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DUAL PAIRS OF STOPPING TIMES FOR RANDOM WALK

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A definition of duality for pairs of stopping times of any random walk is motivated by the duality relation of ascending and descending ladder epochs N, $\bar{N}$ of random walk in R^1. Dual pairs share several of the properties of the pair N, $\bar{N}$.

I. Introduction. Sparre-Andersen [9] introduced a combinatorial method for studying the fluctuations of a random walk. A main idea was that certain sets of paths, viewed with the direction of time reversed, are more easily described or counted, while a set of paths viewed in either direction has the same probability. The relation between a set of paths and the reversed set is called duality. Lindley [7] found that the virtual waiting time in a one-server queue has the same distribution as the maximum of a random walk because of the duality relation. Spitzer used duality to obtain a transform formula for the maximum of a random walk [10] and to give a probabilistic solution of the Wiener–Hopf integral equation on a half-line [11]. The duality relation is the key to Spitzer and Pollaczek's factorization formula for a distribution on R^1 (see, e.g., Feller [2]). The duality relation, which was exploited in these works and in the large literature which followed, can be stated in terms of the first ascending and first descending ladder epochs of a random walk S_n in R^1,

$$N = \min(n > 0 : S_n > 0), \qquad \bar{N} = \min(n > 0 : S_n \leqq 0).$$

A time n is an ascending ladder epoch if S_n reaches a new maximum at n, which is to say that n is obtained by some number of repetitions of N. The duality relation is: the set of paths for which n is an ascending ladder epoch, viewed in reverse, is the set of paths such that $n < \bar{N}$.

In this paper a definition of duality for a pair of stopping times of any random walk is stated, and some properties of dual pairs are explored. In another paper [5], duality for particular pairs of stopping times for multivariate random walks is used to begin a fluctuation theory for these processes, somewhat analogous to the well-known theory for random walk in R^1. The multivariate fluctuation theory will have applications for certain storage and queueing systems.

II. Definitions and notations. Let $S_n = \sum_{i=1}^n X_i$, $S_0 = 0$, where X_i, $i = 1, 2, 3, \cdots$ is a sequence of independent identically distributed random elements. For most of what follows the range of the X_i, denoted by $\mathbf{R}$, may be any topological group. We have in mind R^d. It is convenient to use the sequence

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space $\mathbf{R}^\infty = \{(x_1, x_2, \cdots), x_i \in \mathbf{R}\}$ as the Ω of our probability space $(\Omega, \mathscr{F}, P)$. The words "for each n" will mean for each nonnegative integer.

For each n let r_n denote the map on Ω defined by $r_n(\omega) = r_n(x_1, \cdots, x_n, x_{n+1}, \cdots) = (x_n, \cdots, x_1, x_{n+1}, \cdots)$. The mapping r_n preserves set operations and measure, since $X_1, \cdots, X_n$ are exchangeable. Note that r_n and r_n^{-1} are the same map.

For each k we denote by θ_k the map on Ω defined by $\theta_k(x_1, x_2, \cdots) = (x_{k+1}, x_{k+2}, \cdots)$. If, for instance, τ is a stopping time for S_n, $\tau \circ \theta_k$ is the same stopping time for $S_{n+k} - S_k$. For clarity we sometimes write $\tau\omega$ to denote τ evaluated on the sequence $\omega \in \Omega$.

Let τ be a stopping time for S_n, $\tau_0 = 0$, $\tau_1 = \tau$, $\tau_2 = \tau_1 + \tau'$ where $\tau'\omega = \tau \circ \theta_{\tau_1}\omega$, and so on. We denote by $\mathscr{M}_\tau$ the random set $(\tau_0, \tau_1, \tau_2, \cdots)$. The set $\mathscr{M}_\tau$ is almost surely finite if $P(\tau < \infty) < 1$. By an occurrence of τ at n we mean $(n \in \mathscr{M}_\tau)$. By $\mathscr{M}_\tau \theta_k$ we will mean the random set $\mathscr{M}_\tau$ for $\theta_k(\omega)$, $\omega \in \Omega$.

Let τ and η be stopping times for S_n. We say that τ is dual to η if

$$(1) \qquad (\omega : n \in \mathscr{M}_\tau \omega) = (\omega : n < \eta r_n \omega) \qquad \text{for each} \quad n\,.$$

If τ, τ' are dual to η, then $\mathscr{M}_\tau = \mathscr{M}_{\tau'}$, and $\tau = \tau'$. Also τ is dual to at most one η. Since (1) is not symmetric in τ and η, it is not evident that if (1) holds then η is also dual to τ. This will be proved in Theorem 1.

For each n let $L(\tau, n) = \max(i \leqq n : i \in \mathscr{M}_\tau)$. The notation $=_d$ means that the two random variables have the same distribution. These notations when composed have the obvious meanings.

III. Duality and related properties. J. Pitman suggested proving (2) by looking back from n to $L(\tau, n)$ as we do below. This is a considerable simplification of our original method.

THEOREM 1. *Let τ and η be stopping times for a random walk $S_n = \sum_{i=1}^n X_i$. If τ is dual to η, then η is dual to τ, and for each n*

$$(2) \qquad L(\tau, n) =_d n - L(\eta, n) \qquad \text{and} \qquad \textstyle\sum_{i=1}^{L(\tau,n)} X_i =_d \sum_{i=L(\eta,n)+1}^{n} X_i\,.$$

PROOF. We fix n and ω, and show that the number $n - L(\tau, n)$ is the same as the number $L(\eta, n)$ evaluated for the same sequence reversed from n. Let $q = L(\tau, n)\omega$, $m = n - q$. We wish to show that $m = L(\eta, n)r_n\omega$. Since $q \in \mathscr{M}_\tau\omega$, relation (1) says $q < \eta r_q \omega$. The first $n - m$ terms of the sequences $\theta_m r_n \omega$ and $r_{n-m}\omega$ are the same. Replacing q by $n - m$ and then $r_{n-m}\omega$ by $\theta_m r_n \omega$, we have $n - m < \eta\theta_m r_n \omega$. If also $m \in \mathscr{M}_\eta r_n \omega$, then m must be the largest number in this set, i.e., $m = L(\eta, n)r_n\omega$.

To show that $m \in \mathscr{M}_\eta r_n \omega$ we will prove the apparently stronger statement, $m = L(\eta, m)r_n\omega$. Let $k = L(\eta, m)r_n\omega$. From the definition of $L(\eta, m)$ we have $k \leqq m$ and $m - k < \eta\theta_k r_n \omega$. The first $m - k$ terms of the sequences $\theta_k r_n \omega$ and $r_{m-k}\theta_q\omega$ are the same since $n - k = q + m - k$. Replacing $\theta_k r_n \omega$ by $r_{m-k}\theta_q\omega$ gives us

$$m - k < \eta r_{m-k}\theta_q\omega\,.$$

The hypothesis, (1), with $m - k$ in place of n and $\theta_q \omega$ in place of ω, now says that $m - k \in \mathscr{M}_\tau \theta_q \omega$. But since $k = L(\eta, m) r_n \omega$, $m - k$ must be 0.

Having observed that $n - L(\tau, n)\omega = L(\eta, n) r_n \omega$, we see also that r_n maps the left sum in (2) to the right sum. The distribution equalities hold, since r_n preserves measure. In particular, $(\omega : n \in \mathscr{M}_\eta \omega) = (\omega : L(\eta, n)\omega = n)$ is mapped by r_n to $(\omega : L(\tau, n) = 0) = (\omega : n < \tau\omega)$, so η is dual to τ.

It is useful to know which stopping times have duals. There follows a characterization of such stopping times τ in terms of a property of the random set $\mathscr{M}_\tau$. Often a computation will show for a certain stopping time τ that $\mathscr{M}_\tau$ does not have this property. Some examples are given in Section V.

THEOREM 2. *A stopping time τ has a dual if and only if for each n and $\omega \in \Omega$,*

$$n \in \mathscr{M}_\tau \omega \quad \textit{implies} \quad j \in \mathscr{M}_\tau \theta_{n-j} \omega , \qquad j = 1, \cdots, n . \tag{3}$$

PROOF. Suppose τ has a dual, η. Following Theorem 1, the duality relation holds in both directions. If $n \in \mathscr{M}_\tau \omega$ then $n < \eta r_n \omega$, and for each $j = 1, \cdots, n$, $j < \eta r_n \omega$. The definition of "τ is dual to η," applied to the sequence $\theta_{n-j}\omega$, says $j < \eta r_n \omega$ if and only if $j \in \mathscr{M}_\tau \theta_{n-j} \omega$.

Suppose τ satisfies (3). Define sets $A_n = (\omega : n \in \mathscr{M}_\tau r_n \omega)$. A stopping time η is specified by $A_n = (\omega : n < \eta\omega)$, if the A_n are determined by $X_1, \cdots, X_n$ and are nonincreasing in n. The first condition is clear. Now $A_n = \bigcap_{j=1}^n (\omega : j \in \mathscr{M}_\tau \theta_{n-j} r_n \omega)$, since by (3) each of these sets is contained in the last. Each set is the same if $\theta_{n-j} r_n \omega$ is replaced by $r_j \omega$,

$$A_n = \bigcap_{j=1}^n (\omega : j \in \mathscr{M}_\tau r_n \omega) .$$

A sequence formed by intersections is nonincreasing.

Here is a way to construct new dual pairs from known ones. It is a corollary to Theorem 1 only in that we may now speak of dual pairs of stopping times.

COROLLARY 1. *Suppose that γ, $\bar{\gamma}$ are dual stopping times and that η, $\bar{\eta}$ are dual stopping times. Then $\tau = \min n \in \mathscr{M}_\gamma \cap \mathscr{M}_\eta$ is dual to $\bar{\tau} = \min(\bar{\gamma}, \bar{\eta})$.*

PROOF. From its definition, $\mathscr{M}_\tau = \mathscr{M}_\gamma \cap \mathscr{M}_\eta$. Then

$$\begin{aligned}(\omega : n \in \mathscr{M}_\tau r_n \omega) &= (\omega : n < \min i \in \mathscr{M}_{\bar{\gamma}} \omega) \cap (\omega : n < \min i \in \mathscr{M}_{\bar{\eta}} \omega) \\ &= (\omega : n < \min i \in \mathscr{M}_{\bar{\gamma}} \omega \cup \mathscr{M}_{\bar{\eta}} \omega) \\ &= (\omega : n < \min(\bar{\gamma}\omega, \bar{\eta}\omega)) .\end{aligned}$$

Some examples are constructed using Corollary 1 in Section V.

IV. A Spitzer-Pollaczek factorization and consequences. For any dual pair of stopping times, τ and η, a factorization formula for the distribution F of X_i in terms of the distribution of S_τ and S_η can be obtained by a slight modification of the argument given by Feller [2], XII.3. We prefer to give a new proof which avoids several definitions and sums. The following lemma is similar to what Feller calls the "basic duality lemma" involving N and $\bar{N}$ in [2], XII.2, and the proof is nearly the same. We use A to denote a measurable set in **R**, and δ is a unit mass at 0.

LEMMA 1. *Let η, τ be dual stopping times for a random walk S_n. Let T be an independent geometrically distributed random variable, $P(T \geqq n) = u^n$ where $0 < u \leqq 1$. Let*

$$H_{\eta,u}(A) = P(S_\eta \in A, \eta \leqq T), \qquad G_{\tau,u}(A) = \sum_{n=0}^{\infty} P(S_n \in A, n < \tau, n \leqq T).$$

Then

$$G_{\tau,u} = \sum_{n=0}^{\infty} H_{\eta,u}^{n*}. \qquad \text{If} \quad u < 1, \quad (\delta - H_{\eta,u}) * G_{\tau,u} = \delta. \tag{4}$$

THEOREM 3. *If η and τ are dual stopping times then*

$$\delta - uF = (\delta - H_{\tau,u}) * (\delta - H_{\eta,u}), \qquad 0 < u \leqq 1. \tag{5}$$

Here, $H_{\tau,u}(A) = P(S_\tau \in A, \tau \leqq T)$, $H_{\eta,u}(A) = P(S_\eta \in A, \eta \leqq T)$, T is independent of the random walk, and $P(T \geqq n) = u^n$.

PROOF. Let $u < 1$, and consider the distribution of S_T on the ω-set where $T \leqq \tau$, in terms of whether $T = 0$ or $T \geqq 1$,

$$\begin{aligned} P(S_T \in A, T \leqq \tau) &= (1-u)\,\delta(A) + uP(S_T \in A, T \leqq \tau \mid T \geqq 1) \\ &= (1-u)\,\delta(A) + uP(S_{T+1} \in A, T + 1 \leqq \tau) \\ &= (1-u)\,\delta(A) + uK_{\tau,u} * F(A), \end{aligned}$$

where $K_{\tau,u}(A) = P(S_T \in A, T < \tau)$. On the other hand,

$$P(S_T \in A, T \leqq \tau) = K_{\tau,u}(A) + (1-u)H_{\tau,u}(A).$$

Equate the two versions and rearrange terms to obtain

$$\frac{K_{\tau,u}}{1-u} * (\delta - uF) = \delta - H_{\tau,u}. \tag{6}$$

Notice that $K_{\tau,u}/(1-u) = G_{\tau,u}$ as defined in Lemma 1. Convolve both sides of (6) with $\delta - H_{\eta,u}$ to obtain (5) for $u < 1$. Let u go to 1 to obtain (5) with $u = 1$.

Relation (6), which holds for any stopping time τ, is well known in the form involving $G_{\tau,u}$ and with $u = 1$, e.g., Arjas [1]. The above method may be useful to obtain more easily the fundamental identity of [1] for semi-Markov processes.

Applying transforms to (5) gives

$$(1 - u\phi(\xi)) = (1 - \psi_\tau(u, \xi))(1 - \psi_\eta(u, \xi)), \tag{7}$$

where $\phi(\xi) = E \exp i\xi \cdot X_1$, $\psi_\tau(u, \xi) = Eu^\tau \exp i\xi \cdot S_\tau$, $\psi_\eta(u, \xi) = Eu^\eta \exp i\xi \cdot S_\eta$.

A number of properties of the pair N, $\bar{N}$ follow from identity (5) or (7). Such properties are shared by any dual pair τ, η. The following four corollaries list some of these.

COROLLARY 2. *Let* $g_\eta(u) = Eu^\eta$, $g_\tau(u) = Eu^\tau$. *Then for* $0 \leqq u \leqq 1$,

$$1 - u = (1 - g_\eta(u))(1 - g_\tau(u)) . \tag{8}$$

PROOF. Let $\xi = 0$ in (7).

COROLLARY 3. *Either* τ, η *are both proper and* $E\eta = E\tau = \infty$, *or* η *is defective and* $E\tau = 1/(1 - P(\eta < \infty))$ *or* τ *is defective and* $E\eta = 1/(1 - P(\tau < \infty))$.

PROOF. From (8), $(1 - g_\tau(u))/(1 - u) = (1 - g_\eta(u))^{-1}$. Let $u \to 1$.

COROLLARY 4. *If* S_n *is random walk in* R^1 *and if* ES_τ, ES_η *are finite, then* $EX_1 = 0$ *and* $EX_1^2 = -2ES_\tau ES_\eta$.

PROOF. See Feller XVIII.4, Lemma 3.

COROLLARY 5. *The following are equivalent independent decompositions of the distribution of* S_T:

$$S_T =_d Y_\tau(T_\tau) + Y_\eta(T_\eta) \tag{9}$$

$$S_T =_d S(L(\tau, T)) + S(L(\eta, T)) . \tag{10}$$

In (9), T_τ, T_η are independent, $P(T_\tau \geqq n) = P(\tau \leqq T)^n$, $P(T_\eta \geqq n) = P(\eta \leqq T)^n$, and Y_τ, Y_η are random walks with step distributions $P(S_\tau \in A \mid \tau \leqq T)$, $P(S_\eta \in A \mid \eta \leqq T)$.

PROOF. Relation (9) is equivalent to (5) as for N, $\bar{N}$ in Greenwood [3], equation 5.4. Relation (10) is also equivalent as in Greenwood [4].

The factorization (5) can be repeated. Consider the random walk Y_n which is S_n restricted to the time set $\mathscr{M}_\tau$, i.e., $Y_n = S_{\tau_n}$. The killing time for Y_n, if $F(\mathbf{R}) = u < 1$, is $T_Y + 1$, where $T_Y = \max n : \tau_n \leqq T$, and T is as defined earlier. If γ_1, γ_2 are dual stopping times for Y_n, (5) is

$$\delta - H_\tau = (\delta - H_{\tau,\gamma_1}) * (\delta - H_{\tau,\gamma_2}) \tag{11}$$

where

$$\begin{aligned} H_{\tau,\gamma_i}(A) &= P(Y_{\gamma_i} \in A, \gamma_i \leqq T_Y) \\ &= P(S(\tau_{\gamma_i}) \in A, \tau_{\gamma_i} \leqq T) , \qquad i = 1, 2 . \end{aligned}$$

Repeated factorization is used in [5] to obtain F in terms of distributions concentrated on each element of a partition of R^2 into convex cones.

If τ is defective, the random walk can be evaluated at the last occurrence of τ. Let $L_\tau = \max n \in \mathscr{M}_\tau$. Then

$$S(L_\tau) = \textstyle\sum_{\tau_i \in \mathscr{M}_\tau} S(\tau_{i+1}) - S(\tau_i) .$$

The terms of this sum are independent and distributed like S_τ.

The distribution of $S(L_\tau)$ is $P(\tau = \infty) \sum_{n=0}^\infty H_\tau^{n*}$ where H_τ is the distribution of S_τ. If η is dual to τ, this distribution can be written in terms of η, using Lemma 1 and Corollary 3.

COROLLARY 6. *If τ is defective and has dual η, the distribution of $S(L_\tau)$ is $(E\eta)^{-1}G_\eta$ where $G_\eta(A) = \sum_{n=0}^{\infty} P(S_n \in A, n < \eta)$.*

Suppose now that τ is proper. Recall the notation $L(\tau, n) = \max(i \leqq n : i \in \mathscr{M}_\tau)$. It is known from renewal theory that $(n - L(\tau, n))/n$ has a limiting distribution if and only if the distribution of τ is regularly varying, that is $P(\tau > n) \sim n^{-\alpha}\mathscr{L}(n)$, $0 < \alpha < 1$, where $\mathscr{L}(mn)/\mathscr{L}(n) \to 1$ as $n \to \infty$ (see, e.g., Feller [2], XIV.3.). Then the limiting distribution has arcsine density

$$q_\alpha(x) = \frac{\sin \pi\alpha}{\pi} x^{-\alpha}(1 - x)^{\alpha-1}, \qquad 0 \leqq x \leqq 1 .$$

If τ has a dual, the same kind of statement can be made about $L(\tau, n)/n$. The following corollary has an analogue in [5] for certain finite families of stopping times.

COROLLARY 7. *If τ and η are dual and the distribution of either has a regularly varying tail, then $L(\tau, n)/n$ has a limiting arcsine distribution.*

PROOF. If one of τ, η has a regularly varying tail with exponent α, the other is regularly varying with exponent $1 - \alpha$. This follows from (8) and a Tauberian theorem, e.g., Feller [2], XIII.5, Theorem 5. By the result quoted above, $(n - L(\eta, n))/n$ has a limiting arcsine distribution. But $n - L(\eta, n) =_d L(\tau, n)$, according to (2).

Some properties of the pair N, $\bar{N}$ are not shared by all dual pairs of stopping times. For instance if S_n is in R^d, Baxter's equation,

$$\log(1 - \phi_\tau(u, \xi))^{-1} = \sum_{n=1}^{\infty} \phi_\tau{}^n/n = \sum_{n=1}^{\infty} \frac{u^n}{n} \int_{C_\tau} e^{i\xi\cdot x} F^{n*}(dx) ,$$

is true only when the supports C_τ, C_η, of S_τ, S_η lie in complementary half-spaces.

Kingman [6] observed that Baxter's equation cannot be generalized in any essential way. However, by allowing more than two factors in (7), hence more than two cones, Baxter's equation is generalized in [5].

V. Examples. Let S_n be random walk in R^1.

1. Let $b > 0$. Let $\eta = \min(n: X_n > b)$. Let $\tau = 1$ if $X_1 \leqq b$, $\tau = \infty$ if $X_1 > b$. Then $(n \in \mathscr{M}_\eta) = (X_n > b)$, $(n \in \mathscr{M}_\eta \circ r_n) = (X_1 > b) = (n < \tau)$ for each $n > 0$, so τ and η are dual.

2. Let η and τ be as in Example 1. Then τ and η are dual, also N and $\bar{N}$ are dual. By Corollary 1, $\gamma = \min n \in \mathscr{M}_\tau \cap \mathscr{M}_N$ is dual to $\bar{\gamma} = \eta \wedge \bar{N}$. The stopping time $\bar{\gamma}$ is the first time S_n is $\leqq 0$ or has a step $> b$. Its dual is $\gamma = \min(n: n = N, X_i \leqq b, i = 1, \cdots, N)$, $\gamma = \infty$ otherwise.

3. With the same η and τ, Corollary 1 gives the additional dual pairs $\beta = \min n \in \mathscr{M}_\tau \cap \mathscr{M}_{\bar{N}}$, $\bar{\beta} = \eta \wedge N$ and $\alpha = \min n \in \mathscr{M}_\eta \cap \mathscr{M}_N$, $\bar{\alpha} = \tau \wedge \bar{N}$. The remaining possibility is not interesting.

4. Let $\eta = N$ restricted to the ω-set $(S_i \geqq S_N - b, i = 0, 1, \cdots, N)$. Let

$N_b = \min(n$: there exists $i \leqq n$ such that $S_i - S_{m_i} \geqq b)$ where $m_i = \max(j < i$: S_m is minimal over $(S_m, \cdots, S_n))$. Let $\tau = \bar{N} \wedge N_b$. Then $(n \in \mathscr{M}_\eta r_n) = (n < \tau)$. The verification is notationally, but not conceptually, difficult. The following are some stopping times without duals.

5. Let $\eta > 1$ a.s., e.g., $\eta = \min(n > 0 : S_n = 0)$ for random walk with steps of 1 or -1. If η has a dual τ, then $(\eta > 1) = (\tau = 1)$. But $\tau = 1$ a.s., with duality, implies $n < \eta$ a.s. for all n, so $\eta = \infty$ a.s.

6. Let $N_b = \min(n : S_n > b)$. It happens, with positive probability, that $n \in \mathscr{M}_{N_b}$ but $X_n \leqq b$ so that $N_b \circ \theta_{n-1} > 1$. By Theorem 2, N_b does not have a dual. Examples for random walks in R^2 are in [5].

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