : x(0) = 0, \lim_{t\to\infty} t^{-1}x(t) = 0\} \subset B_0$. Note that $\mathbb{K} \subset C_0$. The following lemma summarizes some of the useful properties of R.

LEMMA. *The time inversion map R is*

(a) *an isometry of the metric space* (C_0, ρ_0); $\rho_0(Rx, Ry) = \rho_0(x, y)$ *for all* $x, y \in C_0$;
(b) *a continuous function from* (C_0, ρ_0) *to* (C_0, ρ_0);
(c) *its own inverse*; $R(R(x)) = x$ *for all* $x \in C_0$;
(d) $\mathbb{K}$-*preserving*; i.e., $R(\mathbb{K}) = \mathbb{K}$.

PROOF. Assertions (a) and (c) are easily verified, and (b) is a consequence of (a). To see that R preserves $\mathbb{K}$, note that R preserves Brownian motion and is continuous; hence, normalizing as in Strassen's Theorem 1, applying that theorem as extended by Wichura, and considering the resulting sets of limit points yields $R(\mathbb{K}) = \mathbb{K}$. □

Suppose that $f(x)$ ($f : C_0 \to \mathbb{R}^1$) is some measure of the "initial" behavior of functions x in C_0. Then the *dual function*, $Df(x)$, defined by

$$(Df)(x) = f(R(x)) , \quad x \in C_0$$

will be a corresponding measure of the "tail" behavior of x (and vice versa). The examples considered in Section 3 illustrate this duality between "initial" and "tail" functions.

In view of (i) the duality connection between "initial" functions and "tail" functions via the time inversion map R; (ii) the fact that R is continuous and preserves $\mathbb{K}$; and (iii) Corollary 1, it is easily seen that iterated logarithm limit theorems for "tail" functions of the processes H_n can easily be deduced from the corresponding results for "initial" functions. The following theorem makes this more precise. If a function f: $(C_0, \rho_0) \to (\mathbb{R}^1, |\ |)$ is continuous at every point of $\mathbb{B} \subset C_0$, we say that f is $\mathbb{B}$-continuous on (C_0, ρ_0).

THEOREM 2. *Suppose that f is a $\mathbb{K}$-continuous function on (C_0, ρ_0) and that $\{H_n\}_{n\geq 1}$ is*, w.p. 1, *relatively compact with respect to ρ_0 with limit set $\mathbb{K}$. Then*

(a) $Df = f \circ R$ *is $\mathbb{K}$-continuous on* (C_0, ρ_0);
(b) w.p. 1 $\{f(H_n)\}_{n\geq 1}$ *and* $\{Df(H_n)\}_{n\geq 1}$ *are relatively compact with the same limit set* $f(\mathbb{K}) = f(R(\mathbb{K})) = Df(\mathbb{K})$; *and*
(c) *if* $\sup_{x\in \mathbb{K}} f(x) = f(x_0)$, $x_0 \in \mathbb{K}$, *then* $\sup_{x\in\mathbb{K}} Df(x) = Df(Rx_0) = f(x_0)$.

Note that if, for some subsequence n', w.p. 1 $\{H_n\}_{n\in\{n'\}}$ is relatively compact with R-invariant set of limit points $\mathbb{K}^* \subset \mathbb{K}$ (so $R(\mathbb{K}^*) = \mathbb{K}^*$), then (ii) and (iii) remain true with $\mathbb{K}$ replaced by $\mathbb{K}^*$; see Example 7 in this connection.

3. Examples. All of the following "initial" functions $f: C[0, 1] \to \mathbb{R}^1$, with the exception of Examples 3 and 6 were considered by Strassen (1964). In view of Theorem 2, we need only to translate his results for "initial" functions f to the "tail" functions Df. We draw freely in this section on Strassen's results concerning $\sup_{x\in\mathbb{K}} f(x)$ and the x's in $\mathbb{K}$ for which the supremum is obtained for the various functions f. Even though the functions considered in Examples 3 and 4 are not $\mathbb{K}$-continuous, one can verify that the asserted lim sup results hold by reasoning as in Example (v), pages 223–224 of Strassen (1964).

EXAMPLE 1. Let $f_1(x) = x(s_0)$ with $0 < s_0 \leqq 1$ fixed. Then $Df_1(x) = t_0^{-1}x(t_0)$ where $1 \leqq t_0 = s_0^{-1} < \infty$, $\sup_{x\in\mathbb{K}} f_1(x) = s_0^{-\frac{1}{2}} = f_1(x_1)$ with $x_1(t) = s_0^{-\frac{1}{2}}t \wedge s_0^{\frac{1}{2}} \in \mathbb{K}$. Hence $\sup_{x\in\mathbb{K}} Df_1(x) = Df_1(Rx_1) = t_0^{-\frac{1}{2}}$ with $Rx_1(t) = t_0^{-\frac{1}{2}}t \wedge t_0^{\frac{1}{2}} \in \mathbb{K}$,

$$\limsup_{n\to\infty} H_n(s_0) = s_0^{\frac{1}{2}} \quad \text{w.p. } 1\,,$$

and

$$\limsup_{n\to\infty} t_0^{-1}H_n(t_0) = t_0^{-\frac{1}{2}} \quad \text{w.p. } 1\,;$$

further, for n large $H_n(s_0)$ and $t_0^{-1}H_n(t_0)$ are close to $s_0^{\frac{1}{2}} = t_0^{-\frac{1}{2}}$ iff H_n is close to x_1 or Rx_1 respectively.

EXAMPLE 2. Let $f_2(x) = \sup_{0\leq t\leq 1} x(t)$. Then $Df_2(x) = \sup_{1\leq t<\infty} (t^{-1}x(t))$, $\sup_{x\in\mathbb{K}} f_2(x) = 1 = f_2(x_2)$ with $x_2(t) = t \wedge 1 \in \mathbb{K}$, and $\sup_{x\in\mathbb{K}} Df_2(x) = 1 = Df_2(x_2)$ since $Rx_2 = x_2$. Note that

$$f_2(H_n) = (\max_{1\leq k\leq n} S_k)/(2n \log\log n)^{\frac{1}{2}}$$

and

$$Df_2(H_n) = (n/2 \log\log n)^{\frac{1}{2}} \max_{n\leq k<\infty} (k^{-1}S_k)\,.$$

Hence Theorem 2 implies that

$$\limsup_{n\to\infty} (2n \log\log n)^{-\frac{1}{2}}(\max_{1\leq k\leq n} S_k) = 1 \quad \text{w.p. } 1\,,$$

and

$$\limsup_{n\to\infty} (n/2 \log\log n)^{\frac{1}{2}} \max_{n\leq k<\infty} (k^{-1}S_k) = 1 \quad \text{w.p. } 1\,;$$

furthermore for large n $f_2(H_n)$ and $Df_2(H_n)$ are close to 1 iff H_n is close to $x_2 = Rx_2$. This example strengthens (3) in the presence of a second moment.

EXAMPLE 3. For $\theta > 0$ let $f_3(x) = (\inf\{t \geqq 0: x(t) = \theta\})^{-1}$ where the infimum equals $+\infty$ if the set is empty (and hence $f_3(x) = 0$). Then $Df_3(x) = \sup\{t \geqq 0: t^{-1}x(t) = \theta\}$, $\sup_{x\in\mathbb{K}} f_3(x) = \theta^{-2} = f_3(x_3)$ with $x_3(t) = \theta^{-1}t \wedge \theta \in \mathbb{K}$, and $\sup_{x\in\mathbb{K}} Df_3(x) = \theta^{-2} = Df_3(Rx_3)$ with $Rx_3(t) = \theta t \wedge \theta^{-1}$. Note that

$$P(\limsup_{n\to\infty} Df_3(H_n) = \limsup_{n\to\infty} n^{-1} \max\{m \geqq 1: m^{-1}S_m \geqq \theta(2 \log\log n/n)^{\frac{1}{2}}\}) = 1$$

and hence

$$\limsup_{n\to\infty} n^{-1} \max\{m \geqq 1: m^{-1}S_m \geqq \theta(2 \log\log n/n)^{\frac{1}{2}}\} = \theta^{-2} \quad \text{w.p. } 1\,;$$

furthermore, for large n $Df_3(H_n)$ is close to θ^{-2} iff H_n is close to $Rx_3(t) = \theta t \wedge \theta^{-1}$. This example strengthens (4) in the presence of a finite second moment.

EXAMPLE 4. For $0 < c < 1$ let $f_4(x) = \lambda\{0 \leqq t \leqq 1 : x(t) \geqq ct^{\frac{1}{2}}\} = \int_0^1 1_{[x(t) \geqq ct^{\frac{1}{2}}]}\, dt$ where λ denotes Lebesgue measure. Then $Df_4(x) = \int_1^\infty 1_{[s^{-1}x(s) \geqq cs^{-\frac{1}{2}}]} s^{-2}\, ds$, $\sup_{x \in \mathbb{K}} f_4(x_4) = 1 - \exp(-4(c^{-2} - 1)) \equiv 1 - s_0 = f_4(x_4)$ where $x_4(t) = cs_0^{-\frac{1}{2}}t, ct^{\frac{1}{2}}$, c according as $0 \leqq t \leqq s_0$, $s_0 \leqq t \leqq 1$, or $1 \leqq t < \infty$ and $Rx_4(t) = ct$, $ct^{\frac{1}{2}}$, $cs_0^{-\frac{1}{2}}$ according as $0 \leqq t \leqq 1$, $1 \leqq t \leqq s_0^{-1}$, or $s_0^{-1} \leqq t < \infty$. Hence

$$\limsup_{n\to\infty} Df_4(H_n) = 1 - \exp(-4(c^{-2} - 1)) \quad \text{w.p. } 1 .$$

EXAMPLE 5. If ϕ is a fixed Riemann integrable real function on $[0, 1]$ let $f_5(x) = \int_0^1 x(t)\phi(t)\, dt$. Then $Df_5(x) = \int_1^\infty t^{-1}x(t)\psi(t)t^{-2}\, dt$ where $\psi(t) = \phi(1/t)$ for $1 \leqq t < \infty$, $\sup_{x \in \mathbb{K}} f_5(x) = \{\int_0^1 \Phi(t)^2\, dt\}^{\frac{1}{2}} \equiv \sigma = f_5(x_5)$ where $x_5(t) = \sigma^{-1}\int_0^1 (s \wedge t)\phi(s)\, ds$, $\sigma^{-1}\int_0^1 s\phi(s)\, ds$ according as $0 \leqq t \leqq 1$ or $1 \leqq t < \infty$, and $\Phi(t) = \int_t^1 \phi(s)\, ds$. Thus $Rx_5(t) = t\sigma^{-1}\int_1^\infty s^{-3}\psi(s)\, ds$, $\sigma^{-1}\int_1^\infty s^{-1}(s \wedge t)\psi(s)s^{-2}\, ds$ according as $0 \leqq t \leqq 1$, or $1 \leqq t < \infty$. Hence it follows from Theorem 2 that

$$\limsup_{n\to\infty} Df_5(H_n) = \sigma \quad \text{w.p. } 1 ;$$

furthermore, for large n $Df_5(H_n)$ is close to σ iff H_n is close to Rx_5.

EXAMPLE 6. Let $f_6(x) = \int_0^1 t^{-1}x(t)\, dt - x(1)$. Then $Df_6(x) = \int_1^\infty t^{-2}x(t)\, dt - x(1) = \int_1^\infty t^{-1}\, dx(t)$, $\sup_{x \in \mathbb{K}} f_6(x) = f_6(x_6)$ with $x_6(t) = t \log(1/t)$, 0 according as $0 \leqq t \leqq 1$ or $1 \leqq t < \infty$; and $\sup_{x \in \mathbb{K}} Df_6(x) = Df_6(Rx_6) = 1$ with $Rx_6(t) = 0$, $\log(t)$ according as $0 \leqq t \leqq 1$ or $1 \leqq t < \infty$. Note that

$$Df_6(H_n) = (n/2 \log\log n)^{\frac{1}{2}} \sum_{i=n+1}^\infty X_i \left(-\log\left(1 - \frac{1}{i}\right)\right),$$

and hence it follows from Theorem 2 that

$$\limsup_{n\to\infty} (n/2 \log\log n)^{\frac{1}{2}} \sum_{i=n+1}^\infty X_i \left(-\log\left(1 - \frac{1}{i}\right)\right) = 1 \quad \text{w.p. } 1 ;$$

and, for large n $Df_6(H_n)$ is close to 1 iff H_n is close to $Rx_6(t) = \log(t)$ on $[1, \infty)$. This example is related to a type of duality studied by Barbour (1974); see the discussion in the following section in this connection.

EXAMPLE 7. This final example illustrates the remark following Theorem 2 concerning R-invariant subsets of $\mathbb{K}$. Fix $\beta \in [-1, +1]$; for every ω in a set with probability one there is a subsequence $\{n'\} = \{n'(\omega)\}$ such that $\lim_{n'\to\infty} H_{n'}(1) = \beta$; further, w.p. 1, the functions $\{H_n\}_{n \in \{n'\}}$ are relatively compact with respect to ρ_0 and have limit set $\mathbb{K}_\beta \equiv \{x \in \mathbb{K} : x(1) = \beta\}$. Note that $R(\mathbb{K}_\beta) = \mathbb{K}_\beta$; i.e., $\mathbb{K}_\beta$ is R-invariant. Let $f_7(x) = \sup_{0 \leqq t \leqq 1} x(t)$ as in Example 2. Then $Df_7(x) = \sup_{1 \leqq t < \infty}(t^{-1}x(t))$ as before, but now we have $\sup_{x \in \mathbb{K}} f_7(x) = \sup_{x \in \mathbb{K}, x(1) = \beta} \{\sup_{0 \leqq t \leqq 1} x(t)\} = \frac{1}{2}(1 + \beta) = f_7(x_7)$ with $x_7(t) = t$, $1 + \beta - t$, β according as $0 \leqq t \leqq (1 + \beta)/2$, $(1 + \beta)/2 \leqq t \leqq 1$, or $1 \leqq t < \infty$. Hence $\sup_{x \in \mathbb{K}_\beta} Df_7(x) = \frac{1}{2}(1 + \beta) = Df_7(Rx_7)$ where $Rx_7(t) = \beta t$, $(1 + \beta)t - 1$, 1 according as $0 \leqq t \leqq 1$,

$1 \leqq t \leqq 2(1+\beta)^{-1}$, or $2(1+\beta)^{-1} \leqq t < \infty$, and it follows from Theorem 2 that

$$\limsup_{n \in \{n'\}} (n/2 \log\log n)^{\frac{1}{2}} \max_{n \leqq k < \infty} (k^{-1}S_k) = \tfrac{1}{2}(1+\beta) \quad \text{w.p. } 1 .$$

Similar questions, involving limit superiors along subsequences and consequent restrictions to the subsets $\mathbb{K}_\beta$ of $\mathbb{K}$, could be raised for each of the five other examples considered above.

4. Related work. A "weak" invariance theorem for the strong law of large numbers was proved by Müller (1968); one of his theorems asserts that the processes (H_n), with the $(2 \log\log n)^{\frac{1}{2}}$ factor omitted, converge weakly to Brownian motion B in (C_0, ρ_0). From this he deduced, via time inversion, that

$$\begin{aligned} \lim_{n\to\infty} P(n^{\frac{1}{2}} \max_{n \leqq k < \infty} k^{-1}S_k \leqq \lambda) &= P(\sup_{1 \leqq t < \infty} t^{-1}B(t) \leqq \lambda) \\ &= P(\sup_{0 \leqq t \leqq 1} B(t) \leqq \lambda) \\ &= (2/\pi)^{\frac{1}{2}} \textstyle\int_0^\lambda \exp(-u^2/2)\, du , \end{aligned}$$

which is the "weak" version of our Example 2; that

$$\begin{aligned} \lim_{n\to\infty} P(\max\{k: k^{-1}S_k \geqq \theta n^{-\frac{1}{2}}\} \leqq \lambda n) &= P(\sup\{t: B(t) \geqq \theta t\} \leqq \lambda) \\ &= P(\inf\{t: B(t) \geqq \theta\} \geqq \lambda^{-1}) \\ &= \theta \textstyle\int_0^\lambda (2\pi u)^{-\frac{1}{2}} \exp(-\theta^2 u/2)\, du , \end{aligned}$$

which is the "weak" version of our Example 3; and that

$$\begin{aligned} &\lim_{n\to\infty} P(n^{\frac{1}{2}} \max_{n \leqq k < \infty} (k^{-1}S_k) \geqq \alpha \mid n^{-1}S_n = n^{-\frac{1}{2}}\beta) \\ &\quad = P(\sup_{1 \leqq t < \infty} (t^{-1}B(t)) \geqq \alpha \mid B(1) = \beta) \\ &\quad = P(\sup_{0 \leqq t \leqq 1} B(t) \geqq \alpha \mid B(1) = \beta) \\ &\quad = \exp(-2\alpha(\alpha - \beta)) \qquad \text{for} \quad \alpha \geqq \beta \geqq 0 , \end{aligned}$$

which is the "weak" version of our Example 7. The present note was largely inspired by Müller's (1968) paper.

If $Y_1, Y_2, \cdots$ are independent rv's with mean 0 and $\mathrm{Var}(Y_n) = \sigma_n^2$, then it is well known that $\sum_{n=1}^\infty \sigma_n^2 < \infty$ implies that $\sum_{n=1}^\infty Y_n < \infty$ w.p. 1. Barbour (1974) studied the relationship between limit theorems for the "initial" sums $S_n = \sum_{i=1}^n Y_i$ and the "tail" sums $T_n = \sum_{i=n}^\infty Y_i$. His results depend on a duality induced by the map G from C_0 (say) to C_0 defined by

$$Gx(t) = tx(1/t) - \textstyle\int_{t^{-1}}^\infty u^{-2}x(u)\, du \qquad t > 0 .$$

G plays much the same role in Barbour's paper that R plays in ours; G preserves $\mathbb{K}$ and Brownian motion, is continuous, and $G \circ G$ is the identity. The duality theme of his paper is similar to ours, but the processes considered by Barbour are different than the processes considered here. It would be interesting to know of other $\mathbb{K}$-preserving mappings, and their interrelations and uses.

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REFERENCES

BARBOUR, A. D. (1974). Tail sums of convergent series of independent random variables. *Proc. Cambridge Philos. Soc.* **75** 361-364.

HARTMAN, P. and WINTNER, A. (1941). On the law of the iterated logarithm. *Amer. J. Math.* **63** 169-176.

HEYDE, C. C. and SCOTT, D. J. (1973). Invariance principles for the law of the iterated logarithm for martingales and processes with stationary increments. *Ann. Probability* **1** 428-443.

MÜLLER, D. W. (1968). Verteilungs-Invarianzprinzipien fur das starke Gesetz der graben Zahl. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **10** 73-92.

PHILIPP, W. and STOUT, W. (1975). Almost sure invariance principles for partial sums of weakly dependent random variables. *Mem. Amer. Math. Soc.* **161**.

STRASSEN, V. (1964). An invariance principle for the law of the iterated logarithm. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **3** 211-226.

WICHURA, M. J. (1974). Functional laws of the iterated logarithm for the partial sums of i.i.d. random variables in the domain of attraction of a completely asymmetric stable law. *Ann. Probability* **2** 1108-1138.

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