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ATTRACTIVE NEAREST NEIGHBOR SPIN SYSTEMS ON THE INTEGERS[1]

By Thomas M. Liggett

University of California, Los Angeles

We prove that an attractive nearest neighbor spin system on the integers has at most two extremal invariant measures, provided that its flip rates satisfy a mild positivity condition.

1. Introduction. Stochastic spin systems have been studied extensively in recent years (see [11] for a survey of results and a bibliography). Many of the results have been concerned with the problem of determining when such a system is ergodic, i.e., when it has a unique invariant measure to which the distribution of the process at time t converges as t tends to infinity. Relatively few of the results have described the behavior of the system when it is not ergodic. Among the papers which do consider the nonergodic case are [1], [3], [4] and [6]. In the present paper, we will prove that certain processes can have at most two extremal invariant measures. The proof will be based on coupling techniques similar to some of those which were used in [10] to determine the set of all invariant measures for certain asymmetric exclusion processes. One of our purposes is to illustrate the application of these techniques to spin systems.

In order to describe the processes we will consider, let $X = \{0, 1\}^Z$ with the product topology, where Z is the set of integers, and let $c(x, \eta)$ be a nonnegative translation invariant function on $Z \times X$ which satisfies:

(1.1) Nearest neighbor assumption: $c(x, \eta)$ depends on η only through $\eta(x)$, $\eta(x-1)$ and $\eta(x+1)$;

(1.2) Attractiveness assumption: if $\eta \leqq \zeta$, then $c(x, \eta) \leqq c(x, \zeta)$ for $\eta(x) = \zeta(x) = 0$, and $c(x, \eta) \geqq c(x, \zeta)$ for $\eta(x) = \zeta(x) = 1$;

(1.3) Positivity assumption: $c(x, \eta) + c(x, \eta_x) > 0$ for each η, where $\eta_x(x) = 1 - \eta(x)$, and $\eta_x(y) = \eta(y)$ for $y \neq x$.

The spin system η_t with flip rates $c(x, \eta)$ is the Feller process on X whose generator is the closure in $C(X)$ of the operator

$$\Omega f(\eta) = \sum_x c(x, \eta)[f(\eta_x) - f(\eta)],$$

which is defined for functions f which depend on finitely many coordinates. The existence and uniqueness of this process was proved in [9]. $S(t)$ will denote

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both the corresponding semigroup on $C(X)$, and the dual semigroup on the space of probability measures on X.

The attractiveness assumption, which was introduced and exploited by Holley in [5], leads to several important monotonicity properties. Let ν_0 and ν_1 be the pointmasses on $\eta \equiv 0$ and $\eta \equiv 1$ respectively. Given two probability measures μ_1 and μ_2 on X, say that $\mu_1 \leqq \mu_2$ if there is a probability measure ν on $X \times X$ with marginals μ_1 and μ_2 respectively such that $\nu\{(\eta, \zeta) : \eta \leqq \zeta\} = 1$. Then Holley used coupling techniques to show that $\nu_0 S(t)$ increases and $\nu_1 S(t)$ decreases in t, and therefore that the weak limits $\bar{\nu}_0 = \lim_{t\to\infty} \nu_0 S(t)$ and $\bar{\nu}_1 = \lim_{t\to\infty} \nu_1 S(t)$ exist and are invariant for the process. Furthermore, he showed that if $\bar{\nu}_0 = \bar{\nu}_1$, then the process is ergodic. In fact, if μ is any invariant measure, then $\bar{\nu}_0 \leqq \mu \leqq \bar{\nu}_1$.

Let $\mathscr{I}$ be the set of probability measures on X which are invariant for the process, and let $\mathscr{I}_e$ be the set of extreme points of $\mathscr{I}$. It is clear that $\bar{\nu}_0, \bar{\nu}_1 \in \mathscr{I}_e$, since for example if $\bar{\nu}_1 = \alpha\mu_1 + (1 - \alpha)\mu_2$ for $\mu_1, \mu_2 \in \mathscr{I}$ and $0 < \alpha < 1$, then $\mu_1 \leqq \bar{\nu}_1$ and $\mu_2 \leqq \bar{\nu}_1$, so $\mu_1 = \mu_2 = \bar{\nu}_1$. Our result then is

THEOREM 1.4. *Under assumptions* (1.1), (1.2) *and* (1.3), $\mathscr{I}_e = \{\bar{\nu}_0, \bar{\nu}_1\}$.

This theorem is of relatively minor interest in case $c(x, \eta) > 0$ for all η, since then it is probably true that $\bar{\nu}_0 = \bar{\nu}_1$, although the proof of this remains an open problem. If $c(x, \eta)$ is zero for some η, on the other hand, then it is possible for $\bar{\nu}_0$ and $\bar{\nu}_1$ to be different, and it is in that case that the result is most interesting. One simple case to which our theorem applies where it is easy to see that $\bar{\nu}_0 \neq \bar{\nu}_1$ is that in which $c(x, \eta) = 0$ for $\eta \equiv 0$ and $\eta \equiv 1$, since then $\bar{\nu}_0 = \nu_0$ and $\bar{\nu}_1 = \nu_1$. The nearest neighbor one dimensional voter model [6] is one such case in which the conclusion of Theorem 1.4 has been obtained using other techniques. A more interesting class of examples is provided by Harris' contact processes [2], in which $c(x, \eta) = 1$ if $\eta(x) = 1$ and $c(x, \eta) = \lambda[\eta(x - 1) + \eta(x + 1)]$ (the symmetric case) or $c(x, \eta) = \lambda\eta(x + 1)$ (the one-sided case) if $\eta(x) = 0$. Then $\bar{\nu}_0 = \nu_0$, of course, and $\bar{\nu}_1 \neq \nu_0$ for sufficiently large values of λ ([2], [4], [7]). It is in this context that Griffeath [1] raised several questions which are partially answered by our result.

In [1] and [4], most results concerning the nonergodic case were proved under the assumption that λ be sufficiently large, and did not cover all values of λ for which the process is nonergodic. (Similar assumptions occur frequently in the statistical mechanics literature.) One interesting feature of our theorem is therefore that it applies to all values of the parameter. In [1], Griffeath proved a stronger result than that in Theorem 1.4 for the symmetric contact process with λ so large that the one-sided contact process is not ergodic. In [3], Harris proved for a class of processes which overlaps but does not contain the ones we are considering, that all invariant measures which are also translation invariant are convex combinations of $\bar{\nu}_0$ and $\bar{\nu}_1$.

It would be very interesting to remove the nearest neighbor assumption in Theorem 1.4. It would have to be replaced by some type of irreducibility assumption, since the conclusion of the theorem is false for the process with $c(x, \eta) = 1$ for $\eta(x) = 1$ and $c(x, \eta) = \lambda[\eta(x-2) + \eta(x+2)]$ for $\eta(x) = 0$, provided that λ is sufficiently large. Our techniques seem not to suffice outside of the nearest neighbor case. Theorem 1.4 cannot be extended to processes on Z^d for $d \geqq 3$, since the voter model [6] and the stochastic Ising model provide counterexamples. It would be very interesting to determine whether the result is true for Z^2. If it is, that would solve a problem in statistical mechanics which is apparently still open. As was pointed out to me by David Griffeath, some type of irreducibility assumption or strengthened positivity assumption would have to be imposed in two dimensions to eliminate the following types of counterexamples: (a) $c(x, \eta) = 0$ whenever three or four neighbors of x are the same as $\eta(x)$, since then configurations which are one on one side of a vertical (or horizontal) line and zero on the other are absorbing; (b) $c(x, \eta)$ depends only on the neighbors directly above or below x, so that the process is not genuinely two dimensional; and (c) the example at the end of [8].

Section 2 is devoted to the proof of Theorem 1.4. In Section 3, we state an analogous result for discrete time processes, and indicate briefly how to modify the proof in Section 2 to obtain it. The discrete time theorem resolves the conjecture stated just before the Theorem in the introduction of [12], and gives some information regarding the problem mentioned at the end of [13]. I want to thank David Griffeath for encouraging me to work out the discrete time version of these results, and for providing me with references [12] and [13].

2. The proofs. Before turning to the proof of Theorem 1.4, we will illustrate our technique by sketching a coupling proof of Griffeath's complete convergence theorem for symmetric contact processes [1]. Consider the symmetric contact process η_t described in the introduction, and let λ be so large that the one-sided contact process with that λ is nonergodic. Griffeath's result is that for any $\eta \in X$, $\delta_\eta S(t)$ converges to $(1-\alpha)\nu_0 + \alpha\bar{\nu}_1$, where $\alpha = P^\eta[\eta_s \not\equiv 0$ for all $s]$ and δ_η is the pointmass at η. We will prove this for an initial configuration consisting of only one particle, since the general case is similar. The standard coupling of two copies η_t and ζ_t of the process with $\eta_0(0) = 1$, $\eta_0(x) = 0$ for $x \neq 0$, and $\zeta_0 \equiv 1$ has the property that for each t, either $\eta_t \equiv 0$ or there are integers L_t and R_t such that $\eta_t(L_t) = \eta_t(R_t) = 1$, $\eta_t(x) = \zeta_t(x)$ for $L_t \leqq x \leqq R_t$, and $\eta_t(x) = 0$ for $x < L_t$ and $x > R_t$. Since the corresponding one-sided contact process is nonergodic, $L_t \to -\infty$ and $R_t \to -\infty$ a.s. on $\{\eta_s \not\equiv 0$ for all $s\}$. Therefore

$$P[\eta_t(x) = \zeta_t(x) \text{ for all large } t \mid \eta_s \not\equiv 0 \text{ for all } s] = 1$$

for each x, and hence the result follows from the fact that the conditional distribution $(\zeta_t \mid \eta_t \not\equiv 0)$ converges to $\bar{\nu}_1$. The key point is that the number of regions into which Z is decomposed by the requirement that $\eta_t \equiv 0$ or $\eta_t \equiv \zeta_t$ on each region is nonincreasing in t, and therefore is at most three for all t. It is this type of monotonicity which we will exploit.

Throughout the rest of this section, we will assume that

$$(2.1)\qquad \varepsilon = \min\{\min_\eta [c(x,\eta) + c(x,\eta_x)],\ 2c(x,1) + c(x,0_{x-1}) + c(x,0_{x+1}),\ 2c(x,0) + c(x,1_{x-1}) + c(x,1_{x+1})\} > 0\,,$$

where 0 and 1 are the configurations which are identically zero or one respectively. If (1.3) holds and $\varepsilon = 0$, then either the second or third term in the definition of ε is zero. If $c(x,1) = c(x,0_{x-1}) = c(x,0_{x+1}) = 0$, for example, then it is easy to see directly that if there are ever two consecutive zeros in η_t, then the distribution of the process converges to ν_0, while if there are never two consecutive zeros in η_t, then the distribution of the process converges to ν_1. Therefore Theorem 1.4 holds in that case, and we may assume that $\varepsilon > 0$.

The technique of proof involves coupling together several copies of the spin system in an appropriate way, and then studying the resulting coupled process. This will be done twice, and in each case, the coupling is chosen in such a way that the copies which agree at a given x at a given time, will flip together as much as possible, subject to the constraint that each copy be Markovian with semigroup $S(t)$. The coupling is a simple extension to several processes of the "basic coupling" which is described in [11] and which has been used extensively by several authors. In our main application, three processes η_t, γ_t, and ζ_t are coupled in such a way that $\eta_t \leqq \gamma_t \leqq \zeta_t$ for all $t \geqq 0$. The coupled process is then a Feller process $(\eta_t, \gamma_t, \zeta_t)$ on $X_3 = \{(\eta,\gamma,\zeta) \in X^3 : \eta \leqq \gamma \leqq \zeta\}$ whose flip rates are given by the following table.

	(0, 0, 0)	(0, 0, 1)	(0, 1, 1)	(1, 1, 1)
(0, 0, 0)	—	$c(x,\zeta) - c(x,\gamma)$	$c(x,\gamma) - c(x,\eta)$	$c(x,\eta)$
(0, 0, 1)	$c(x,\zeta)$	—	$c(x,\gamma) - c(x,\eta)$	$c(x,\eta)$
(0, 1, 1)	$c(x,\zeta)$	$c(x,\gamma) - c(x,\zeta)$	—	$c(x,\eta)$
(1, 1, 1)	$c(x,\zeta)$	$c(x,\gamma) - c(x,\zeta)$	$c(x,\eta) - c(x,\gamma)$	—

For example, the third row is to be read as follows: if $\eta_t(x) = 0$, $\gamma_t(x) = 1$, $\zeta_t(x) = 1$, then $\gamma_t(x)$ and $\zeta_t(x)$ will flip to zero together at rate $c(x,\zeta_t)$, $\gamma_t(x)$ will flip to zero alone at rate $c(x,\gamma_t) - c(x,\zeta_t)$, and $\eta_t(x)$ will flip to one at rate $c(x,\eta_t)$. Note that the attractiveness assumption (1.2) guarantees that all the entries in the table are nonnegative. Also, the marginal processes η_t, γ_t, and ζ_t are separately Markovian with the transition law of the spin system with flip rates $c(x,\eta)$. Let $\tilde{\Omega}$ be the generator of the coupled process $(\eta_t, \gamma_t, \zeta_t)$, and put

$$W_1 = \{(\eta,\gamma,\zeta\} \in X_3 : \eta \equiv \gamma\}\,, \qquad W_2 = \{(\eta,\gamma,\zeta) \in X_3 : \gamma \equiv \zeta\}$$

$$W_3 = \{(\eta,\gamma,\zeta\} \in X_3 : \text{there is an } x \in Z \text{ so that } \eta(y) = \gamma(y) \text{ for } y \leqq x \text{ and } \gamma(y) = \zeta(y) \text{ for } y > x\}\,,$$

$$W_4 = \{(\eta,\gamma,\zeta) \in X_3 : \text{there is an } x \in Z \text{ so that } \eta(y) = \gamma(y) \text{ for } y > x \text{ and } \gamma(y) = \zeta(y) \text{ for } y \leqq x\}\,.$$

In what follows, it is suggested that the reader keep in mind the case in which

$c(x, \eta) \equiv 0$ if $\eta \equiv 0$ or $\eta \equiv 1$, and set $\eta_t \equiv 0$ and $\zeta_t \equiv 1$. The key ideas of the proof are much easier to see in this case, and of course, $\bar{\nu}_0 = \nu_0$ and $\bar{\nu}_1 = \nu_1$.

LEMMA 2.2. *If the probability measure ν on X_3 is invariant for $(\eta_t, \gamma_t, \zeta_t)$, then $\nu(W_1 \cup W_2 \cup W_3 \cup W_4) = 1$. If ν is extremal invariant, then $\nu(W_i) = 1$ for some $i = 1, 2, 3, 4$.*

PROOF. For $m \leqq n$ and $l \geqq 1$, define functions $f_{m,n}$ and $g^l_{m,n}$ on X_3 in the following way: Let $m \leqq x_1 < x_2 < \cdots < x_k \leqq n$ be all those x's between m and n for which $\zeta(x) = 1$ and $\eta(x) = 0$. Then

$$\begin{aligned} f_{m,n}(\eta, \gamma, \zeta) &= 0 && \text{if} \quad k = 0 \\ &= 1 + \text{number of} \quad i \quad \text{such that} \quad \gamma(x_{i+1}) \neq \gamma(x_i) && \text{if} \quad k \geqq 1\,, \end{aligned}$$

and

$$\begin{aligned} g^l_{m,n}(\eta, \gamma, \zeta) = \text{number of} \quad i \quad \text{such that} \quad i \geqq 1\,, \quad i + l + 1 \leqq k\,, \quad \text{and} \\ \gamma(x_i) \neq \gamma(x_{i+1}) = \gamma(x_{i+2}) = \cdots = \gamma(x_{i+l}) \neq \gamma(x_{i+l+1})\,. \end{aligned}$$

Thus $f_{m,n}$ is the number of strings of zeros or ones in γ between m and n, where γ is only observed at points at which ζ and η differ, and $g^l_{m,n}$ is the number of interior strings of length l. Note that

$$f_{m,n}(\eta, \gamma, \zeta) \leqq 2 + \textstyle\sum_{l \geqq 1} g^l_{m,n}(\eta, \gamma, \zeta)\,, \tag{2.3}$$

$$\textstyle\sum_{l \geqq 1} l g^l_{m,n}(\eta, \gamma, \zeta) \leqq n - m + 1\,, \tag{2.4}$$

and $f_{m,n}$ and $g^l_{m,n}$ are increasing functions of n and decreasing functions of m for all $(\eta, \gamma, \zeta) \in X_3$. The first conclusion of the lemma can then be restated as $\int g^l_{m,n}\, d\nu = 0$ for $l \geqq 1$, $m \leqq n$. Since ν is invariant and $f_{m,n}$ and $g^l_{m,n}$ are in the domain of $\tilde{\Omega}$, $\int \tilde{\Omega} f_{m,n}\, d\nu = 0$ and $\int \tilde{\Omega} g^l_{m,n}\, d\nu = 0$. A somewhat tedious computation of these expressions leads respectively to the following two inequalities, where ε is as in (2.1) and $K = \max_\eta c(x, \eta)$:

$$2\varepsilon \int g^1_{m,n}\, d\nu \leqq K \int [f_{m-1,n} + f_{m,n+1} - 2f_{m,n}]\, d\nu\,, \tag{2.5}$$

$$\varepsilon \int g^{l+1}_{m,n}\, d\nu \leqq 4Kl \int g^l_{m,n}\, d\nu \qquad \text{for} \quad l \geqq 1\,. \tag{2.6}$$

While the complete computations are tedious, it is not hard to see why they imply these inequalities. To obtain (2.5), one should observe (a) that $f_{m,n}(\eta_t, \gamma_t, \zeta_t)$ can only increase via a flip at m or n, and it increases via a flip at m only if $f_{m-1,n} > f_{m,n}$, and via a flip at n only if $f_{m,n+1} > f_{m,n}$, and (b) that $f_{m,n}(\eta_t, \gamma_t, \zeta_t)$ can decrease by two via a flip at each x_i for $1 < i < k$ for which $\gamma(x_{i-1}) \neq \gamma(x_i)$ and $\gamma(x_{i+1}) \neq \gamma(x_i)$, and that the rate at which that flip occurs is at least ε. In fact, the reason that $f_{m,n}$ is a particularly useful function to consider is that it can only increase via flips at m and n, while it can decrease via flips at all x between m and n. It is here that we need to make the nearest neighbor assumption (1.1). To obtain (2.6), note (a) that $g^l_{m,n}(\eta_t, \gamma_t, \zeta_t)$ can only decrease via flips at at most $lg^l_{m,n}$ x_i's or their neighbors, and that the rate at which this occurs is at most $2K$ at an x_i and at most K at the neighbor of an x_i, and (b) that

$g^l_{m,n}(\eta_t, \gamma_t, \zeta_t)$ can increase at at least $g^{l+1}_{m,n}$ sites or pairs of sites at a total rate for each pair which is at least ε. To check the last statement, observe for example that if $u < v$ are such that $\gamma(y) = 1$ and $\eta(y) = 0$ for $u \leqq y \leqq v$ and $\eta(u-1) = \eta(v+1) = 1$, then there is a flip at u or v at rate $2c(x, 1) + c(x, 0_{x-1}) + c(x, 0_{x+1})$.

Now, since $f_{m-1,n} + f_{m,n+1} \leqq 2f_{m,n} + 2$, (2.5) gives $\sup_{m\leqq n} \int g^1_{m,n}\, d\nu < \infty$, and then repeated use of (2.6) gives $\sup_{m\leqq n} \int g^l_{m,n}\, d\nu < \infty$ for all $l \geqq 1$. Inequalities (2.3) and (2.4) together yield

$$\frac{f_{m,n}}{n-m+1} \leqq \frac{1}{L} + \frac{1}{n-m+1}\left[2 + \sum_{l=1}^{L} g^l_{m,n}\right]$$

for all $L \geqq 1$, so $\lim_{n-m\to\infty} 1/(n-m) \int f_{m,n}\, d\nu = 0$. Therefore

$$\lim_{N\to\infty} \frac{1}{N^2} \sum_{m=-N+1}^{0} \sum_{n=0}^{N-1} \int [f_{m-1,n} + f_{m,n+1} - 2f_{m,n}]\, d\nu$$

$$\leqq \lim_{N\to\infty} \frac{1}{N^2}\left[\sum_{n=0}^{N-1} \int f_{-N,n}\, d\nu + \sum_{m=-N+1}^{0} \int f_{m,N}\, d\nu\right] = 0\,,$$

so $\lim_{m\to-\infty;n\to+\infty} \int g^1_{m,n}\, d\nu = 0$ by (2.5) and the monotonicity of $g^1_{m,n}$ in m and n. Using this monotonicity again gives $\int g^1_{m,n}\, d\nu = 0$ for all $m \leqq n$, and then (2.6) implies $\int g^l_{m,n}\, d\nu = 0$ for all $m \leqq n$ and $l \geqq 1$, which completes the proof of the first part of the lemma. The second part follows from the fact that for each i, $P^{(\eta,\gamma,\zeta)}[(\eta_t, \gamma_t, \zeta_t) \in W_i] = 1$ if $(\eta, \gamma, \zeta) \in W_i$.

PROOF OF THEOREM 1.4. Let $\mu_1 \in \mathscr{I}_e$, and let μ_2 be the translate of μ_1 defined by

$$\mu_2\{\eta\colon \eta(x) = 1 \quad \text{for} \quad x \in T\} = \mu_1\{\eta\colon \eta(x+1) = 1 \quad \text{for} \quad x \in T\}$$

for finite $T \subset Z$. Then $\mu_2 \in \mathscr{I}_e$ also. Consider the coupled process $(\eta_t, \gamma_t^1, \gamma_t^2, \zeta_t)$ on $X_4 = \{(\eta, \gamma^1, \gamma^2, \zeta) \in X^4\colon \eta \leqq \gamma^1, \gamma^2 \leqq \zeta\}$ which is constructed in a manner analogous to the one described at the beginning of this section. Then $(\eta_t, \gamma_t^i, \zeta_t)$ is Markovian for $i = 1, 2$ and has generator $\tilde{\Omega}$. Since $\bar{\nu}_0, \mu_1, \mu_2, \bar{\nu}_1 \in \mathscr{I}_e$ and $\bar{\nu}_0 \leqq \mu_1, \mu_2 \leqq \bar{\nu}_1$, there is a probability measure ν on X_4 with marginals $\bar{\nu}_0, \mu_1, \mu_2, \bar{\nu}_1$ which is extremal invariant for $(\eta_t, \gamma_t^1, \gamma_t^2, \zeta_t)$. The proof of this is the same as the proof of Lemma 2.3 of [10]. Let $\nu^{(1)}$ and $\nu^{(2)}$ be the measures on X_3 which are obtained from ν via the projections $(\eta, \gamma^1, \gamma^2, \zeta) \to (\eta, \gamma^1, \zeta)$ and $(\eta, \gamma^1, \gamma^2, \zeta) \to (\eta, \gamma^2, \zeta)$ respectively. Then by Lemma 2.2, $\nu^{(1)}(W_i) = 1$ for some i and $\nu^{(2)}(W_i) = 1$ for some i. Since μ_1 and μ_2 are translates of one another, it is easy to see that if $\bar{\nu}_0 \neq \bar{\nu}_1$, then there is one i for which both $\nu^{(1)}(W_i) = 1$ and $\nu^{(2)}(W_i) = 1$. (If $\bar{\nu}_0 = \bar{\nu}_1$, then the entire theorem is trivial, of course.) Suppose $i = 3$ or 4. Then $\nu\{(\eta, \gamma^1, \gamma^2, \zeta)\colon \sum_x |\gamma^1(x) - \gamma^2(x)| < \infty\} = 1$. Note that $P^{(\eta,\gamma,\gamma,\zeta)}\{\gamma_t^1 = \gamma_t^2\} = 1$, and that (2.1) implies that $P^{(\eta,\gamma^1,\gamma^2,\zeta)}\{\gamma_t^1 = \gamma_t^2\} > 0$ for $t > 0$ whenever $\sum_x |\gamma^1(x) - \gamma^2(x)| < \infty$. Therefore, since ν is invariant, it follows that $\nu\{(\eta, \gamma^1, \gamma^2, \zeta)\colon \gamma^1 = \gamma^2\} = 1$, which is impossible if $i = 3$ or 4 and $\bar{\nu}_0 \neq \bar{\nu}_1$. Hence $i = 1$ or 2, so $\mu_1 = \bar{\nu}_0$ or $\mu_1 = \bar{\nu}_1$, which completes the proof of the theorem.

3. Discrete time. Let $\rho_x(\eta)$ be translation invariant on $Z \times X$ and satisfy $0 \leqq \rho_x(\eta) \leqq 1$. The discrete time spin system corresponding to $\{\rho_x(\eta)\}$ is the discrete time Markov chain η_n on X with transition law given by

$$P^\eta[\eta_1(x) = 1, x \in T] = \prod_{x \in T} \rho_x(\eta),$$

for finite subsets T of Z. We will make the following assumptions:

(3.1) Nearest neighbor assumption: either (a) $\rho_x(\eta)$ depends on η only through $\eta(x)$ and $\eta(x+1)$, or (b) $\rho_x(\eta)$ depends on η only through $\eta(x-1)$ and $\eta(x+1)$ if $\eta(x) = 0$, and is constant if $\eta(x) = 1$.

(3.2) Attractiveness assumption: $\rho_x(\eta) \leqq \rho_x(\zeta)$ for $\eta \leqq \zeta$.

(3.3) Positivity assumption: either $\rho_x(\eta) > 0$ for all η or $\rho_x(\eta) < 1$ for all η.

THEOREM 3.4. *Under assumptions* (3.1), (3.2) *and* (3.3), $\mathscr{I}_e = \{\bar\nu_0, \bar\nu_1\}$.

The proof of this result is very similar to that of Theorem 1.4, so most of it will be omitted. The main change is that relations (2.5) and (2.6) are replaced by

$$2 \sum_{l \geq 1} \varepsilon^{3l} \int g^l_{m,n}\, d\nu \leqq \int [f_{m-1,n} + f_{m,n+1} - 2f_{m,n}]\, d\nu \tag{3.5}$$

where $\varepsilon = 1 - \max_\eta \rho_x(\eta) + \min_\eta \rho_x(\eta)$, which is positive by (3.3). Inequality (3.5) follows from

$$P^{(\eta,\gamma,\zeta)}[\eta_1(x) = \gamma_1(x) = \zeta_1(x)] \geqq \varepsilon$$

where $(\eta_n, \gamma_n, \zeta_n)$ is the analogous coupled discrete time process.

Assumption (3.1) above appears to be somewhat unnatural, so some remarks may be helpful. The role of this assumption is to guarantee that $f_{m,n}$ can only increase via a flip at m or at n. To see this, suppose $\eta \equiv 0$, $\zeta \equiv 1$, and $\gamma(x) = 1$ if $x \leqq 0$ and $\gamma(x) = 0$ if $x \geqq 1$. Then $f_{m,n} = 2$ for $m \leqq 0 < n$. If $\rho_x(\eta)$ is allowed to depend on η through $\eta(x-1)$, $\eta(x)$ and $\eta(x+1)$, then after one transition, it is possible to have flips at 0 and 1 in such a way that $f_{m,n}(\eta_1, \gamma_1, \zeta_1) = 4$ for negative m and positive n. Assumption (3.1) makes this impossible. In continuous time, (1.1) suffices because only one flip occurs at a time.

In fact, as was pointed out in [12], Theorem 3.4 does not hold if (3.1) is replaced by the requirement that $\rho_x(\eta)$ depend only on $\eta(x-1)$, $\eta(x)$ and $\eta(x+1)$. Their counterexample is given by

$$\rho_x(\eta) = (1-\theta)[\eta(x-1) + \eta(x+1) - \eta(x-1)\eta(x+1)].$$

For small θ, there is an invariant measure which is not translation invariant, but does have period two under translations in Z. Our techniques can be used, however, to show that for all θ in this example, either the process is ergodic or there are exactly three extremal invariant measures: $\bar\nu_0$, $\bar\nu_1$, and the measure which is the limit in t of $\nu S(t)$, where

$$\nu\{\eta(2x) = 1, \eta(2x+1) = 0 \ \text{ for all } \ x\} = \nu\{\eta(2x) = 0, \eta(2x+1) = 1 \ \text{ for all } \ x\} = \tfrac{1}{2}.$$

The conjecture in [12] which is resolved by our result pertains to the spin system with $\rho_x(\eta) = (1 - \theta)[\eta(x) + \eta(x+1) - \eta(x)\eta(x+1)]$. There it is proved that the process has at most two extremal invariants which are translation invariant.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA, LOS ANGELES
LOS ANGELES, CALIFORNIA 90024