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ON THE LOCAL LIMIT THEOREM FOR INDEPENDENT NONLATTICE RANDOM VARIABLES

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Let $(X_n : n \geqq 1)$ be a sequence of independent random variables, each having mean 0 and a finite variance. Under the Lindeberg condition and uniformity conditions on the characteristic functions, it is shown that the local limit theorem holds, i.e., if S_n is the nth partial sum of the sequence, then $(2\pi \operatorname{Var} S_n)^{\frac{1}{2}} P(S_n \in (a, b)) \to b - a$.

Under the assumption that the local limit theorem holds for each tail of (X_n), and one other condition, it is then shown that the random walk generated by (X_n) is recurrent if $\sum (\operatorname{Var} S_n)^{-\frac{1}{2}} = \infty$.

1. Introduction and notation. Let $(X_n : n \geqq 1)$ be a sequence of independent random variables with $E(X_n) = 0$ and $E(X_n^2) = \sigma_n^2$. Let $(S_n : n \geqq 1)$ be the sequence of partial sums and $s_n^2 = E(S_n^2)$. The aim of this note is to give conditions under which the local limit theorem holds, i.e., $(2\pi)^{\frac{1}{2}} s_n P(S_n \in (a, b)) \to b - a$. In the nonlattice case the first such theorem was given by Shepp (1964) for i.i.d. variables. Stone (1965) considered local limit theorems for i.i.d. nonlattice random vectors and Mineka and Silverman (1970) proved a local limit theorem for nonlattice random variables in the nonidentically distributed case.

We prove a local limit theorem from which follow two corollaries. In the first corollary we allow the individual variances, σ_n^2, to be unbounded from above, by considering weighted sums of i.i.d. variables. Condition (α) of Mineka and Silverman (1970) requires the variances to be bounded from above. In the second corollary, $s_n^2 = O(n)$. This corollary is not a full generalization of Corollaries 1 and 2 of Mineka and Silverman since the convergence is not uniform over as large a class of intervals. However, the conditions of our Corollary 2 are weaker since (4) and (5) together are weaker than (α) of Mineka and Silverman, and (6) here is implied both by the conditions (β_1) of Corollary 1 and (β_2) of Corollary 2 there. Condition (6) is a uniformity condition on the characteristic functions which we believe will be easier to check than (β) of [3]. As an example we prove Corollary 3 where it is assumed that the random variables have densities. The conditions of Corollary 2 also allow a given fraction, less than 1, of the variables to be distributed on the same lattice.

In Section 4 we use the lemma of Mineka and Silverman to show that if $E(X_n) = 0$, $E(X_n^2) = \sigma_n^2 < \infty$ and each tail of the sequence (X_n) satisfies the local limit theorem, then, under one other condition, (X_n) generates a recurrent random walk if $\sum (\operatorname{Var} S_n)^{-\frac{1}{2}} = \infty$. We then show that under the conditions of Corollaries 1 and 3 the random walk is recurrent.

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Let F_n be the distribution function of X_n and φ_n its characteristic function. Denote by F_n^s the distribution function of the symmetrization of X_n and by G_n the distribution function of S_n. Following Shepp, the method of proof is to show that the sequence $((2\pi)^{\frac{1}{2}} s_n G_n : n \geqq 1)$ converges weakly to Lebesgue measure on $(-\infty, \infty)$. To do this it suffices to show that $(2\pi)^{\frac{1}{2}} s_n E(h(S_n)) \to \int_{-\infty}^{\infty} h(x)\,dx$ for each function h which is integrable and the Fourier transform of a function $\hat{h}$ which is continuous with compact support (Breiman (1968), Theorem 10.7).

2. Local limit theorem.

THEOREM 1. *We assume the following conditions:*

(1) *The Lindeberg condition: for each* $t > 0$.

$$s_n^{-2} \sum_{k=1}^{n} \int_{|x| > ts_n} x^2\, dF_k(x) \to 0 .$$

For (2) *and* (3) *assume that* $(\varepsilon_n : n \geqq 1)$ *is a sequence of positive constants with* $\varepsilon_n s_n \to \infty$.

(2) *There is a nonnegative integrable function* $f(t)$ *so that for each* n, *on* $[-\varepsilon_n s_n, \varepsilon_n s_n]$,

$$\prod_{k=1}^{n} |\varphi_k(t/s_n)| \leqq f(t) .$$

(3) *For any* $b > 0$, *there exists* $\alpha \in (0, 1)$ *so that for each* k,

$$|\varphi_k(t)| \leqq \alpha , \qquad t \in [\varepsilon_k, b] .$$

Then, for any $A > 0$ *and nonnegative integer* N,

$$\lim_{n\to\infty} \sup_{|x| \leqq A} |(2\pi(s_{N+n}^2 - s_N^2))^{\frac{1}{2}} P(S_{N+n} - S_N \in (a + x, b + x)) - (b - a)| = 0 .$$

PROOF. We first prove:

(i) for each integer $N \geqq 0$, $\lim_n (2\pi(s_{N+n}^2 - s_N^2))^{\frac{1}{2}} P(S_{N+n} - S_N \in (a, b)) = b - a$. Let h be integrable on $(-\infty, \infty)$ and the Fourier transform of a continuous function $\hat{h}$ with compact support. Say $\hat{h}$ vanishes outside $[-z, z]$. By the remarks before the statement of Theorem 1 and the fact that $\lim_n s_{N+n}/(s_{N+n}^2 - s_N^2)^{\frac{1}{2}} = 1$, it suffices to show that

$$(2\pi)^{\frac{1}{2}} s_{N+n} E(h(S_{N+n} - S_N)) \to \int_{-\infty}^{\infty} h(x)\,dx .$$

By the definition of h and Fubini's theorem the left side is

(ii) $(2\pi)^{\frac{1}{2}} s_{N+n} \int_{-z}^{z} (\prod_{N+1}^{N+n} \varphi_k(t)) \hat{h}(t)\,dt$.

Choose an integer p so large that

(iii) $\prod_{k=1}^{N} |\varphi_k(t)| \geqq \frac{1}{2}$ on $[-2^{-p}\varepsilon_N, 2^{-p}\varepsilon_N]$. Set $\delta_n = 2^{-p}\varepsilon_{N+n}$, $\psi_n(t) = \prod_{N+1}^{N+n} \varphi_k(t)$ and rewrite the integral (ii) as

$$(2\pi)^{\frac{1}{2}} s_{N+n} \int_{|t| \leqq \delta_n} \psi_n(t)\hat{h}(t)\,dt + (2\pi)^{\frac{1}{2}} s_{N+n} \int_{\delta_n \leqq |t| \leqq z} \psi_n(t)\hat{h}(t)\,dt = I_n + J_n .$$

We first show that $\lim I_n = \int_{-\infty}^{\infty} h(x)\,dx$. By change of variables,

$$I_n = (2\pi)^{\frac{1}{2}} \int_{-\delta_n s_{N+n}}^{\delta_n s_{N+n}} \psi_n(t/s_{N+n}) \hat{h}(t/s_{N+n})\,dt .$$

But $|\psi_n(t/s_{n+N})| \leqq 2 \prod_{k=1}^{N+n} |\varphi_k(t/s_{N+n})| \leqq 2f(t)$ on $|t| \leqq \delta_n s_{N+n}$.

The first inequality follows from (iii) and the second from (2). Thus the integrand of I_n is bounded above by $2f(t)\max(|\hat{h}(u)|: -z \leqq u \leqq z)$, an integrable function. Since $\psi_n(t/S_{N+n})$ is the characteristic function of $(S_{N+n} - S_N)/s_{N+n}$, the dominated convergence theorem, together with (1), yields

$$I_n \to (2\pi)^{\frac{1}{2}} \int_{-\infty}^{\infty} e^{-t^2}\hat{h}(0)\,dt = 2\pi\hat{h}(0) = \int_{-\infty}^{\infty} h(x)\,dx .$$

The last equality follows from the inversion theorem for Fourier transforms.

If $\delta_k \downarrow \delta \geqq z$, then $J_n \equiv 0$ and the proof is complete. Otherwise $\lim J_n = 0$ must be shown. Since $\hat{h}$ is bounded and $|\varphi_k(-t)| = |\varphi_k(t)|$, it suffices to show that

$$\lim_n s_{N+n} \int_{\delta_n}^{z} |\psi_n(t)|\,dt = 0 .$$

It follows from (1) that $\lim_n s_{N+n+1}/s_{N+n} = 1$. For each n, put $a_n = \sup_{k\geqq n} s_{N+k+1}/s_{N+k}$, so that $\lim_n a_n = 1$. Put $b_n = \inf_{k\geqq n} \delta_{k+1}s_{N+k}$. By the fact that $\delta_{k+1} = 2^{-p}\varepsilon_{N+k+1}$ and the assumption that $\lim_n \varepsilon_n s_n = \infty$, $\lim_n b_n = \infty$. Set $b = \max(z, 2^p\varepsilon_1)$; by (3) there exists $\alpha \in (0, 1)$ so that for each $k \geqq 1$, $|\varphi_k(t)| \leqq \alpha$ on $[\varepsilon_k, b]$. By repeated application of the formula

$$\operatorname{Re}\varphi_j(t) \leqq \tfrac{3}{4} + \operatorname{Re}\varphi_j(2t)/4 ,$$

valid for characteristic functions, and the fact that $|\varphi_j(t)|^2$ is a characteristic function, we have

$$|\varphi_j(t)| \leqq (1 - 4^{-p}(1 - |\varphi_j(2^p t)|^2))^{\frac{1}{2}} , \qquad \text{all} \quad j \geqq 1 .$$

As $2^p t \in [\varepsilon_{N+k}, b]$ whenever $t \in [\delta_k, \varepsilon_{N+k}]$, it follows that

(iv) $\prod_{N+k}^{N+n} |\varphi_j(t)| \leqq \beta^{n-k+1}$, all $n \geqq k$ and all $t \in [\delta_k, z]$ where $\alpha < \beta = (1 - 4^{-p}(1 - \alpha^2))^{\frac{1}{2}} < 1$.

Let $\eta > 0$ be arbitrary and choose n_0 so large that $a_{n_0}\beta < 1$ and $[2a_{n_0}\beta/(1 - a_{n_0}\beta)] \int_{b_{n_0}}^{\infty} f(t)\,dt < \eta$. We have

$$\begin{aligned} &\limsup_n s_{N+n} \int_{\delta_n}^{z} |\psi_n(t)|\,dt \\ &\qquad \leqq \limsup_n s_{N+n} \int_{\delta_n}^{\delta_{n_0}} |\psi_n(t)|\,dt + \limsup_n s_{N+n} \int_{\delta_{n_0}}^{z} |\psi_n(t)|\,dt . \end{aligned}$$

By (iv), $s_{N+n} \int_{\delta_{n_0}}^{z} |\psi_n(t)|\,dt \leqq s_{N+n} z\beta^{n-n_0+1}$. Since $s_{N+n+1}/s_{N+n} \to 1$ as $n \to \infty$ and $0 < \beta < 1$, it follows that $\limsup_n s_{N+n} \int_{\delta_{n_0}}^{z} |\psi_n(t)|\,dt \leqq \limsup_n s_{N+n} z\beta^{n-n_0+1} = 0$. It remains to show that $\limsup_n s_{N+n} \int_{\delta_n}^{\delta_{n_0}} |\psi_n(t)|\,dt = 0$. For $n \geqq n_0$, $s_{N+n} \int_{\delta_n}^{\delta_{n_0}} |\psi_n(t)|\,dt = \sum_{k=n_0}^{n-1} s_{N+n} \int_{\delta_{k+1}}^{\delta_k} |\psi_n(t)|\,dt$. As $s_{N+n} \leqq s_{N+k} a_{n_0}^{n-k}$ for $n \geqq k \geqq n_0$, it follows that

$$\begin{aligned} s_{N+n} \int_{\delta_{k+1}}^{\delta_k} |\psi_n(t)|\,dt &\leqq a_{n_0}^{n-k} s_{N+k} \int_{\delta_{k+1}}^{\delta_k} |\psi_n(t)|\,dt \\ &\leqq (a_{n_0}\beta)^{n-k} \int_{\delta_{k+1}s_{N+k}}^{\delta_k s_{N+k}} \left(\textstyle\prod_{N+1}^{N+k} |\varphi_j(t/s_{N+k})|\right) dt \\ &\leqq 2(a_{n_0}\beta)^{n-k} \int_{\delta_{k+1}s_{N+k}}^{\delta_k s_{N+k}} f(t)\,dt . \end{aligned}$$

The last inequality follows from (iii) and (2). Thus,

$$\begin{aligned} s_{N+n} \int_{\delta_n}^{\delta_{n_0}} |\psi_n(t)|\,dt &\leqq 2 \textstyle\sum_{k=n_0}^{n-1} (a_{n_0}\beta)^{n-k} \int_{b_{n_0}}^{\infty} f(t)\,dt \\ &\leqq [2a_{n_0}\beta/(1 - a_{n_0}\beta)] \int_{b_{n_0}}^{\infty} f(t)\,dt < \eta . \end{aligned}$$

Since η was arbitrarily chosen, (i) holds. To complete the proof it suffices to note that if $(\mu_n : n \geqq 1)$ is any sequence of positive measures on the Borel sets of the real line which are finite on compacta and converge weakly to Lebesgue measure, then, for any $A > 0$, $\lim_{n\to\infty} \sup_{|x|\leqq A} |\mu_n(a+x, b+x) - (b-a)| = 0$. See Problem 1, page 227 of [1]. The proof there only requires that the measures $\mu_n(A) = (2\pi)^{\frac{1}{2}} s_n P(S_n \in A)$ be finite on compacta.

REMARK. If for each n, $\varepsilon_n = \varepsilon > 0$, then (3) may be weakened to

(3′) For any fixed $0 < a < b$ and integer $N \geqq 0$ there exists $\alpha \in (0, 1)$ with $\prod_{N+1}^{N+n} |\varphi_j(t)| \leqq \alpha^n$ for $t \in [a, b]$, and n sufficiently large.

It is this version of (3) that is satisfied by the conditions of Corollary 2.

COROLLARY 1. *Let $(X_n : n \geqq 1)$ be* i.i.d. *with mean* 0, *variance* 1 *and characteristic function φ satisfying*

$$\limsup_{|t|\to\infty} |\varphi(t)| < 1 .$$

Let $(\sigma_n : n \geqq 1)$ be positive numbers with $m_n = \max(\sigma_k : 1 \leqq k \leqq n)$ and assume (a) $m_n s_n^{-1} \to 0$, *and* (b) $d = \inf(\sigma_k m_k^{-1} : k \geqq 1) > 0$. *Then, for any positive A, and $N \geqq 0$,*

$$\lim_n \sup_{|x|\leqq A} |(2\pi \textstyle\sum_{N+1}^{N+n} \sigma_k^2)^{\frac{1}{2}} P(\sum_{N+1}^{N+n} \sigma_k X_k \in (a+x, b+x) - (b-a)| = 0 .$$

PROOF. Condition (a) implies that the Lindeberg condition, (1) of Theorem 1, holds. Choose $\varepsilon > 0$ so that $|\varphi(t)| < e^{-t^2/4}$ whenever $|t| \leqq \varepsilon$ and put $\varepsilon_k = \varepsilon m_k^{-1}$. Then on $|t| \leqq \varepsilon_n s_n$,

$$\textstyle\prod_{k=1}^{n} |\varphi_k(t/s_n)| = \prod_{k=1}^{n} |\varphi(\sigma_k t/s_n)| \leqq e^{-t^2/4}$$

and (2) holds since $\varepsilon_n s_n \to \infty$. Finally, on $[\varepsilon_k, b]$,

$$|\varphi_k(t)| = |\varphi(\sigma_k t)| \leqq \sup_{|u|\geqq \varepsilon d} |\varphi(u)| < 1 .$$

REMARK. Another way to let the individual variances be unbounded would be to assume that, for each k, $|\varphi_k|$ is nonincreasing and that there exists $c > 0$ with $|\varphi_k(t)| \leqq \exp(-ct^2\sigma_k^2)$ whenever $|t| \leqq \varepsilon_k$. The integrable function then becomes $f(t) = e^{-ct^2}$. Then if $\inf_k \varepsilon_k \sigma_k > 0$, (2) and (3) hold.

For example, assume that for some $A > 0$, $E|X_k|^3 \leqq A\sigma_k^3$ for each k and that (1) holds. If we put $\varepsilon_k = 3/2Am_k$, $m_k = \max(\sigma_j : j \leqq k)$, then $\varepsilon_n s_n \to \infty$ by (1), and (2) holds with $c = \frac{1}{4}$ since $|\varphi_k(t)| \leqq 1 - \sigma_k^2 t^2/2 + |t|^3 E|X_k|^3/6$ whenever $|t| \leqq (2)^{\frac{1}{2}}/\sigma_k$. Finally, since $|\varphi_k|$ is nonincreasing, $|\varphi_k(t)| \leqq |\varphi_k(\varepsilon_k)| \leqq 1 - \sigma_k^2 \varepsilon_k^2/4$ if $t \in [\varepsilon_k, b]$ and (3) holds if $\inf_k \sigma_k/m_k > 0$.

3. $s_n = O(n)$. Throughout this section $(X_k : k \geqq 1)$ is a sequence of independent random variables with mean 0 and finite variances. We will assume that (1), the Lindeberg condition, holds and

(4) $(F_k : k \geqq 1)$ is tight, i.e., given $\varepsilon > 0$, there exists $A > 0$ so that $F_k([-A, A]^c) \leqq \varepsilon$ for all k.

Before we state Corollary 2, we prove 3 lemmas.

LEMMA 1. *Let* $(X_k\colon k \geqq 1)$ *be as above. Then if* (4) *holds and there exists constants* $M > 0$ *and* $\eta > 0$ *so that, for each* $n \geqq 1$,

$$s_n^{-2} \sum_{k=1}^n \int_{|x|\leqq M} x^2\, dF_k(x) \geqq \eta ,$$

then there exists $A > 0$ *so that for each* $n \geqq 1$,

$$s_n^{-2} \sum_{k=1}^n \int_{|x|\leqq 2A} x^2\, dF_k^s(x) \geqq \eta .$$

PROOF. Choose $A \geqq M$ so large that $m = \min_k P(|X_k| \leqq A) > (2+\eta)/(2+2\eta)$ and let $(\hat{X}_k\colon k \geqq 1)$ be independent of $(X_k\colon k \geqq 1)$ and have the same distribution. Then

$$\begin{aligned} s_n^{-2} &\sum_{k=1}^n \int_{|x|\leqq 2A} x^2\, dF_k^s(x) \\ &\geqq s_n^{-2} \sum_{k=1}^n \int_{\{|X_k|\leqq A, |\hat{X}_k|\leqq A\}} (X_k - \hat{X}_k)^2\, dP \\ &= s_n^{-2} \sum_{k=1}^n 2P(|X_k| \leqq A) \int_{|x|\leqq A} x^2\, dF_k(x) - s_n^{-2} \sum_{k=1}^n 2\Big(\int_{|x|\leqq A} x\, dF_k(x)\Big)^2 \\ &\geqq 2(m\eta - 1 + m) > \eta . \end{aligned}$$

Here the Cauchy–Schwarz inequality was applied to get

$$s_n^{-2} \sum_{k=1}^n \Big(\int_{|x|\leqq A} x\, dF_k(x)\Big)^2 = s_n^{-2} \sum_{k=1}^n \Big(\int_{|x|> A} x\, dF_k(x)\Big)^2 \leqq 1 - m .$$

LEMMA 2. *Let* $(F_n\colon n \geqq 1)$ *be a sequence of distribution functions with characteristic functions* $(\varphi_n\colon n \geqq 1)$. *Suppose that* (F_n) *is tight and for* $t \neq 0$, $\limsup (1/n) \sum_{k=1}^n |\varphi_k(t)| < 1$. *Then, if* $[a, b]$ *does not contain* 0, *there exists* $\alpha < 1$ *and* n_0 *satisfying*:

$$n \geqq n_0 \quad \textit{implies} \quad n^{-1} \sum_{k=1}^n |\varphi_k(t)| \leqq \alpha \qquad \textit{for} \quad t \in [a, b] .$$

PROOF. Since $(F_n\colon n \geqq 1)$ is tight, the sequence $(\varphi_n\colon n \geqq 1)$, restricted to $[a, b]$, is relatively compact in the topology of uniform convergence on the continuous functions on $[a, b]$. It follows from the Arzelà–Ascoli theorem that $(\varphi_n\colon n \geqq 1)$ is equicontinuous on $[a, b]$. Hence, if we define $\zeta_N(t) = \sup_{n\geqq N} (1/n) \sum_{k=1}^n |\varphi_k(t)|$, $(\zeta_N\colon N \geqq 1)$ is equicontinuous and since the sequence is nonincreasing, $\zeta_N(t) \to \limsup (1/n) \sum_{k=1}^n |\varphi_k(t)|$ uniformly on $[a, b]$. The lemma follows.

LEMMA 3. *There exists a universal constant* $c > 0$ *so that for any random variable* X *with distribution function* F *and characteristic function* φ,

$$|\varphi(t)| \leqq \exp\Big(-ct^2 \int_{|x|\leqq 1/|t|} x^2\, dF^s(x)\Big) .$$

PROOF.

$$\begin{aligned} (1 - |\varphi(t)|^2)/t^2 &= \int_{-\infty}^{\infty} (1 - \cos(tx))/t^2\, dF^s(x) \\ &\geqq \int_{|xt|\leqq 1} (x^2(1 - \cos(tx))/t^2x^2)\, dF^s(x) \geqq \alpha \int_{|xt|\leqq 1} x^2\, dF^s(x) \end{aligned}$$

where $\alpha = \inf((1 - \cos(u))/u^2\colon |u| \leqq 1)$. Take $c = \alpha/2$ and the result follows from the fact that $1 - x \leqq e^{-x}$.

COROLLARY 2. *Let* $(X_k : k \geqq 1)$ *be as above. Suppose that conditions* (1), (4) *and the following hold*:

(5) *There exist constants* $M > 0$ *and* $\eta > 0$ *satisfying*

$$s_n^{-2} \sum_{k=1}^n \int_{|x| \leqq M} x^2 \, dF_k(x) \geqq \eta .$$

(6) *If* $t \neq 0$, *then* $\limsup (1/n) \sum_{k=1}^n |\varphi_k(t)| < 1$.

Then,

$(2\pi)^{\frac{1}{2}} s_n P(S_n \in (a + x, b + x)) \to b - a$ *uniformly for* x *in bounded intervals.*

PROOF. By Lemma 1, there exists a constant A so that

$$s_n^{-2} \sum_{k=1}^n \int_{|x| \leqq A} x^2 \, dF_k^s(x) \geqq \eta .$$

By Lemma 3, $\prod_{k=1}^n |\varphi_k(t/s_n)| \leqq \exp(-ct^2/s_n^2 \sum_{k=1}^n \int_{|x| \leqq s_n/|t|} x^2 \, dF_k^s(x))$. But for $t \in [-s_n A^{-1}, s_n A^{-1}]$, the right-hand side is dominated by $e^{-c\eta t^2}$. Thus we may choose $\varepsilon_n = A^{-1}$ for each n. Finally, by Lemma 2, given N, there exist n_0 and $\alpha < 1$ satisfying $\prod_{N+1}^{N+n} |\varphi_k(t)| \leqq (n^{-1} \sum_{N+1}^{N+n} |\varphi_k(t)|)^n \leqq \alpha^n$ for $t \in [A^{-1}, b]$, all $n \geqq n_0$.

COROLLARY 3. *Suppose that* $s_n^2 = O(n)$ *and that* (1) *and* (4) *hold. Moreover, assume that each* X_k *has density function* f_k *and that* $\sup_k \int_{-\infty}^{\infty} f_k^2(x) \, dx < \infty$. *Then the local limit theorem holds.*

PROOF. By the inversion theorem for Fourier transforms (Chung (1968), Theorem 6.2.1), $P(|X_k^s| \leqq \varepsilon) = (\varepsilon/\pi) \int_{-\infty}^{\infty} (\sin(t\varepsilon)/t\varepsilon) |\varphi_k(t)|^2 \, dt$. It follows from the Plancherel theorem that

$$\sup_k P(|X_k^s| \leqq \varepsilon) = O(\varepsilon) . \tag{7}$$

By (4), for each $\delta > 0$ there exists $\alpha = \alpha(\delta)$ and a bounded sequence $(a_k : k \geqq 1)$, depending on δ, with the property that $\inf_k P(|X_k - a_k| \leqq \delta) > \alpha$. Thus

(8) for each $\delta > 0$, there exists $\alpha = \alpha(\delta)$ with $\inf_k P(|X_k^s| \leqq \delta) > \alpha$.

Combining (7) and (8) we have

(9) for each $\delta > 0$, there exists $\varepsilon \in (0, \delta)$ and $\alpha > 0$ so that

$$\inf_{k \geqq 1} P(|X_k^s| \in [\varepsilon, \delta]) \geqq \alpha .$$

Fix $\delta > 0$ and choose ε and α as in (9). Then if $t \in [-\delta^{-1} s_n, \delta^{-1} s_n]$ we have, by Lemma 3,

$$\begin{aligned} \prod_{k=1}^n |\varphi_k(t/s_n)| &\leqq \exp(-ct^2 s_n^{-2} \sum_{k=1}^n \int_{|x| \leqq s_n/|t|} x^2 \, dF_k^s(x)) \\ &\leqq \exp(-ct^2 s_n^{-2} \sum_{k=1}^n \int_{|x| \leqq \delta} x^2 \, dF_k^s(x)) \\ &\leqq \exp(-ct^2 s_n^{-2} n \varepsilon^2 \alpha) \leqq \exp(-c\zeta \varepsilon^2 \alpha t^2) . \end{aligned}$$

Here we choose ζ so that $\zeta s_n^2 \leqq n$. Thus (2) is satisfied with $\varepsilon_n = \delta^{-1}$. Fix $b > 0$ and choose $\varepsilon \in (0, b^{-1})$ and α as in (9) so that

$$\inf_k P(|X_k^s| \in [\varepsilon, b^{-1}]) \geqq \alpha .$$

Then

$$|\varphi_k(t)| \leqq \exp(-ct^2 \int_{|x|\leqq 1/|t|} x^2 \, dF_k(x))$$
$$\leqq \exp(-ct^2 \int_{|x|\leqq b^{-1}} x^2 \, dF_k(x)) \leqq \exp(-ct^2\varepsilon^2\alpha)$$

whenever $|t| \leqq b$. Thus (3) holds. This corollary now follows from Theorem 1.

REMARK. The same proof shows that under (1), (4), (5) and

$$\limsup_{n\to\infty} n^{-1} \sum_{k=1}^{n} \int_{-\infty}^{\infty} f_k^2(x)\, dx < \infty ,$$

then the local limit theorem holds.

4. Recurrence. In this section we use the lemma of Mineka and Silverman (1970) to give conditions under which a random walk with nonidentically distributed components is recurrent.

Throughout the section $(X_n : n \geqq 1)$ will be a sequence of independent mean 0 random variables with finite variances. For a real number z and $\Delta > 0$ set $\Delta_z = (z, z + \Delta)$. The walk is recurrent if and only if for each z and Δ,

$$P(S_n \in \Delta_z, \text{ i.o.}) = 1 . \tag{10}$$

A condition equivalent to (10) is

(11) for each integer $k \geqq 0$, $\Delta > 0$ and z,

$$P(S_{n+k} - S_k \in \Delta_z, \text{ for some } n) = 1 ,$$

i.e., every interval is visited at least once with probability 1 by every tail.

As in [3] set ${}_kU_n(z) = P(S_n - S_k \in \Delta_z)$ and, for $\gamma \in [0, \Delta]$,

$$a_{nk}(\gamma) = \sum_{m=k}^{n} {}_kU_m(-\gamma)[\sum_{m=1}^{n} {}_0U_m(z)]^{-1} .$$

If

(12) for each γ and k, $\lim_n a_{nk}(\gamma) = 1$, and

(13) there exists $L > 0$ so that for all n sufficiently large $a_{nk}(\gamma) < L$, all $k \geqq 1$ and all $\gamma \in [0, \Delta]$;

then,

(14) $\sum_{k=1}^{\infty} f_k(\Delta_z) = 1$. Here $f_k(\Delta_z) = P(S_k \in \Delta_z, S_j \notin \Delta_z, \text{ all } j < k)$.

This result follows from the dominated convergence theorem. We now state the lemma of [3] which implies (11).

LEMMA 4 (Mineka and Silverman). *If $(X_n : n \geqq 1)$ is such that for all tails of the sequence, and for all z and $\Delta > 0$,* (12) *and* (13) *hold, then the random walk generated by (X_n) is recurrent.*

In Lemma 6 we need the following lemma which is stated without proof.

LEMMA 5. *Let $(a_n : n \geqq 1)$ and $(y_n : n \geqq 1)$ be sequences of positive numbers with $\lim_n a_n y_n = t$ and $\sum_{n=1}^{\infty} y_n = \infty$. Then, for fixed M,*

$$\lim_n \sum_{k=M}^{n} a_k^{-1} (\sum_{k=M}^{n} y_k)^{-1} = t^{-1} .$$

LEMMA 6. *Let* $(X_n: n \geqq 1)$ *be as above. Moreover, suppose that* $\sum_{n=1}^{\infty} s_n^{-1} = \infty$ *and that for each* $k \geqq 1$, $(X_n: n \geqq k)$ *satisfies the local limit theorem. Then for each* $k \geqq 1$,

$$\lim_n a_{nk}(\gamma) = 1 , \qquad \text{all} \quad \gamma \in [0, \Delta] .$$

PROOF. Fix $k \geqq 1$ and adopt the following notation:

$$a_n = (\mathrm{Var}\,(S_n - S_k))^{\frac{1}{2}} , \qquad s_n = (\mathrm{Var}\, S_n)^{\frac{1}{2}} , \qquad x_n(\gamma) = {}_kU_n(-\gamma) \qquad \text{and}$$
$$y_n = P(S_n \in \Delta_z) .$$

From Theorem 1, $\sup_{\gamma \leqq \Delta} |(2\pi)^{\frac{1}{2}} a_m x_m(\gamma) - \Delta| \to 0$ as $m \to \infty$. Moreover, $|(2\pi)^{\frac{1}{2}} s_m y_m - \Delta| \to 0$ as $m \to \infty$.

Let $\varepsilon > 0$ and $\eta > 0$ be given and choose M so large that $m \geqq M$ implies

(i) $\sup_{\gamma \leqq \Delta} |(2\pi)^{\frac{1}{2}} a_m x_m(\gamma) - \Delta| < \varepsilon/2$,
(ii) $|(2\pi)^{\frac{1}{2}} s_m y_m - \Delta| < \varepsilon/2$,
(iii) $1 \leqq s_m/a_m \leqq 1 + \eta$.

By Lemma 5 choose $N > M$ so large that when $n \geqq N$,

(iv) $(\sum_{m=M}^{n} a_m^{-1})(\sum_{m=M}^{n} y_m)^{-1} \leqq 2(2\pi)^{\frac{1}{2}}\Delta^{-1}$.

From (i) and (ii) follow

(v) $-\varepsilon < a_m x_m(\gamma) - s_m y_m < \varepsilon$,
(vi) $y_m - \varepsilon a_m^{-1} < x_m(\gamma) < s_m a_m^{-1} y_m + \varepsilon a_m^{-1} < (1 + \eta) y_m + \varepsilon a_m^{-1}$.

Summing (vi) over $m = M, M + 1, \cdots, n$ and dividing by $\sum_{m=M}^{n} y_m$ yields, by (iv)

(vii) $1 - \varepsilon 2(2\pi)^{\frac{1}{2}}\Delta^{-1} \leqq \sum_M^n x_m(\gamma)(\sum_M^n y_m)^{-1} \leqq (1 + \eta) + \varepsilon 2(2\pi)^{\frac{1}{2}}\Delta^{-1}$.

Now note that for $n \geqq N$,

$$\begin{aligned} a_{nk}(\gamma) &\leqq \textstyle\sum_{m=k}^{M-1} x_m(\gamma)(\sum_{m=1}^{n} y_m)^{-1} + \sum_{m=M}^{n} x_m(\gamma)(\sum_{m=M}^{n} y_m)^{-1} \\ &\leqq (M - k)(\textstyle\sum_1^n y_m)^{-1} + 1 + \eta + \varepsilon 2(2\pi)^{\frac{1}{2}}\Delta^{-1} . \end{aligned}$$

Since ε and η were arbitrary, $\limsup_{n\to\infty} a_{nk}(\gamma) = 1$.

To show that $\liminf_{n\to\infty} a_{nk}(\gamma) = 1$, note that

$$a_{nk}(\gamma) \geqq (1 - \varepsilon 2(2\pi)^{\frac{1}{2}}\Delta^{-1}) \textstyle\sum_M^n y_m(\sum_1^n y_m)^{-1} \qquad \text{so that}$$
$$\liminf_{n\to\infty} a_{nk}(\gamma) \geqq 1 - \varepsilon 2(2\pi)^{\frac{1}{2}}\Delta^{-1} .$$

This completes the proof since ε was arbitrary.

LEMMA 7. *Let* $(X_n: n \geqq 1)$ *satisfy the local limit theorem. Fix* $z \in (-\infty, \infty)$ *and assume* (C_Δ): *there exists* $M \geqq 1$ *and* $C > 0$ *so that* $\sup_k s_n P(S_{n+k} - S_k \in (-\Delta, \Delta)) \leqq C\Delta$, $n \geqq M$. *Then there exists* $L > 0$ *and integer* N *so that whenever* $n \geqq N$, $k \geqq 1$ *and* $\gamma \in [0, \Delta]$

$$a_{nk}(\gamma) < L .$$

PROOF. Let $\zeta \in (0, (2\pi)^{-\frac{1}{2}})$ and choose $N \geqq M$ so large that $P(S_n \in \Delta_z) \geqq \zeta\Delta s_n^{-1}$

whenever $n \geqq N$. For $n \geqq N$,

$$\begin{aligned} &\textstyle\sum_{m=k}^{n} {}_kU_m(-\gamma) \\ &\quad\leqq \textstyle\sum_{m=k}^{n} P(S_m - S_k \in (-\Delta, \Delta)) \leqq N + \sum_{m=N+k}^{n} P(S_m - S_k \in (-\Delta, \Delta)) \\ &\quad\leqq N + \textstyle\sum_{N+k}^{n} C\Delta s_{m-k}^{-1} = N + \sum_{m=N}^{n-k} C\Delta s_m^{-1} \\ &\quad\leqq N + (C/\zeta) \textstyle\sum_{m=1}^{n} P(S_m \in \Delta_z) . \end{aligned}$$

Thus

$$a_{nk}(\gamma) \leqq N/(\textstyle\sum_{m=1}^{n} P(S_m \in \Delta_z)) + C/\zeta \leqq N/(\sum_{m=1}^{N} P(S_m \in \Delta_z)) + C/\zeta = L .$$

THEOREM 2. *Let $(X_n: n \geqq 1)$ be a sequence of independent mean 0 random variables with finite variances and suppose that for each $k \geqq 1$, $(X_n: n \geqq k)$ satisfies the local limit theorem. Moreover, suppose that for each $\Delta > 0$, (X_n) satisfies (C_Δ). Then the random walk generated by (X_n) is recurrent if $\sum_{n=1}^{\infty} s_n^{-1} = \infty$.*

PROOF. Let $z \in (-\infty, \infty)$ and $\Delta > 0$ be arbitrary. By Lemma 7, there exist constants $N \geqq 1$ and $L > 0$ with $a_{nk}(\gamma) < L$ whenever $n \geqq N$, $k \geqq 1$ and $\gamma \in [0, \Delta]$. By Lemma 6, $a_{nk}(\gamma) \to 1$ since each tail of (X_n) satisfies the local limit theorem. Thus by (14),

$$P(S_n \in \Delta_z, \text{ some } n) = 1 .$$

Let $K \geqq 1$ and define $a_{nk}^K(\gamma) = \sum_{m=K+k}^{n} {}_{K+k}U_m(-\gamma)(\sum_{m=K}^{n} {}_KU_m(z))^{-1}$. Since each tail of $(X_n: n \geqq K)$ is a tail of $(X_n: n \geqq 1)$, it follows that $a_{nk}^K(\gamma) \to 1$ by Lemma 6.

Moreover, $\lim_n (s_{n+K}^2 - S_K^2)/s_n^2 = 1$ by (1). Thus (C_Δ) holds for $(X_n: n \geqq K)$ also. We conclude that for each $k \geqq K$, $P(S_n - S_k \in \Delta_z, \text{ some } n > K) = 1$. Since z and Δ were arbitrary this completes the proof by (11).

COROLLARY 4. *Suppose that each tail of (X_n) satisfies the local limit theorem, (4) holds, $s_n^2 = O(n)$ and*

$$\lim_{\varepsilon \downarrow 0} \lim_{n \to \infty} n^{-1} \sup_{k \geqq 0} \textstyle\sum_{j=k+1}^{n+k} P(|X_j^s| \leqq \varepsilon) = 0 . \tag{15}$$

Then, if $\sum_{n=1}^{\infty} s_n^{-1} = \infty$, the random walk generated by (X_n) is recurrent.

PROOF. By Theorem 2 it suffices to show that, for each $\Delta > 0$, (C_Δ) holds. Fix $\Delta > 0$ and define

$$\begin{aligned} \hat{h}(x) &= (2\Delta^{-1} - |x|)/4\Delta^{-2} \qquad &&\text{if } |x| \leqq 2\Delta^{-1} \\ &= 0 &&\text{otherwise.} \end{aligned}$$

The transform of $\hat{h}$ is $h(t) = \sin^2(t/\Delta)/t^2\Delta^{-2}$. On $[-\Delta, \Delta]$, $h(t) \geqq \beta^2$ where $\beta = \inf(t^{-1}\sin(t): 0 \leqq t \leqq 1)$. Thus,

$$\begin{aligned} &s_n P(S_{n+k} - S_k \in (-\Delta, \Delta)) \\ &\qquad \leqq \beta^{-2} s_n E(h(S_{n+k} - S_k)) \\ &\qquad = \beta^{-2} \textstyle\int_{-\delta s_n}^{\delta s_n} \prod_{k+1}^{n+k} \varphi_j(ts_n^{-1}) \hat{h}(ts_n^{-1})\, dt \qquad \text{where } \delta = 2\Delta^{-1} . \end{aligned}$$

Now by Lemma 3, for $|t| \leqq \delta s_n$,

$$\begin{aligned} \textstyle\prod_{k+1}^{n+k} |\varphi_j(t/s_n)| &\leqq \exp(-ct^2 s_n^{-2} \textstyle\sum_{k+1}^{n+k} \int_{|x| \leqq s_n/|t|} x^2\, dF_j^s(x)) \\ &\leqq \exp(-ct^2 s_n^{-2} \textstyle\sum_{k+1}^{n+k} \int_{|x| \leqq \Delta/2} x^2\, dF_j^s(x) . \end{aligned}$$

Choose $\alpha = \alpha(\Delta/2)$ as in (8) and pick $\varepsilon < \Delta/2$ so small that, for n sufficiently large,

$$\sup_k n^{-1} \sum_{k+1}^{n+k} P(|X_j^s| \leqq \varepsilon) < \alpha/2 .$$

This is possible by (15). Then the last exponential is no more than

$$\exp(-ct^2 s_n^{-2} n\varepsilon^2(\alpha - n^{-1} \sum_{k+1}^{n+k} P(|X_j^s| \leqq \varepsilon))) \leqq \exp(-ct^2\zeta\varepsilon^2\alpha/2)$$

where ζ is chosen so that $ns_n^{-2} \geqq \zeta$. Set $K = c\zeta\varepsilon^2\alpha$. We have

$$s_n E(h(S_{n+k} - S_k)) \leqq \hat{h}(0) \int_{-\delta s_n}^{\delta s_n} e^{-Kt^2/2}\, dt \leqq \hat{h}(0) \int_{-\infty}^{\infty} e^{-Kt^2/2}\, dt$$
$$= \hat{h}(0)(2\pi/K)^{\frac{1}{2}} = \Delta(\pi/2K)^{\frac{1}{2}} .$$

This corollary now follows from Theorem 2.

From Corollary 4 we have

COROLLARY 5. *Under the conditions of Corollary* 3, *the random walk generated by* (X_n) *is recurrent if* $\sum_{n=1}^{\infty} s_n^{-1} = \infty$.

COROLLARY 6. *Under the conditions of Corollary* 1, *the random walk generated by* (X_n) *is recurrent if* $\sum_{n=1}^{\infty} s_n^{-1} = \infty$.

PROOF. By Theorem 1, each tail of the sequence $(\sigma_n X_n : n \geqq 1)$ satisfies the local limit theorem. Thus, the proof will be complete, by Theorem 2, if we show that for each $\Delta > 0$, (C_Δ) holds. The proof of this fact again uses $\hat{h}$ as defined in Corollary 4. The tedious computations are similar to those of Theorem 1 and will be omitted.

REMARKS. If (X_n) satisfies the local limit theorem and $\sum_n s_n^{-1} < \infty$, then by the Borel–Cantelli lemma, the random walk generated by (X_n) is transient. We thus have the result that if each tail of (X_n) satisfies the local limit theorem, and (C_Δ) holds for each Δ, then recurrence is equivalent to the divergence of the series $\sum s_n^{-1}$.

In Theorem 1 we had to prove the local limit theorem for each tail directly as it is possible for $(X_n : n \geqq 1)$ to satisfy the local limit theorem and $(X_n : n \geqq 2)$ not to satisfy the local limit theorem. For example, let c be the constant for which $ct^{-4} \sin^4(t/4)$ is a density on $(-\infty, \infty)$. Its Fourier transform, $\varphi_1(t)$, vanishes outside the interval $[-1, 1]$. Let X_1 be a random variable with the above density and let $X_2, X_3, \cdots$ be i.i.d. and independent of X_1 with $P(X_2 = 1) = P(X_2 = -1) = \frac{1}{2}$. The sequence $(X_n : n \geqq 2)$ does not satisfy the local limit theorem. However, if h is integrable and the Fourier transform of $\hat{h}$ which is continuous with compact support then

$$(2\pi)^{\frac{1}{2}} s_n E(h(S_n)) = (2\pi)^{\frac{1}{2}} \int_{-s_n}^{s_n} \varphi_1(t/s_n) \cos^{n-1}(t/s_n) \hat{h}(t/s_n)\, dt .$$

On $[-s_n, s_n]$ the integrand is dominated by Ce^{-2kt^2/π^2} where C and k are constants. Thus as in Theorem 1 we conclude that $(X_n : n \geqq 1)$ satisfies the local limit theorem.

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