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ON STOPPING TIMES FOR n DIMENSIONAL BROWNIAN MOTION[1]

By Burgess Davis

Purdue University

Let $\bar{X}(t) = (X_1(t), \cdots, X_n(t))$ be standard n dimensional Brownian motion. Results of the following nature are proved. If τ is a stopping time for $\bar{X}(t)$ then $|\bar{X}(\tau)|$ and $(n\tau)^{\frac{1}{2}}$ are relatively close in L^p if n is large. Also, if n is large most of the moments $EX_i(\tau)^k$, $i = 1, 2, \cdots, n$, are about what they would be if $\bar{X}(t)$ were independent of τ.

1. Introduction. Let $\bar{X}(t) = (X_1(t), X_2(t), \cdots, X_n(t))$, $t \geqq 0$, be standard n dimensional Brownian motion, that is, $X_1(t), X_2(t), \cdots, X_n(t)$ are independent Wiener processes. This paper will be concerned with inequalities involving stopping times for $\bar{X}(t)$, usually with an eye for what happens as the dimension n approaches infinity. Several such results are already known. Let $\hat{X}_n(t) = \hat{X}(t) = |\bar{X}(t)/n^{\frac{1}{2}}|$. Then the law of large numbers gives that, for each fixed t, $\hat{X}(t) \to t^{\frac{1}{2}}$ in probability as $n \to \infty$, and it is easily shown that, for all $p > 0$, $E|\hat{X}(t) - t^{\frac{1}{2}}|^p \to 0$ as $n \to \infty$. Next we mention some recent results of D. L. Burkholder in [2], which were the motivation for this paper.

Let $\hat{X}(t)^* = \sup_{0 \leqq s \leqq t} \hat{X}(s)$. Burkholder shows that if n is large, then for any stopping time τ the L^p norms of $\tau^{\frac{1}{2}}$, $\hat{X}(\tau)^*$, and $\hat{X}(\tau)$ are all relatively close. That is, given $p > 0$, $\varepsilon > 0$, there is an integer N such that if τ is a stopping time for $\bar{X}(t)$ and $0 < E\tau^{p/2} < \infty$ then

$$(1.1) \qquad 1 - \varepsilon \leqq \frac{E(\hat{X}(\tau)^*)^p}{E\tau^{p/2}} \leqq 1 + \varepsilon\,, \qquad n \geqq N\,,$$

and

$$(1.2) \qquad 1 - \varepsilon \leqq \frac{E\hat{X}(\tau)^p}{E\tau^{p/2}} \leqq 1 + \varepsilon\,, \qquad n \geqq N\,.$$

Theorem 1.1, stated below, can be considered a refinement of these results since it can be used to show that the random variables $\tau^{\frac{1}{2}}$, $\hat{X}(\tau)^*$, and $\hat{X}(\tau)$ are all relatively close in L^p if n is large, that is, the L^p norm of the difference of any two is much smaller than the L^p norm of either one. This is discussed more fully in Section 2.

Theorem 1.1. *For each positive number p there is a constant C_p, not depending on n, such that for all stopping times τ,*

$$E \sup_{0 \leqq t \leqq \tau} |\hat{X}(t) - t^{\frac{1}{2}}|^p \leqq C_p n^{-p/2} E\tau^{p/2}\,.$$

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Let $\lambda(k) = 2^k k!/(2k)!$ be the inverse of the $2k$th moment of the standard normal distribution, k a nonnegative integer. It is easily shown that if $v \geqq 0$ is a random variable which is independent of $\bar{X}(t)$, $t \geqq 0$, then $\lambda(k)EX_i(v)^{2k} = Ev^k$, $i = 1, 2, \cdots, n$, and that $EX_i(v)^{2k+1} = 0$ provided $Ev^{(2k+1)/2} < \infty$. The following theorem says that, in some sense, stopping times for $\bar{X}(t)$ do not pay much attention to most of the components $X_i(t)$ of $\bar{X}(t)$, if n is large.

THEOREM 1.2. *Let k be a positive integer, and let $\varepsilon > 0$. There is an integer $j(k, \varepsilon) = j$ which does not depend on n such that, if τ is a stopping time for $\bar{X}(t)$, at least $n - j$ of the ratios $\lambda(k/2)EX_i(\tau)^k/E\tau^{k/2}$, $i = 1, 2, \cdots, n$, are within ε of 1, if k is even and $0 < E\tau^{k/2} < \infty$, and at least $n - j$ of the ratios $EX_i(\tau)^k/E\tau^{k/2}$ are within ε of 0 if k is odd and $0 < E\tau^{k/2} < \infty$.*

Of course, it is well known that if τ is a stopping time for $\bar{X}(t)$ then $E\tau^{\frac{1}{2}} < \infty$ implies $EX_i(\tau) = 0$ for each i, and that if $E\tau < \infty$ then $EX_i(\tau)^2 = E\tau$ for each i, so there is nothing new here for $k = 1,2$.

The following inequalities, due to Burkholder and P. W. Millar (for the exponents $p > 1$) and to Burkholder and R. F. Gundy (for the exponents $0 < p \leqq 1$) are well known. There are positive constants k_p, K_p, for each $p > 0$, such that if $Z(t)$, $t \geqq 0$, is standard Brownian motion and T is a stopping time for $Z(t)$ then

$$k_p ET^{p/2} \leqq E \sup_{0 \leqq t \leqq T} |Z(t)|^p \leqq K_p ET^{p/2} . \tag{1.3}$$

The paper [1] is a good reference for these and related inequalities. In Section 4 a simple proof of the left side of (1.3) for the exponents of $0 < p \leqq 2$ and the right-hand side for the exponents $0 < p \leqq 1$ is given. We remark that there are proofs of (1.3) for the exponents $p > 1$, including the original one, that do not extend to the exponents $0 < p \leqq 1$.

The proofs of Theorems 1.1 and 1.2 are based on Ito's stochastic calculus and the methods and results of Burkholder and Gundy which deal with one dimensional Brownian motion.

2. Inequalities for $|\bar{X}(t)|$. The integer n will always stand for the dimension of the Brownian motion $\bar{X}(t)$. If $Z(t)$, $t \geqq 0$, is a stochastic process (possibly multi-dimensional) we let $Z(t)^* = \sup_{0 \leqq s \leqq t} |Z(s)|$. We will always be operating with respect to the σ-fields $\mathscr{A}(s) = \sigma(\bar{X}(t), t \leqq s)$, and the statement that a process is a martingale will mean with respect to these σ-fields. Every Brownian motion $b(t)$, $t \geqq 0$, to be considered has the property that for each fixed s, $b(t + s) - b(s)$, $t \geqq 0$, is a standard Brownian motion independent of $\mathscr{A}(s)$. We will make frequent use of Ito's stochastic calculus. A reference for this is chapter two of McKean's book, [3].

For future reference we state now a generalization of (1.3), also due to Burkholder and Gundy (see [1]). Let Φ be any positive nondecreasing function on $[0, \infty)$ satisfying $\Phi(0) = 0$ and $\Phi(2\lambda) \leqq \alpha\Phi(\lambda)$ for some constant $\alpha_\Phi = \alpha$ and all positive numbers λ. Then there are constants k and K depending only on α

such that if $Z(t)$, $t \geqq 0$, is a Wiener process and τ is a stopping time for $Z(t)$ then

$$kE\Phi(\tau^{\frac{1}{2}}) \leqq E\Phi(Z(\tau)^*) \leqq KE\Phi(\tau^{\frac{1}{2}}) . \tag{2.1}$$

We will be particularly interested in the right hand side. The crucial inequality involved in the proof of this is (6.4) on page 26 of Burkholder's paper [2], which says that if $\beta > 1$, $\delta > 0$, and τ is a stopping time for $Z(t)$, $t \geqq 0$, then

$$P(Z(\tau)^* > \beta\lambda, \tau^{\frac{1}{2}} \leqq \delta\lambda) \leqq \frac{\delta^2}{(\beta-1)^2} P(Z(\tau)^* > \lambda) , \qquad \lambda > 0 . \tag{2.2}$$

The main theorem in this section is the following:

THEOREM 2.1. *Let Φ be as above. Then there is a constant A_Φ independent of n such that, if τ is a stopping time for $\bar{X}(t)$,*

$$E\Phi(\gamma(\tau)^*) \leqq A_\Phi E\Phi(\tau^{\frac{1}{2}}) , \qquad \textit{where} \quad \gamma(t) = |\bar{X}(t)| - ((n-1)t)^{\frac{1}{2}} . \tag{2.3}$$

PROOF. We will prove that, if $\beta > 1$ and $\delta > 0$,

$$P(\gamma(\tau)^* \geqq \beta\lambda, \tau^{\frac{1}{2}} \leqq \delta\lambda) \leqq \frac{\delta^2}{(\beta-1)^2} P(\gamma(\tau)^* \geqq \lambda) , \qquad \lambda > 0 . \tag{2.4}$$

If γ is replaced by Z this is identical to (2.2) except that some inequalities are strict in (2.2) but not in (2.4), which is of no consequence in the resulting integral inequality. Thus (2.3) follows from (2.4) exactly as the right side of (2.1) follows from (2.2).

For $n = 1$, (2.3) is the right side of (2.1). Thus we assume $n \geqq 2$. We use the notation of McKean's book [3]. See especially page 47. Thus let $r(t) = |\bar{X}(t)| = (\sum_{i=1}^n X_i(t)^2)^{\frac{1}{2}}$. The stochastic differential of r is

$$dr = da + (n-1)(2r)^{-1}\, dt ,$$

where $a(t)$ is the one dimensional Brownian motion given by

$$a(t) = \int_0^t \frac{\sum X_i(s)\, dX_i(s)}{r(s)} . \tag{2.5}$$

This is Problem 6 on page 47 of [3]. Of course the stochastic differential of the nonrandom function $((n-1)t)^{\frac{1}{2}}$ is

$$d((n-1)t)^{\frac{1}{2}} = ((n-1)^{\frac{1}{2}}/2t^{\frac{1}{2}})\, dt .$$

Thus $\gamma(t) = r(t) - ((n-1)t)^{\frac{1}{2}}$ satisfies

$$\begin{aligned} d\gamma &= da + [(n-1)(2r)^{-1} - (n-1)^{\frac{1}{2}}(2t^{\frac{1}{2}})^{-1}]\, dt \\ &= da - [(n-1)^{\frac{1}{2}}\gamma/2rt^{\frac{1}{2}}]\, dt . \end{aligned} \tag{2.6}$$

Now define, for $\lambda > 0$, $q_\lambda(x) = (|x| - \lambda)^2 I(|x| > \lambda)$. We will prove that the process $Q(t) = q_\lambda(\gamma(t)) - t$, $t \geqq 0$, is a supermartingale. To show this fix λ and define, for $0 < \varepsilon < \lambda$,

$$P_\varepsilon(x) = \frac{1}{2\varepsilon} \int_{\lambda-\varepsilon}^{\lambda+\varepsilon} q_\eta(x)\, d\eta .$$

It is readily checked that P_ε has continuous first and second derivatives. Now, using (2.6), Ito's lemma gives

$$dP_\varepsilon(\gamma) = P_\varepsilon'(\gamma)\,da + \tfrac{1}{2}P_\varepsilon''(\gamma)\,dt - P_\varepsilon'(\gamma)[(n-1)^{\frac{1}{2}}\gamma/2rt^{\frac{1}{2}}]\,dt\,,$$

so that, for $t > 0$,

$$d[P_\varepsilon(\gamma) - t] = P_\varepsilon'(\gamma)\,da + [\tfrac{1}{2}P_\varepsilon''(\gamma) - 1]\,dt + [-P_\varepsilon'(\gamma)(n-1)^{\frac{1}{2}}\gamma/2rt^{\frac{1}{2}}]\,dt\,.$$

Since both terms in brackets are never positive this gives that $P_\varepsilon(\gamma(t)) - t$, $t \geqq 0$, is a supermartingale, and since $P_\varepsilon(x) \to q_\lambda(x)$ uniformly on compact sets we have that $Q(t)$ is a supermartingale.

Now the proof of (2.4) and thus of Theorem 2.1 will be completed. If $a > 0$, let $\tau_a = \inf\{t > 0 : |\gamma(t)| = a\}$. Let $\sigma > 0$ and $y > 0$ be fixed numbers. Let $T = \tau_{\beta\lambda} \wedge y$, where $\wedge$ stands for minimum. Then $\hat{Q}(s) = Q(s \wedge T)$, $0 \leqq s < \infty$, is a bounded supermartingale, and so

$$E\hat{Q}(\tau \wedge (\tau_\lambda + \sigma)) \leqq E\hat{Q}(\tau \wedge \tau_\lambda)\,,$$

implying

$$\begin{aligned} &Eq_\lambda(\gamma(\tau \wedge \tau_{\beta\lambda} \wedge (\tau_\lambda + \sigma) \wedge y)) - Eq_\lambda(\gamma(\tau \wedge \tau_\lambda \wedge y)) \\ &\qquad \leqq E(\gamma(\tau \wedge \tau_{\beta\lambda} \wedge (\tau_\lambda + \sigma) \wedge y) - (E(\tau \wedge \tau_\lambda \wedge y)\,. \end{aligned}$$

Now $q_\lambda(\gamma(\tau_{\beta\lambda})) = ((\beta - 1)\lambda)^2$ while $q_\lambda(\gamma(t)) = 0$ if $t \leqq \tau_\lambda$, so that the above inequality gives

$$((\beta-1)\lambda)^2 P(\tau_{\beta\lambda} \leqq y,\, \tau_{\beta\lambda} \leqq \tau,\, \tau_{\beta\lambda} \leqq \tau_\lambda + \sigma) \leqq \sigma P(\tau_\lambda \leqq \tau \wedge y)\,.$$

Let $y \to \infty$. This yields

$$(\beta-1)^2\lambda^2 P(\tau_{\beta\lambda} \leqq \tau,\, \tau_{\beta\lambda} \leqq \tau_\lambda + \sigma) \leqq \sigma P(\tau_\lambda \leqq \tau)\,.$$

Now $\{\tau_a \leqq \tau\} = \{\gamma(\tau)^* \geqq a\}$ so that we have $(\beta-1)^2\lambda^2 P(\gamma(\tau)^* \geqq \beta\lambda, \tau \leqq \sigma) \leqq \sigma P(\gamma(\tau)^* \geqq \lambda)$, and letting $\sigma = \delta^2\lambda^2$ we get (2.4), completing the proof of the theorem.

We also have

$$\begin{aligned} (2.7) \qquad E\Phi(\sup\nolimits_{0\leqq t\leqq\tau} ||\bar{X}(t)| - (nt)^{\frac{1}{2}}|) &\leqq E\Phi(\gamma(\tau)^* + \tau^{\frac{1}{2}}) \\ &\leqq E\Phi(2\gamma(\tau)^*) + E\Phi(2\tau^{\frac{1}{2}}) \\ &\leqq \alpha_\Phi E\Phi(\gamma(\tau)^*) + \alpha_\Phi E\Phi(\tau^{\frac{1}{2}}) \\ &\leqq \alpha_\Phi(A_\Phi + 1)E\Phi(\tau^{\frac{1}{2}})\,. \end{aligned}$$

Taking $\Phi(x) = x^p$, (2.7) gives Theorem 1.1.

Note that

$$(2.8) \qquad ||\bar{X}(\tau)| - (n\tau)^{\frac{1}{2}}| \leqq \sup\nolimits_{0\leqq t\leqq\tau} ||\bar{X}(t)| - (nt)^{\frac{1}{2}}|\,,$$

and

$$(2.9) \qquad |\bar{X}(\tau)^* - (n\tau)^{\frac{1}{2}}| \leqq \sup\nolimits_{0\leqq t\leqq\tau} ||\bar{X}(t)| - (nt)^{\frac{1}{2}}|\,.$$

Using these inequalities and (2.7) it is easy to show that N exists so that (1.1) and (1.2) hold. For other interesting results related to these, see [2].

3. Proof of Theorem 1.2. Unless otherwise indicated, sums will be taken over $i = 1, 2, \cdots, n$, so that $\sum_{i=1}^{n}$ will be shortened to $\sum$. As before, n is the dimension of the Brownian motion. For each nonnegative integer k we define $G_{k,n}(s) = G_k(s)$ and $H_{k,n}(s) = H_k(s)$ by

$$G_k(s) = \quad = \sum X_i(s)^k\,, \qquad \text{and} \qquad H_k(s) = \int_0^s \sum X_i(t)^k\, dX_i(t)\,.$$

The following extension of (1.3), due to the same people responsible for (1.3), will be needed. Only the right hand side will be stated. Let $Z(t)$ be a Wiener process and $f(t, \omega)$ be a nonanticipating functional. Then, for each number $p > 0$, and all stopping times τ,

$$E \sup_{0 \leqq s \leqq \tau} \left|\int_0^t f(s, \omega)\, dZ(s)\right|^p \leqq K_p E\left(\int_0^\tau f(t, \omega)^2\, dt\right)^{p/2}, \tag{3.1}$$

where K_p is the same constant as in (1.3).

By Ito's lemma, for each integer $k \geqq 2$,

$$G_k(t) = kH_{k-1}(t) + \frac{k(k-1)}{2} \int_0^t G_{k-2}(s)\, ds\,. \tag{3.2}$$

The next lemma estimates the first of these components. The constants K_q are those of (1.3).

LEMMA 3.1. *Let j be a positive integer and $p \geqq 1$. Then, if τ is a stopping time for $\bar{X}(t)$,*

$$E \sup_{0 \leqq t \leqq \tau} |H_j(t)|^p \leqq g(p, n, j) E\tau^{p(j+1)/2}\,, \tag{3.3}$$

where $g(p, n, j) = K_{p(j+1)}^{j/(j+1)} n^{p/2}$ *if* $p \geqq 2$,

$$g(p, n, j) = K_{p(j+1)}^{j/(j+1)} n\,, \quad \text{if} \quad 1 < p \leqq 2\,, \qquad \text{and}$$

$$g(1, n, j) = (1 + K_{j+1}) n^{(2j+1)/(2j+2)}\,.$$

PROOF. It can be shown, using the same reasoning suggested by McKean to prove that the process $a(t)$ of (2.5) is a Brownian motion and that the equation preceding (2.5) holds, that $W(s) = \int_0^s (\sum X_i(t)^j dX_i(t)/G_{2j}(t)^{\frac{1}{2}})$ is a standard Brownian motion and that

$$H_j(t) = \int_0^t G_{2j}(s)^{\frac{1}{2}}\, dW(s)\,.$$

Applying (3.1) to the functional $G_{2j}(s)^{\frac{1}{2}}$ gives

$$E \sup_{0 \leqq t \leqq \tau} |H_j(t)|^p \leqq K_p E\left(\int_0^\tau G_{2j}(s)\, ds\right)^{p/2}. \tag{3.4}$$

Let $\gamma_i = \sup_{0 \leqq t \leqq \tau} |X_i(t)|$. Then

$$\begin{aligned} E\left(\int_0^\tau G_{2j}(s)\, ds\right)^{p/2} &= E\left(\sum \int_0^\tau X_i(s)^{2j}\, ds\right)^{p/2} \\ &\leqq E\left(\sum \tau\gamma_i^{2j}\right)^{p/2}. \end{aligned} \tag{3.5}$$

If $p \geqq 2$ we have, using (1.3),

$$\begin{aligned} E\left(\sum \tau\gamma_i^{2j}\right)^{p/2} &\leqq \left(\sum [E(\tau\gamma_i^{2j})^{p/2}]^{2/p}\right)^{p/2} \\ &\leqq n^{p/2} \max_{1 \leqq i \leqq n} E(\tau\gamma_i^{2j})^{p/2} \\ &\leqq n^{p/2} \max \left(E\tau^{p(j+1)/2}\right)^{1/(j+1)} \left(E\gamma_i^{p(j+1)}\right)^{j/(j+1)} \\ &\leqq n^{p/2} \left(E\tau^{p(j+1)/2}\right)^{1/(j+1)} \left(K_{p(j+1)} E\tau^{p(j+1)/2}\right)^{j/(j+1)} \\ &= n^{p/2} K_{p(j+1)}^{j/(j+1)} E\tau^{p(j+1)/2}\,. \end{aligned}$$

If $1 < p < 2$

$$\begin{aligned} E(\sum \tau\gamma_i^{2j})^{p/2} &\leqq \sum E(\tau\gamma_i^{2j})^{p/2} \\ &\leqq \sum (E\tau^{p(j+1)/2})^{1/(j+1)}(K_{p(j+1)}E\tau^{p(j+1)/2})^{j/(j+1)} \\ &= nK_{p(j+1)}^{j/(j+1)}E\tau^{p(j+1)/2} . \end{aligned}$$

If $p = 1$, we let $M = M(n)$ be for the moment an arbitrary positive number. Define $\alpha_i = \gamma_i I(\gamma_i \leqq M\tau^{\frac{1}{2}})$ and $\beta_i = \gamma_i I(\gamma_i > M\tau^{\frac{1}{2}})$. Then

$$\begin{aligned} E(\sum \tau\gamma_i^{2j})^{\frac{1}{2}} &= E(\sum \tau(\alpha_i^{2j} + \beta_i^{2j}))^{\frac{1}{2}} \\ &\leqq E(\sum \tau\alpha_i^{2j})^{\frac{1}{2}} + E(\sum \tau\beta_i^{2j})^{\frac{1}{2}} . \end{aligned} \tag{3.6}$$

Now

$$E(\sum \tau\alpha_i^{2j})^{\frac{1}{2}} \leqq E(\sum M^{2j}\tau^{j+1})^{\frac{1}{2}} = M^j n^{\frac{1}{2}} E\tau^{(j+1)/2} .$$

Also, for each $q > 0$,

$$M^q E\tau^{q/2} I(\gamma_i > M\tau^{\frac{1}{2}}) \leqq E\gamma_i^q \leqq K_q E\tau^{q/2} ,$$

so that

$$\begin{aligned} E(\sum \tau\beta_i^{2j})^{\frac{1}{2}} &= E(\sum \tau I(\gamma_i > M\tau^{\frac{1}{2}})\gamma_i^{2j})^{\frac{1}{2}} \\ &\leqq \sum E(\tau I(\gamma_i > M\tau^{\frac{1}{2}})\gamma_i^{2j})^{\frac{1}{2}} \\ &\leqq \sum [E(\tau I(\gamma_i > M\tau^{\frac{1}{2}}))^{(j+1)/2}]^{1/(j+1)}(E\gamma_i^{j+1})^{j/(j+1)} \\ &\leqq \sum K_{j+1}E\tau^{(j+1)/2}/M^{j+1})^{1/(j+1)}(K_{j+1}E\tau^{(j+1)/2})^{j/(j+1)} \\ &= nK_{j+1}E\tau^{(j+1)/2}/M . \end{aligned}$$

Taking $M = n^{1/(2j+2)}$ and using (3.6) we get the desired value for $g(1, n, j)$.

Define $\lambda(k)$ as in the introduction, and let $\lambda(0) = 1$.

LEMMA 3.2. *For each nonnegative integer k and each $p \geqq 1$ there are constants $C(p, n, k)$ which approach 0 as n approaches infinity such that if τ is a stopping time for $\bar{X}(t)$ then*

$$E \sup_{0\leqq t\leqq\tau} |\lambda(k)G_{2k}(t)n^{-1} - t^k|^p \leqq C(p, n, 2k)E\tau^{kp} , \tag{3.7}$$

and

$$E \sup_{0\leqq t\leqq\tau} |G_{2k+1}(t)/n|^p \leqq C(p, n, 2k+1)E\tau^{(2k+1)p/2} . \tag{3.8}$$

PROOF. Let $\Gamma(p, n, j)$ denote the smallest possible value for $C(p, n, j)$ such that (3.7) and (3.8) hold for all stopping times τ. Clearly $\Gamma(p, n, 0) = 0$ for all p and n. Using equation (3.2),

$$\begin{aligned} &(E \sup_{0\leqq t\leqq\tau} |\lambda(k)G_{2k}(t)n^{-1} - t^k|^p)^{1/p} \\ &\quad = (E \sup_{0\leqq t\leqq\tau} |2k\lambda(k)H_{2k-1}(t)n^{-1} + k(2k-1)\lambda(k)\textstyle\int_0^t G_{2k-2}(s)\,ds n^{-1} - t^k|^p)^{1/p} \\ &\quad \leqq (E \sup_{0\leqq t\leqq\tau} |2k\lambda(k)H_{2k-1}(t)/n|^p)^{1/p} \\ &\qquad + (E \sup_{0\leqq t\leqq\tau} |k(2k-1)\lambda(k) \textstyle\int_0^t G_{2k-2}(s)\,ds n^{-1} - t^k|^p)^{1/p} \\ &\quad = I + II . \end{aligned}$$

Using Lemma 3.1

$$I \leqq 2k\lambda(k)g(p, n, 2k-1)^{1/p}(E\tau^{kp})^{1/p}/n ,$$

while

$$\begin{aligned} II^p &= E\sup_{0\le t\le\tau} |k(2k-1)\lambda(k)\textstyle\int_0^t G_{2k-2}(s)\,dsn^{-1} - \int_0^t ks^{k-1}\,ds|^p \\ &\leqq E(\textstyle\int_0^\tau |k(2k-1)\lambda(k)G_{2k-2}(s)n^{-1} - ks^{k-1}|\,ds)^p \\ &\leqq E(\tau\sup_{0\le s\le\tau}|k(2k-1)\lambda(k)G_{2k-2}(s)n^{-1} - ks^{k-1}|)^p \\ &\leqq k^p(E\tau^{kp})^{1/k}(E\sup_{0\le s\le\tau}|\lambda(k-1)G_{2k-2}(s)n^{-1} - s^{k-1}|^{kp/k-1)})^{(k-1)/k} \\ &\leqq k^p(E\tau^{kp})^{1/k}\left(\Gamma\left(\frac{kp}{k-1}, n, 2k-2\right)E\tau^{kp}\right)^{(k-1)/k} \\ &= k^p\Gamma\left(\frac{kp}{k-1}, n, 2k-2\right)^{(k-1)/k}E\tau^{kp}, \quad k>1\,. \end{aligned}$$

Thus if $k=1$, $(I+II)^p = I^p \leqq (2k\lambda(k))^p g(p, n, 1)n^{-p}E\tau^{kp} = M(p, n, 2)E\tau^{kp}$, while

$$\begin{aligned} (I+II)^p &\leqq 2^p(I^p + II^p) \\ &\leqq \Big[2^p(2k\lambda(k))^p g(p, n, 2k-1)n^{-p} \\ &\qquad + 2^p k^p \Gamma\left(\frac{kp}{k-1}, n, 2k-2\right)^{(k-1)/k}\Big]E\tau^{kp} \\ &= M(p, n, 2k)E\tau^{kp}, \quad k>1\,. \end{aligned}$$

The rest of the proof of (3.7) is by induction. Since as $n\to\infty$, $g(p, n, 1)/n^p\to 0$ for each $p\geqq 1$, we have $M(p, n, 2)\to 0$ as $n\to\infty$, so that $\Gamma(p, n, 2)\to 0$ for each $p\geqq 1$, as $n\to\infty$. This fact and the fact that $g(p, n, 3)/n^p\to 0$ as $n\to\infty$ give that $\Gamma(p, n, 4)\to 0$ as $n\to\infty$ for each $p\geqq 1$, and so on.

The proof of (3.8) is similar. Since $G_1(s)/n^{\frac12}$ is standard Brownian motion,

$$E\sup_{0\le s\le\tau}\left|\frac{G_1(s)}{n}\right|^p \leqq n^{-p/2}K_pE\tau^{p/2},$$

so that $\Gamma(p, n, 1)\to 0$ as $n\to\infty$. If $k\geqq 1$,

$$\begin{aligned} &\left(E\sup_{0\le t\le\tau}\left|\frac{G_{2k+1}(t)}{n}\right|^p\right)^{1/p} \\ &\quad\leqq \frac{(2k+1)}{n}(E(H_{2k}(\tau)^*)^p)^{1/p} + \frac{(2k+1)k}{n}(E\sup_{0\le t\le\tau}|\textstyle\int_0^t G_{2k-1}(s)\,ds|^p)^{1/p} \\ &\quad= I + II\,. \end{aligned}$$

The first of these terms goes to 0 as $n\to\infty$ for each fixed k and p by Lemma 3.1, and

$$\begin{aligned} E&\sup_{0\le t\le\tau}|\textstyle\int_0^t G_{2k-1}(s)\,dsn^{-1}|^p \\ &\leqq E(\tau G_{2k-1}(\tau)^*/n)^p \\ &\leqq \Gamma((2k+1)p/(2k-1), n, 2k-1)^{(2k-1)/(2k+1)}E\tau^{(2k+1)p/2}, \end{aligned}$$

and (3.8) follows by induction.

PROOF OF THEOREM 1.2. Inequality (3.7) with $p = 1$ implies

$$|\lambda(k)EG_{2k}(\tau)n^{-1} - E\tau^k| \leqq C(1, n, 2k)E\tau^k .$$

If $j(1), j(2), \cdots, j(m)$ is any subset of $1, 2, \cdots, n$ then $\bar{Y}(t) = (X_{j(1)}(t), \cdots, X_{j(m)}(t))$ is standard m-dimensional Brownian motion, so that

$$(3.9) \qquad |m^{-1}\lambda(k) \textstyle\sum_{i=1}^m EX_{j(i)}(\tau)^{2k} - E\tau^k| \leqq C(1, m, 2k)E\tau^k .$$

This holds even though τ is a stopping time for $\bar{X}(t)$ and not $\bar{Y}(t)$, since $\bar{Y}(t+s) - \bar{Y}(s)$ is standard m-dimensional Brownian motion independent of $\mathscr{A}(s) = \sigma(\bar{X}(t), t \leqq s)$. (An alternative argument is to make rigorous the intuitively obvious fact that (3.9) holds conditionally given $\sigma(\bar{Z}(t), t \geqq 0)$, where $\bar{Z}(t)$ is the $n - m$ dimensional Brownian motion generated by those components of $\bar{X}(t)$ not in $\bar{Y}(t)$.) Thus if $N(\varepsilon)$ is the number of i such that $\lambda(k)EX_i(\tau)^{2k} > (1 + \varepsilon)E\tau^k$, we have by picking these i for $j(1), \cdots, j(m)$ in (3.9),

$$|(1 + \varepsilon)E\tau^k - E\tau^k| \leqq C(1, N(\varepsilon), 2k)E\tau^k .$$

Thus $N(\varepsilon)$ can be at most the largest integer l such that $C(1, l, 2k) \geqq \varepsilon$. Since $C(1, n, 2k) \to 0$ as $n \to \infty$ this is a finite integer, and we get the same estimate on the number of i such that $\lambda(k)EX_i(\tau)^{2k} < (1 - \varepsilon)E\tau^k$. The proof of the rest of Theorem 1.2 is similar.

4. A proof of some of (1.3). We first prove the right hand side. Let $\beta_p = E \sup_{0 \leqq t \leqq 1} |Z(t)|^p$. Note β_p is a lower bound for K_p. Let $\nu = \inf\{t \geqq T\colon t = 2^k$ for some integer $k\}$. Then $\nu \leqq 2T$, and $\sum_{k=-\infty}^{\infty} P(\nu = 2^k) = P(\nu > 0)$. We have

$$\begin{aligned} E(Z(T)^*)^p &\leqq E(Z(\nu)^*)^p \\ &\leqq \textstyle\sum_{k=-\infty}^{\infty} E \sup_{2^k \leqq s \leqq 2^{k+1}} |Z(s) - Z(2^k)|^p I(\nu > 2^k) \\ &\leqq \textstyle\sum_{k=-\infty}^{\infty} E(Z(2^k)^*)^p P(\nu > 2^k) \\ &= \textstyle\sum_{k=-\infty}^{\infty} \beta_p 2^{kp/2} P(\nu > 2^k) \\ &= \beta_p \textstyle\sum_{k=-\infty}^{\infty} (\sum_{j=-\infty}^{k-1} 2^{jp/2}) P(\nu = 2^k) \\ &= \beta_p (2^{-p/2})(1 - 2^{-p/2})^{-1} E\nu^{p/2} \\ &\leqq \beta_p (1 - 2^{-p/2})^{-1} ET^{p/2} . \end{aligned}$$

The left hand side is similar. Let $\mu(\lambda) = \inf\{t > 0\colon |Z(t)| = \lambda\}$ and let $\alpha_p = E\mu(1)^{p/2}$. Note $(\alpha_p)^{-1}$ is an upper bound for k_p. Let $\tau_i = \inf\{t > 0\colon |Z(t)| = 2^i\}$, and let $\eta = \inf\{t \geqq T\colon t = \tau_i$ for some integer $i\}$. Then $Z(\eta)^* \leqq 2Z(T)^*$ and $\sum_{i=-\infty}^{\infty} P(Z(\eta)^* = 2^i) = P(Z(\eta)^* > 0)$. Also, $\tau_{i+1} - \tau_i$ is smaller in distribution than $\mu(2^i 3)$. We have

$$\begin{aligned} ET^{p/2} &\leqq E\eta^{p/2} \\ &\leqq \textstyle\sum_{i=-\infty}^{\infty} E(\tau_{i+1} - \tau_i)^{p/2} I(\eta > \tau_i) \leqq \sum_{i=-\infty}^{\infty} E\mu(2^i 3)^{p/2} P(\eta > \tau_i) \\ &= \textstyle\sum_{i=-\infty}^{\infty} \alpha_p (2^i 3)^p P(Z(\eta)^* > 2^i) = \alpha_p 3^p 2^{-p} (1 - 2^{-p})^{-1} E(Z(\eta)^*)^p \\ &\leqq \alpha_p 3^p (1 - 2^{-p})^{-1} E(Z(T)^*)^p . \end{aligned}$$

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STATISTICS DEPARTMENT
PURDUE UNIVERSITY
WEST LAFAYETTE
INDIANA 47907