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# CRITERIA FOR RECURRENCE AND EXISTENCE OF INVARIANT MEASURES FOR MULTIDIMENSIONAL DIFFUSIONS[1]

BY R. N. BHATTACHARYA

*University of Arizona*

Let $L = \frac{1}{2}\sum_{i,j=1}^{k} a_{ij}(x)(\partial^2/\partial x_i\,\partial x_j) + \sum_{i=1}^{k} b_i(x)(\partial/\partial x_i)$ be an elliptic operator such that $a_{ij}(\cdot)$ are continuous and $b_i(\cdot)$ are measurable and bounded on compacts. Criteria for transience, null recurrence, and positive recurrence of diffusions on $R^k$ governed by $L$ are derived in terms of the coefficients of $L$.

**1. Introduction.** The main objective of this article is to obtain criteria for transience, positive recurrence, and null recurrence of diffusions on $R^k$ governed by elliptic operators $L = \frac{1}{2}\sum_{i,j=1}^{k} a_{ij}(x)(\partial^2/\partial x_i\,\partial x_j) + \sum_{i=1}^{k} b_i(x)(\partial/\partial x_i)$ in terms of the coefficients of $L$. The matrix $((a_{ij}(x)))$ is assumed to be nonsingular for each $x$; the functions $a_{ij}(\cdot)$ are continuous, and the functions $b_i(\cdot)$ are Borel measurable and bounded on compacts. For the case $k = 1$ complete characterizations are known (see, e.g., Mandl (1968)). For $k > 1$ important criteria were announced without proof by Khas'minskii (1960) in a supplement to his paper [3], under the hypothesis that the coefficients of $L$ are thrice continuously differentiable. The first derivation of criteria for recurrence and transience analogous to Khas'minskii's is due to Friedman (1973), who assumed the coefficients to be Lipschitzian on compacts and to satisfy certain growth conditions at infinity. As far as we know there has not appeared in the literature any proof of Khas'minskii's criteria (or analogous ones) for positive and null recurrence. Since positive recurrence is essentially equivalent to the existence of a (unique) invariant probability measure determining the ergodic behavior of the diffusion (see, e.g., Khas'minskii (1960), Maruyama and Tanaka (1959)), such criteria are of importance. Theorem 3.5 provides a criterion for positive recurrence which implies the corresponding criterion of Khas'minskii (1960) (Theorem III of his Supplement). It also provides a criterion for null recurrence which is comparable to Khas'minskii's (when specialized to Khas'minskii's assumptions), although neither implies the other. We are unable to verify Khas'minskii's criterion for null recurrence. Theorem 3.3 is an improvement upon Friedman's criteria for transience and recurrence. The criteria derived in this article are exact if $L$ is radial near infinity. Among other results we mention Theorem 3.2 establishing a dichotomy (into transience and recurrence) in the class of all diffusions considered here.

*Throughout this article we assume* $k \geqq 2$.

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**2. Notation and preliminaries.** This section is devoted to background material. Proofs are given only when results are not readily available in the desired form.

Let $L = \frac{1}{2}\sum_{i,j=1}^{k} a_{ij}(x)(\partial^2/\partial x_i\,\partial x_j) + \sum_{i=1}^{k} b_i(x)(\partial/\partial x_i)$ be an elliptic operator on $R^k$. More precisely, we assume

(A) The matrix $((a_{ij}(x)))$ is real symmetric and positive definite for each $x$ in $R^k$, the functions $a_{ij}(\cdot)$ are continuous. The functions $b_i(\cdot)$ are real valued, Borel measurable and bounded on compacts.

For $N = 1, 2, \cdots, 1 \leqq i, j \leqq k$, define

$$\begin{aligned} a_{ij,N}(x) &= a_{ij}(x) \\ b_{i,N}(x) &= b_i(x) \qquad \text{if} \quad |x| \leqq N\,, \end{aligned}$$

$$\begin{aligned} a_{ij,N}(x) &= a_{ij}(x_0) \\ b_{i,N}(x) &= b_i(x_0) \qquad \text{if} \quad |x| = cx_0 \quad \text{for some} \quad x_0\,, \quad |x_0| = N\,, \\ &\qquad\qquad\quad \text{and some} \quad c > 1\,. \end{aligned}$$

Let $L_N = \frac{1}{2}\sum_{i,j=1}^{k} a_{ij,N}(x)(\partial^2/\partial x_i\,\partial x_j) + \sum_{i=1}^{k} b_{i,N}(x)(\partial/\partial x_i)$.

For $A \subset R^k$, $\bar{A}$ denotes the *closure* of $A$ and $A^c$ denotes the *complement of* $A$. Also, $\partial A$ denotes the *boundary of* $A$. The symbol $|x|$ stands for the *Euclidean norm of* $x$.

Denote the space $C([0, \infty) : R^k)$ of all continuous functions on $[0, \infty)$ into $R^k$ by $\Omega'$. Endow $\Omega'$ with the topology of uniform convergence on compact subsets of $[0, \infty)$. Let $\mathscr{M}'$ denote the Borel sigma field of $\Omega'$. Let $X(t) = X(t, \cdot)$ be the $t$th coordinate map: $X(t, \omega) = \omega(t)$ for $\omega \in \Omega'$. The sigma field generated by $\{X(s) : 0 \leqq s \leqq t\}$ is denoted by $\mathscr{M}_t'$ $(0 \leqq t < \infty)$. A function $\tau$ on $\Omega'$ into $[0, \infty]$ is a *stopping time* if $\{\tau \leqq t\} \in \mathscr{M}_t'$ for all $t \geqq 0$. If $\tau$ is a stopping time then the map $X_\tau^-$ on $\Omega'$ into $\Omega'$ defined by

$$X_\tau^-(t) = X(\tau \wedge t) \qquad t \geqq 0$$

is measurable and is called the *process stopped at* $\tau$. The *pre-*$\tau$ *sigma field* $\mathscr{M}_\tau'$ is generated by $\{X(\tau \cap t) : t \geqq 0\}$. Also measurable on the restriction of $(\Omega', \mathscr{M}')$ to $\{\tau < \infty\}$ is the map $X_\tau^+$ defined by

$$X_\tau^+(t) = X(\tau + t) \qquad t \geqq 0\,.$$

Let $\{P_x : x \in R^k\}$ be a family of probability measures on $(\Omega', \mathscr{M}')$ such that for every a.s. $(P_x)$ finite stopping time $\tau$ a regular conditional distribution of $X_\tau^+$ given $\mathscr{M}_\tau'$ is $P_{X(\tau)}$. We then say that $X$ is a *strong Markov process* under $P_x$. Such a process is said to be *strong Feller* if for every bounded real measurable function $f$ on $R^k$ the function: $x \to E_x f(X(t))$ is continuous on $R^k$ for each $t > 0$. Here $E_x$ denotes *expectation under* $P_x$. The following result due to Stroock and Varadhan ([7]—[9]) will be frequently used in this article.

THEOREM 2.1. *If, in addition to the hypothesis* (A), $a_{ij}(\cdot)$ *and* $b_i(\cdot)$ *are bounded on* $R^k$, *then for each* $x$ *in* $R^k$ *there exists a unique probability measure* $P_x$ *on* $(\Omega', \mathscr{M}')$

*such that* (i) $P_x(X(0) = x) = 1$, (ii) *for every bounded real f on $R^k$ having bounded and continuous first and second order derivatives, the process*

$$f(X(t)) - \int_0^t Lf(X(s))\,ds \qquad t \geqq 0$$

*is a martingale under $P_x$. Further,* (a) *X is strong Markov and strong Feller, and* (b) *support of $P_x$* is $\Omega_x' = \{\omega \in \Omega' : \omega(0) = x\}$.

Let $P_{x,N}$ denote the probability measure in Theorem 2.1 with $L = L_N$.

The following simple result will also be needed. For any set $A$, $\chi_A$ is the *indicator function* of $A$.

LEMMA 2.2. *Let U be a nonempty bounded open subset of $R^k$. Let*

$$\tau_U = \inf\{t \geqq 0 : X(t) \notin U\}\,. \tag{2.1}$$

*Under the hypothesis of Theorem* 2.1, *for every bounded real Borel measurable function f on U, the function $E_x(\chi_{\{\tau_U > t\}} f(X(t)))$ is continuous on U. If $\phi$ is a real valued bounded measurable function on $\partial U$, then the function $E_x(\phi(X(\tau_U)))$ is continuous on U.*

PROOF. The first assertion is proved in Dynkin (1965) (Volume II, page 30, relation (13.4)). It is of course necessary to check Dynkin's hypothesis that

$$\lim_{t \downarrow 0} \sup_{x \in D} P_x(|X(t) - x| > \varepsilon) = 0 \tag{2.2}$$

for every compact subset $D$ of $U$.

But (2.1) follows from the inequality (see Stroock and Varadhan (1969), page 355)

$$\sup_{x \in R^k} P_x(|X(t) - x| > \varepsilon) \leqq 2k \exp\{-(\varepsilon - \beta t)^2/2\alpha t\} \qquad \varepsilon > 0,\ t > 0 \tag{2.3}$$

where $\beta^2$ is an upper bound of $\sum b_i^2(x)$ for all $x$, and $\alpha$ is an upper bound for the largest eigenvalue of $((a_{ij}(x)))$, $x \in R^k$. To prove the second assertion define $\tilde{\phi}$ on $\bar{U}$ by letting $\tilde{\phi} = \phi$ on $\partial U$ and $\tilde{\phi} = c$ on $U$ where $c \leqq \phi(x)$ for all $x$ in $\partial U$. Then, according to Dynkin (1965) (Volume II, page 30, relation (13.5)), the function $h_t(x) \equiv E_x \tilde{\phi}(X(\tau_U \wedge t))$ is continuous on $U$ for every $t > 0$. Letting $t \uparrow \infty$, one has $h_t(x) \uparrow \phi(x)$. Hence $\phi$ is lower semicontinuous. Similarly, $-\phi$ is lower semicontinuous. □

To construct probability measures $P_x$ under the hypothesis (A) replace the "state space" $R^k$ by its *one point compactification* $R^k \cup \{\infty\}$. Let $\Omega = C([0, \infty) : R^k \cup \{\infty\})$ be the set of all continuous functions on $[0, \infty)$ into $R^k \cup \{\infty\}$ and endow $\Omega$ with the topology of uniform convergence (relative to some metric metrizing $R^k \cup \{\infty\}$) on compact subsets of $[0, \infty)$. Let $\mathscr{M}$ be the Borel sigma field of $\Omega$. We continue to denote by $X(t)$ the $t$th *coordinate map* (this time on $\Omega$ into $R^k \cup \{\infty\}$). Also, $\mathscr{M}_t$ will denote the sigma field generated by $\{X(s) : 0 \leqq s \leqq t\}$, and $\mathscr{M}_\tau$ will denote the *pre-$\tau$ sigma field* for any stopping time $\tau$ (relative to $\mathscr{M}_t$, $t \geqq 0$) on $\Omega$. Denote by $P_\infty$ the *probability measure degenerate at* $\omega_\infty$ where $\omega_\infty(t) = \infty$ for all $t \geqq 0$. For $x \neq \infty$, one way to construct $P_x$ is to introduce the product probability space $(E, \mathscr{E}, \mu)$, where $E$ is the Cartesian product $\times_{x,N} \Omega'_{x,N}$ (each $\Omega'_{x,N}$ being a copy of $\Omega'$), $\mathscr{E}$ is the product sigma field, and $\mu$ is the product probability $\times_{x,N} P_{x,N}$. If $x_0$, $N_0$ are such that $|x_0| < N_0$, define a map $Y$ on $E$ into $\Omega$ by requiring that $Y(t) = X_{x_0,N_0}(t)$ (here $X_{x,N}(t)$ is the $t$th coordinate map on $\Omega'_{x,N}$) for $t \leqq \eta_1 \equiv \inf\{s \geqq 0 : |X_{x_0,N_0}(s)| = N_0\}$, $Y(t) = X_{X(\eta_i),N_0+i}(t)$ for $\eta_i < t \leqq \eta_{i+1}$, where $\eta_{i+1} - \eta_i = \inf\{s \geqq 0 : X_{X(\eta_i),N_0+i}(s)| = N_0 + i\}$; let $\eta_\infty = \lim_{i\uparrow\infty} \eta_i$ and define $Y(t) = \infty$ for $\eta_\infty \leqq t < \infty$. We denote by $P_{x_0}$ the probability measure on $(\Omega, \mathscr{M})$ induced by $Y$, i.e., $P_{x_0} = \mu \circ Y^{-1}$. It is simple to check from this construction that the *coordinate process* $X = \{X(t) : 0 \leqq t < \infty\}$ on $\Omega$ *is a strong Markov process under* $P_x$, $x \in R^k \cup \{\infty\}$.

On $\Omega$ define the stopping times $\tau_U$ for open subsets $U$ of $R^k$ as in (2.1), and define the *explosion time* $\zeta$ by

$$\zeta = \lim_{N\uparrow\infty} \tau_{B(0:N)} \tag{2.4}$$

where $B(0:N) = \{x \in R^k : |x| < N\}$. The probability measure $P_x$ $(x \in R^k)$ is said to be *conservative* if $P_x(\zeta = \infty) = 1$. A Borel measurable real valued function $f$ on $R^k$ will be said to be *L-harmonic* on an open subset $G$ of $R^k$ if it is bounded on compacts, and for all $x$ in $G$

$$f(x) = E_x f(X(\tau_U)) \tag{2.5}$$

for every neighborhood $U$ of $x$ having compact closure $\bar{U}$ in $G$. It may be remarked at this stage that the notation $P_x$, $E_x$ used here is consistent with that used earlier. For it follows immediately from the construction that under the hypothesis of Theorem 2.1 the present $P_x$ has support $\Omega_x'$ and coincides with the corresponding $P_x$ in Theorem 2.1 on $\Omega_x'$. Also, if (A) holds, then for $|x| < N$ the present measure $P_x$ agrees with the earlier $P_{x,N}$ on $\mathscr{M}_{\tau_{B(0:N)}}$ (i.e., on the trace of this sigma field on $\Omega_x'$). From now on we regard all measures $P_x$, $P_{x,N}$ to be defined on $(\Omega, \mathscr{M})$, and $E_x$, $E_{x,N}$ are corresponding expectations.

Part (a) of the following lemma may also be obtained from Dynkin (1965), Volume II, Theorem 13.2, page 31.

LEMMA 2.3. *Assume* (A) *holds.* (a) *Every L-harmonic function on an open subset G of $R^k$ is continuous in G.* (b) (*Maximum principle.*) *Let f be a nonnegative L-harmonic function on a connected open subset G of $R^k$. Then f is either strictly positive or identically zero.*

PROOF. (a) If $f$ is $L$-harmonic in $G$, $x \in G$, and $U$ is a neighborhood of $x$ such that the closure $\bar{U}$ of $U$ is compact in $G$, then

$$f(x) = E_x f(X(\tau_U)) = E_{x,N} f(X(\tau_U)) , \tag{2.6}$$

provided $U \subset B(0:N)$. By Lemma 2.2 the last expression in (2.6) is continuous in $U$.

(b) Suppose $f(x_0) = 0$. Let $B = B(x_0 : \varepsilon)$ be the open ball with center $x_0$ and radius $\varepsilon$ such that $\bar{B} \subset G$. Then

$$0 = f(x_0) = E_{x_0} f(X(\tau_B)) = \int_{\partial B} f(y)\Pi(x_0, dy) ,$$

where $\Pi(x_0, dy)$ is the distribution of $X(\tau_B)$ under $P_{x_0}$ and, hence, under $P_{x_0,N}$ if $N > |x_0| + \varepsilon$. By Theorem 2.1 the support of $\Pi(x_0, dy)$ is $\partial B$. Since $f \geqq 0$ and continuous it follows that $f \equiv 0$ on $\partial B$. Therefore, $f \equiv 0$ on $G$. □

LEMMA 2.4. *Assume* (A) *holds.* (a) *If $U$ is a nonempty open subset of $R^k$, $U \neq R^k$, then $x \to P_x(\tau_U < \infty)$ is positive and continuous on $U$.* (b) *If $U_1$, $U_2$ are two nonempty open subsets of $R^k$ such that $\bar{U}_1 \cap \bar{U}_2 = \phi$, $\bar{U}_2^c = R^k \backslash \bar{U}_2$ is connected, then $x \to P_x(\tau_{\bar{U}_1^c} < \tau_{\bar{U}_2^c})$ is positive and continuous on $\bar{U}_1^c \cap \bar{U}_2^c$.*

PROOF. It follows from the strong Markov property that both the functions in question are $L$-harmonic and, therefore, continuous. To prove positivity in (b) (which implies positivity in (a)) let $x \in \bar{U}_1^c \cap \bar{U}_2^c$. Take an open ball $B \subset U_1$ and let $B_1$ be a bounded open set such that $x \in B_1$, $B \subset B_1 \subset \{|x| < N\}$, $B_1 \cap \bar{U}_2 = \phi$. Then $P_x(\tau_{\bar{U}_1^c} < \tau_{\bar{U}_2^c}) \geqq P_x(\tau_{\bar{B}^c} < \tau_{\bar{B}_1^c}) = P_{x,N}(\tau_{\bar{B}^c} < \tau_{\bar{B}_1^c})$. The last expression is positive, since the support of $P_{x,N}$ is $\Omega_x'$. □

LEMMA 2.5. *Assume* (A) *holds. If $P_{x_0}$ is conservative for some $x_0 \in R^k$, then $P_x$ is conservative for all $x \in R^k$ and the process $\{X(t) : t \geqq 0\}$ has the strong Feller property.*

PROOF. Since $x \to P_x(\zeta < \infty)$ is harmonic, the first assertion follows. To prove the second let $f$ be a real valued bounded Borel measurable function on $R^k$. Assume $P_x$ is conservative for all $x \in R^k$. Fix $x_0$ in $R^k$. One has

$$(2.7) \qquad \begin{aligned} |E_x f(X(t)) - E_{x,N} f(X(t))| &= |\textstyle\int \chi_{\{\tau_{B(0:N)} \leq t\}} f(X(t))[P_x(d\omega) - P_{x,N}(d\omega)]| \\ &\leqq 2\|f\|[1 - P_{x,N}(\tau_{B(0:N)} > t)] \qquad t > 0\,, \end{aligned}$$

where $\|f\| = \sup |f(x)|$. Choose $\varepsilon > 0$ and fix $N$ such that the last expression in (2.7) is less than $\varepsilon/3$ if $x = x_0$. Since $x \to P_{x,N}(\tau_{B(0:N)} > t)$ is continuous on $R^k$ (by Lemma 2.2), and $x \to E_{x,N} f(X(t))$ is continuous (since $X$ is strongly Feller under $P_{x,N}$, $x \in R^k$), there exists $\delta > 0$ such that if $|x - x_0| < \delta$, then

$$\begin{aligned} |E_x f(X(t)) - E_{x_0} f(X(t))| &\leqq 2\|f\|[1 - P_{x,N}(\tau_{B(0:N)} > t) + 1 - P_{x_0,N}(\tau_{B(0:N)} > t)] \\ &\quad + |E_{x,N} f(X(t)) - E_{x_0,N} f(X(t))| < \varepsilon\,. \end{aligned}$$ □

LEMMA 2.6. *Assume* (A) *holds, $U$ is a bounded open subset of $R^k$. Then* $\sup_{x \in U} E_x(\tau_U) < \infty$.

PROOF. Let $N$ be such that $B(0 : N) \supset \bar{U}$. Then $E_x \tau_U = E_{x,N} \tau_U$ for $x \in U$. Fix $t_0 > 0$. Since $\Omega_x' = \{\omega \in \Omega' : \omega(0) = x\}$ is the support of $P_{x,N}$, $P_{x,N}(\tau_U > t_0) \leqq P_{x,N}(|X(t_0)| < N) < 1$ for all $x \in \bar{U}$. Since $x \to P_{x,N}(|X(t_0)| < N)$ is continuous, $\sup_{x \in \bar{U}} P_{x,N}(\tau_U > t_0) < 1$. Now use the inequality (see, e.g., Dynkin (1965), Volume I, Lemma 4.3, page 111)

$$E_{x,N} \tau_U \leqq \frac{t_0}{1 - \sup_{x \in U} P_{x,N}(\tau_u > t_0)}\,.$$ □

Finally, a nonzero measure $m$ on the Borel sigma-field $\mathscr{B}^k$ of $R^k$ is said to be *invariant* for the Markov process $P_x$, $x \in R^k$, if for all $B \in \mathscr{B}^k$ and all $t > 0$,

$$(2.10) \qquad m(B) = \textstyle\int_{R^k} P_x(\{X(t) \in B\}) m(dx)\,.$$

We shall henceforth refer to $P_x$, $x \in R^k$, or to the coordinate process under $P_x$, $x \in R^k$, *as the diffusion with generator L.*

**3. Criteria for recurrence and transience.** Assume (A) holds, and consider the diffusion $X$ (under $P_x$, $x \in R^k$) with generator $L$. *A point x in $R^k$ is said to be recurrent* for this diffusion if given any $\varepsilon > 0$

$$P_x(X(t) \in B(x:\varepsilon) \text{ for a sequence of } t\text{'s increasing to infinity}) = 1\,, \tag{3.1}$$

where $B(x:\varepsilon) = \{y: |y - x| < \varepsilon\}$. It follows that $x$ is a recurrent point if and only if for every $\varepsilon > 0$ and every a.s. ($P_x$) finite random variable $\tau$

$$P_x(X(t) \in B(x:\varepsilon) \text{ for some } t > \tau) = 1\,. \tag{3.2}$$

*A point x is transient if*

$$P_x(|X(t)| \to \infty \text{ as } t \to \infty) = 1\,. \tag{3.3}$$

If all points of a diffusion are recurrent, the *diffusion itself is called recurrent.* If all points of a diffusion are transient, the *diffusion is called transient.* It will be presently shown (see Theorem 3.2) that if (A) holds every diffusion is either recurrent or transient. Since different authors often use different definitions of recurrence and transience (see, e.g., Maruyama and Tanaka (1959), Khas'minskii (1960), and Friedman (1973)), it is useful to show that these definitons are equivalent.

PROPOSITION 3.1. *Assume that* (A) *holds. The following statements are equivalent.*

(a) *The diffusion is recurrent.*

(b) $P_x(X(t) \in U$ *for some* $t \geqq 0) = 1$ *for all* $x \in R^k$ *and all nonempty open U.*

(c) *There exists a compact set K of* $R^k$ *such that* $P_x(X(t) \in K$ *for some* $t \geqq 0) = 1$ *for all* $x \in R^k$.

(d) $P_x(X(t) \in U$ *for a sequence of t's increasing to infinity*$) = 1$ *for all* $x \in R^k$ *and all nonempty open U.*

(e) *There exist a point z in* $R^k$, *a pair of numbers* $r_0, r_1$, $0 < r_0 < r_1$, *and a point* $y \in \partial B(z:r_1) = \{y': |y' - z| = r_1\}$ *such that* $P_y(\tau_{\bar{B}^c(z:r_0)} < \infty) = 1$.

PROOF. The implications (b) $\Rightarrow$ (c), (b) $\Rightarrow$ (e), (d) $\Rightarrow$ (a), are obvious. We prove (a) $\Rightarrow$ (b), (b) $\Rightarrow$ (d), (c) $\Rightarrow$ (b), (e) $\Rightarrow$ (c).

(a) $\Rightarrow$ (b). Assume (a), $x \in R^k$, $U$ nonempty open, $x \notin U$. Let $B$ be an open ball such that $\bar{B} \subset U$. Choose $\varepsilon > 0$ such that $\overline{B(x:\varepsilon)} \cap \bar{B} = \phi$. Let $U_1$ be a bounded open set containing $\overline{B(x:\varepsilon)} \cup \bar{B}$. Define $\eta_1 = \tau_{U_1}$, $\eta_{2i} = \inf\{t > \eta_{2i-1}: X(t) \in \partial B(x:\varepsilon)\}$, $\eta_{2i+1} = \inf\{t > \eta_{2i}: X(t) \in \partial U_1\}$ $(i = 1, 2, \cdots)$. By Lemma 2.6 and recurrence of $x$, $\eta_i$'s are a.s. ($P_x$) finite stopping times. Consider the events $A_0 = \{X(t) \in \bar{B} \text{ for some } t \in [0, \eta_1)\}$, $A_i = \{X(t) \in \bar{B} \text{ for some } t \in [\eta_{2i-1}, \eta_{2i})\}$ $(i = 1, 2, \cdots)$. Since $y \to P_y(\tau_{\bar{B}^c} < \tau_{\overline{B(x:\varepsilon)}^c})$ is positive and continuous on $\bar{B}^c \cap \overline{B(x:\varepsilon)}^c$ (Lemma 2.4 (b)).

$$\delta \equiv \inf_{y \in \partial U_1} P_y(\tau_{\bar{B}^c} < \tau_{\overline{B(x:\varepsilon)}^c}) > 0\,. \tag{3.4}$$

Using the strong Markov property and induction on $n$ one obtains $P_x(\bigcap_{i=0}^n A_i^c) \leqq (1-\delta)^n$. Thus

$$P_x(X(t) \in U \text{ for no } t \geqq 0) \leqq P_x(X(t) \in \bar{B} \text{ for no } t \geqq 0) \leqq \lim_{n\to\infty} P_x(\textstyle\bigcap_{i=0}^n A_i^c) = 0 . \tag{3.5}$$

(b) $\Rightarrow$ (d). Let $x \in R^k$, $U$ nonempty open, $B$ an open ball and $\varepsilon > 0$ such that $\bar{B} \cap \overline{B(x:\varepsilon)} = \phi$ and $\bar{B} \subset U$. Define $\theta_1 = \inf\{t \geqq 0 : X(t) \in \partial B(x:\varepsilon)\}$, $\theta_{2i} = \inf\{t > \theta_{2i-1} : X(t) \in \partial B\}$, $\theta_{2i+1} = \inf\{t > \theta_{2i} : X(t) \in \partial B(x:\varepsilon)\}$ $(i = 1, 2, \cdots)$. By (b) and the strong Markov property, $\theta_i$'s are a.s. $(P_x)$ finite. Also, $\theta_i \uparrow \infty$ a.s. $(P_x)$ as $i \uparrow \infty$; otherwise, with positive $P_x$ probability the sequences $\{X(\theta_{2i-1}) : i = 1, 2, \cdots\}$ and $\{X(\theta_{2i}) : i = 1, 2, \cdots\}$ converge to a common limit, which is impossible since $\partial B(x:\varepsilon)$ and $\partial B$ are disjoint.

(c) $\Rightarrow$ (b). Let $K$ be as in (c), $B$ an arbitrary open ball, $x \in R^k$. Let $U$ be an open ball containing $\bar{B} \cup K$. Define $\eta_1' = \tau_{K^c}$, $\eta_{2i}' = \inf\{t > \eta_{2i-1}' : X(t) \in \partial U\}$, $\eta_{2i+1}' = \inf\{t > \eta_{2i}' : X(t) \in K\}$ $(i = 1, 2, \cdots)$. By (c), the strong Markov property and Lemma 2.6, the $\eta_i'$'s are a.s. $(P_x)$ finite. Now proceed as in the proof of (a) $\Rightarrow$ (b), i.e., define $A_i$'s with $\eta_i'$'s in place of $\eta_i$'s, and define $\delta = \inf_{y \in K} P_y(A_1)$; by Lemma 2.4 (b), $\delta > 0$; and $P_x(X(t) \in \bar{B}$ for no $t \geqq 0) \leqq P_x(\bigcap_{i=1}^n A_i^c) \leqq (1-\delta)^n$ for all $n$.

(e) $\Rightarrow$ (c). Follows from Lemma 2.4 (a) and the maximum principle (Lemma 2.3 (b)), if one takes $K = \overline{B(z:r_0)}$. □

The next result establishes a dichotomy in the class of all diffusions for which (A) holds.

THEOREM 3.2. *Assume* (A) *holds.* (a) *If there exists a recurrent point, then the diffusion is recurrent.* (b) *If there exists no recurrent point, then the diffusion is transient.*

PROOF. (a) Suppose $y$ is a recurrent point. Choose $r_0$, $r_1$ $(0 < r_0 < r_1)$, $z$ such that $|y - z| = r_1$. It has been shown in the course of the proof of Proposition 3.1 ((a) $\Rightarrow$ (b), (3.5)) that $P_y(X(t) \in \partial B(z:r_0)$ for some $t \geqq 0) = P_y(X(t) \in \overline{B(z:r_0)}$ for some $t \geqq 0) = 1$. By Proposition 3.1 (e), the diffusion is recurrent.

(b) Suppose no point in $R^k$ is recurrent. Fix $x \in R^k$. Let $r$ be an arbitrary positive number such that $r > |x|$. By Proposition 3.1 (e) and the maximum principle (Lemma 2.3 (b)), for each $r_1 > r$ one has

$$\delta_{r_1} \equiv \sup_{|y|=r_1} P_y(\tau_{\overline{B(0:r)}^c} < \infty) < 1 .$$

Define $\eta_1 = \inf\{t \geqq 0 : X(t) \in \partial B(0:r_1)\}$, $\eta_{2i} = \inf\{t > \eta_{2i-1} : X(t) \in \overline{B(0:r)}\}$, $\eta_{2i+1} = \inf\{t > \eta_{2i} : X(t) \in \partial B(0:r_1)\}$ $(i = 1, 2, \cdots)$. By Lemma 2.6 and the strong Markov property, for all $i \geqq 1$

$$\begin{aligned} &P_x(X(t) \in \overline{B(0:r)} \text{ for some sequence of } t\text{'s increasing to infinity}) \\ &\quad \leqq P_x(\eta_{2i+1} < \infty) = E_x(\chi_{\{\eta_{2i-1}<\infty\}} P_{X(\eta_{2i-1})}(\tau_{\overline{B(0:r)}^c} < \infty)) \\ &\quad \leqq \delta_{r_1} P_x(\eta_{2i-1} < \infty) \leqq \cdots \leqq \delta_{r_1}^i . \end{aligned} \tag{3.6}$$

Hence the left side of (3.6) is zero and

$$P_x(\liminf_{t\to\infty} |X(t)| > r) = 1 .$$

Since this holds for all $r > 0$, the proof is complete. □

The next theorem improves and extends a result of Friedman (1973). The criteria for recurrence and transience derived in Theorem 3.3 were announced without proof earlier by Khas'minskii (1960) (Theorem II of his Supplement) under the additional assumption that the coefficients of $L$ are thrice continuously differentiable. To prove it we introduce some notation.

Let $F$ be a real-valued twice continuously differentiable function on $(0, \infty)$. Let $z \in R^k$. Consider the function

$$f(x) = F(|x - z|) \qquad x \in R^k , \quad |x - z| > 0 . \tag{3.7}$$

A straightforward differentiation yields

$$\begin{aligned} \frac{\partial f(x)}{\partial x_i} &= \frac{(x_i - z_i)}{|x - z|} F'(|x - z|) , \\ \frac{\partial^2 f(x)}{\partial x_i^2} &= \frac{(x_i - z_i)^2}{|x - z|^2} F''(|x - z|) - \frac{(x_i - z_i)^2}{|x - z|^3} F'(|x - z|) + \frac{F'(|x - z|)}{|x - z|} , \\ \frac{\partial^2 f(x)}{\partial x_i \partial x_j} &= \frac{(x_i - z_i)(x_j - z_j)}{|x - z|^2} F''(|x - z|) - \frac{(x_i - z_i)(x_j - z_j)}{|x - z|^3} F'(|x - z|) \\ & \qquad i \neq j,\ |x - z| > 0 . \end{aligned} \tag{3.8}$$

Now fix $r_0 > 0$ and write $x' = x - z$ and

$$\begin{aligned} & A_z(x) = \textstyle\sum_{i,j=1}^k a_{ij}(x' + z) x_i' x_j' / |x'|^2 , \qquad B(x) = \sum_{i=1}^k a_{ii}(x' + z) , \\ & C_z(x) = 2 \textstyle\sum_{i=1}^k x_i' b_i(x' + z) , \qquad \bar\beta_z(r) = \sup_{|x'|=r} \frac{B(x) - A_z(x) + C_z(x)}{A_z(x)} , \\ & \underline{\beta}_z(r) = \inf_{|x'|=r} \frac{B(x) - A_z(x) + C_z(x)}{A_z(x)} , \qquad \bar\alpha_z(r) = \sup_{|x'|=r} A_z(x) , \\ & \underline{\alpha}_z(r) = \inf_{|x'|=r} A_z(x) , \qquad \bar I_z(r) = \textstyle\int_{r_0}^r \frac{\bar\beta_z(u)}{u} du , \qquad \underline{I}_z(r) = \int_{r_0}^r \frac{\underline{\beta}_z(u)}{u} du . \end{aligned} \tag{3.9}$$

It is easy to check that

$$2Lf(x) = A_z(x) F''(|x - z|) + \frac{F'(|x - z|)}{|x - z|} [B(x) - A_z(x) + C_z(x)] . \tag{3.10}$$

THEOREM 3.3. *Assume* (A) *holds.* (a) *If for some* $r_0 > 0$ *and* $z$

$$\textstyle\int_{r_0}^\infty \exp\{-\bar I_z(r)\}\, dr = \infty , \tag{3.11}$$

*then the diffusion with generator L is recurrent.* (b) *If for some* $r_0 > 0$ *and* $z$

$$\textstyle\int_{r_0}^\infty \exp\{-\underline{I}_z(r)\}\, dr < \infty , \tag{3.12}$$

*then the diffusion with generator L is transient.*

PROOF. (a) Assume (3.11) holds. Define

$$F(r) = -\int_{r_0}^{r} \exp\{-\bar{I}_z(u)\}\, du\,, \qquad f(x) = F(|x-z|) \qquad |x-z| \geqq r_0\,. \tag{3.13}$$

Let $x$ be such that $r \equiv |x - z| > r_0$. Define stopping times

$$\eta = \inf\{t \geqq 0 : X(t) \in \partial B(z : r_0)\}\,, \qquad \eta_N = \eta \wedge \tau_{B(z:N)}\,. \tag{3.14}$$

By Theorem 2.1, and optional sampling (see Nevue (1965), page 142), $f(X(t \wedge \eta_N)) - \int_0^{t\wedge\eta_N} Lf(X(s))\,ds$ $(t \geqq 0)$ is a $P_x$-martingale, provided $|x - z| < N$. Hence, for $r \equiv |x - z| > r_0$,

$$\begin{aligned} &2E_x F(|X(t \wedge \eta_N) - z|) - 2F(r) \\ &\quad = E_x \int_0^{t\wedge\eta_N} 2Lf(X(s))\, ds \\ &\quad \geqq E_x \int_0^{t\wedge\eta_N} A_z(X(s)) \left[ F''(|X(s) - z|) + \frac{F'(|X(s)-z|)}{|X(s)-z|}\, \bar{\beta}_z(|X(s) - z|) \right] ds \\ &\quad = 0\,, \end{aligned} \tag{3.15}$$

by the relations

$$F'(u) \leqq 0\,, \qquad F''(u) + \frac{1}{u} F'(u)\bar{\beta}_z(u) = 0 \qquad u \geqq r_0\,. \tag{3.16}$$

Letting $t \uparrow \infty$ in (3.15) and remembering that $\eta_N < \infty$ a.s. $(P_x)$, one obtains

$$-E_x F(|X(\eta_N) - z|) \leqq -F(r) = \int_{r_0}^{r} \exp\{-\bar{I}_z(u)\}\, du\,. \tag{3.17}$$

On evaluating the left side of (3.17) one has

$$P_x(\eta > \tau_{B(z:N)}) \int_{r_0}^{N} \exp\{-\bar{I}_z(u)\}\, du \leqq \int_{r_0}^{r} \exp\{-\bar{I}_z(u)\}\, du\,. \tag{3.18}$$

Letting $N \uparrow \infty$ one gets

$$P_x(\eta = \infty) \leqq \lim_{N\uparrow\infty} \frac{\int_{r_0}^{r} \exp\{-\bar{I}_z(u)\}\, du}{\int_{r_0}^{N} \exp\{-\bar{I}_z(u)\}\, du} = 0\,. \tag{3.19}$$

Hence $P_x(\eta < \infty) = 1$ and the diffusion is recurrent by Proposition 3.1 (e).

(b) Assume (3.12) holds. Define

$$G(r) = \int_{r_0}^{r} \exp\{-\underline{I}_z(u)\}\, du\,, \qquad g(x) = G(|x - z|) \qquad |x - z| \geqq r_0\,.$$

Since $G'(u) \geqq 0$ and $G''(u) + (1/u)G'(u)\underline{\beta}_z(u) = 0$ for $u \geqq r_0$, one obtains, as above,

$$E_x G(|X(\eta_N) - z|) - G(|x - z|) \geqq 0 \tag{3.20}$$

or,

$$P_x(\eta > \tau_{B(z:N)}) \int_{r_0}^{N} \exp\{-\underline{I}_z(u)\}\, du \geqq \int_{r_0}^{|x-z|} \exp\{-\underline{I}_z(u)\}\, du\,. \tag{3.21}$$

Hence, letting $N \uparrow \infty$,

$$P_x(\eta = \infty) \geqq \int_{r_0}^{|x-z|} \exp\{-\underline{I}_z(u)\}\, du \Big/ \int_{r_0}^{\infty} \exp\{-\underline{I}_z(u)\}\, du > 0\,. \tag{3.22}$$

Hence the diffusion is not recurrent (by Proposition 3.1) and therefore, it is transient (by Theorem 3.2). □

A recurrent diffusion admits a unique (up to a constant multiple) sigma finite invariant measure. This fact was proved by Maruyama and Tanaka (1959) in a more abstract setting and by Khas'minskii (1960). Khas'minskii's proof applies immediately to the present context. The following fact, which is easily deduced from Theorem 3.3 of Khas'minskii (1960) in conjunction with Lemma 2.6 and Proposition 3.1, will be needed.

LEMMA 3.4. *Assume* (A) *holds.* (a) *The diffusion is recurrent and admits a finite invariant measure if there exists z in $R^k$ such that*

$$\sup_{y \in \partial B(z:r_1)} E_y(\tau_{\overline{B(z:r_0)}^c}) < \infty \tag{3.23}$$

*for some $r_0$, $r_1$ satisfying $0 < r_0 < r_1$.* (b) *If there exist some z in $R^k$ and positive numbers $r_0$, $r_1$ $(0 < r_0 < r_1)$ satisfying*

$$E_y(\tau_{\overline{B(z:r_0)}^c}) = \infty \tag{3.24}$$

*for all $y \in \partial B(z : r_1)$, then there does not exist a finite invariant measure.*

Our final result is

THEOREM 3.5. *Assume* (A) *holds.* (a) *The diffusion with generator L is recurrent and admits a finite invariant measure* (*unique up to a constant multiple*) *if there exists z in $R^k$ and $r_0 > 0$ such that*

$$\int_{r_0}^{\infty} \exp\{-\bar{I}_z(u)\}\, du = \infty\,, \tag{3.25}$$

$$\int_{r_0}^{\infty} \frac{1}{\underline{\alpha}_z(u)} \exp\{\bar{I}_z(u)\}\, du < \infty\,. \tag{3.26}$$

(b) *If there exist z in $R^k$ and $r_0 > 0$ such that* (3.25) *holds and*

$$\lim_{N\to\infty} \frac{\int_{r_0}^{N} \exp\{-\bar{I}_z(s)\}(\int_{r_0}^{s} [\exp\{\bar{I}_z(u)\}/\bar{\alpha}_z(u)]\, du)\, ds}{\int_{r_0}^{N} \exp\{-\underline{I}_z(u)\}\, du} = \infty\,, \tag{3.27}$$

*then the recurrent diffusion does not admit a finite invariant measure.*

PROOF. (a) Assume (3.25), (3.26). Define

$$F(r) = -\int_{r_0}^{r} \exp\{-\bar{I}_z(s)\}\left(\int_{s}^{\infty} \frac{1}{\underline{\alpha}_z(u)} \exp\{\bar{I}_z(u)\}\, du\right) ds \qquad r > r_0\,. \tag{3.28}$$

Then

$$F'(r) = -\exp\{-\bar{I}_z(r)\} \int_{r}^{\infty} \frac{1}{\underline{\alpha}_z(u)} \exp\{\bar{I}_z(u)\}\, du < 0\,, \tag{3.29}$$

$$F''(r) = -\frac{\bar{\beta}_z(r)}{r} F'(r) + \frac{1}{\underline{\alpha}_z(r)} \qquad r \geqq r_0\,.$$

Let

$$f(x) = F(|x - z|) \qquad |x - z| \geqq r_0\,.$$

Then, using (3.29),

$$2Lf(x) \geqq A_z(x)/\underline{\alpha}_z(|x - z|) \geqq 1 \qquad |x - z| \geqq r_0\,. \tag{3.30}$$

If $\eta$, $\eta_N$ are as in (3.14) then, as in the proof of Theorem 3.3,

$$\begin{aligned}(3.31)\qquad & 2E_x f(|X(t \wedge \eta_N) - z|) - 2F(|x - z|) \\ & \geqq E_x \int_0^{t \wedge \eta_N} 2Lf(X(s))\, ds \\ & \geqq E_x(t \wedge \eta_N) \qquad r_0 \leqq |x - z| \leqq N .\end{aligned}$$

First letting $t \uparrow \infty$ in (3.31) and then letting $N \uparrow \infty$, one has

$$(3.32)\qquad E_x(\tau_{\overline{B(z:r_0)}^c}) = E_x \eta \leqq -2F(|x - z|) ,$$

since $\eta_N \uparrow \eta$ a.s. $(P_x)$ as $N \uparrow \infty$ (due to recurrence). Now apply Lemma 3.4(a).

(b) Assume (3.25), (3.27) hold. Define

$$\begin{aligned}(3.33)\qquad G(r) &= \int_{r_0}^{r} \exp\{-\bar{I}_z(s)\} \left( \int_{r_0}^{s} \frac{1}{\bar{\alpha}_z(u)} \exp\{\bar{I}_z(u)\}\, du \right) ds , \\ g(x) &= G(|x - z|) \qquad |x - z| \geqq r_0 .\end{aligned}$$

Since $G'(r) \geqq 0$ and $G''(r) = -(1/r)\bar{\beta}_z(r)G'(r) + 1/\bar{\alpha}_z(r)$, $2Lg(x) \leqq 1$. Therefore, one has

$$(3.34)\qquad E_x(t \wedge \eta_N) \geqq 2E_x G(|X(t \wedge \eta_N) - z|) - 2G(|x - z|) .$$

Letting $t \uparrow \infty$ in (3.34) one gets

$$\begin{aligned}(3.35)\qquad E_x(\eta_N) &\geqq 2E_x G(|X(\eta_N) - z|) - 2G(|x - z|) \\ &= 2P_x(\tau_{B(z:N)} < \eta)G(N) - 2G(|x - z|) .\end{aligned}$$

Let $N \uparrow \infty$ to obtain, using (3.21),

$$E_x(\eta) \geqq 2 \lim_{N \to \infty} \frac{\int_{r_0}^{|x-z|} \exp\{-\underline{I}_z(u)\}\, du}{\int_{r_0}^{N} \exp\{-\underline{I}(u)\}\, du} \cdot G(N) - 2G(|x - z|) = \infty .$$

The proof is now complete by Lemma 3.4(b). □

Following the terminology used for Markov chains one may call a point $x$ in $R^k$ *positive recurrent* for the diffusion with generator $L$ if for every $r_0$, $r_1$ $(0 < r_0 < r_1)$ one has $E_x(\theta(x; r_0, r_1)) < \infty$, where

$$\theta(x; r_0, r_1) = \inf\{t > \tau_{B(x:r_1)} : |X(t) - x| = r_0\} .$$

If $x$ is recurrent but not positive recurrent then $x$ is called *null recurrent.* If all points in $R^k$ are positive (null) recurrent, the *diffusion is called positive* (respectively, *null*) *recurrent.* In this terminology, Theorem 3.5 provides criteria for positive and null recurrence. The criterion for positive recurrence extends a criterion announced by Khas'minskii (1960; Theorem III of his Supplement) to more general coefficients. The criterion for null recurrence given here is comparable in strength to Khas'minskii's (1960; Theorem III of Supplement), when specialized to Khas'minskii's hypothesis; however, neither implies the other.

An indication of the sensitivity of the criteria provided by Theorems 3.3, 3.5 are afforded by the fact that if for some $z$ the functions $A_z(x)$, $B(x) + C_z(z)$, defined by (3.10), are functions of $|x'|$ for sufficiently large $|x'|$, then the criteria are exact.

For in this case one may delete the "bars" and assert: (i) the diffusion is recurrent if and only if $\int_{r_0}^{\infty} \exp\{-I_z(u)\}\,du = \infty$ for some $r_0 > 0$; (ii) a recurrent diffusion is positive or null according as $\int_{r_0}^{\infty} [\exp\{I_z(u)\}/\alpha_z(u)]\,du$ is finite or infinite.

**4. Some remarks.** This section is devoted to some miscellaneous comments on the material in the preceding sections.

First, the definition of transience used in this article leaves open the possibility that a transient diffusion may be nonconservative. In order to restrict oneself to conservative diffusions one may use the following criterion for explosion essentially proved by McKean (1969), pages 102–104, and earlier stated by Khas'minskii (1960) (Theorem I of his Supplement), extending a one-dimensional result of Feller: *Assume that, in addition to* (A), *the coefficients* $a_{ij}$, $b_i$ *are Lipschitzian on compacts.* (a) *If, for some* $z \in R^k$ *and some* $r > 0$,

$$\int_r^{\infty} \exp\{-\underline{I}_z(s)\} \left(\int_r^s \frac{1}{\bar{\alpha}_z(u)} \exp\{\bar{I}_z(u)\}\,du\right) ds = \infty , \tag{4.1}$$

*then the diffusion is conservative.* (b) *If, for some* $z \in R^k$ *and some* $r > 0$,

$$\int_r^{\infty} \exp\{-\underline{I}_z(s)\} \left(\int_r^s \frac{1}{\underline{\alpha}_z(u)} \exp\{\underline{I}_z(u)\}\,du\right) ds < \infty , \tag{4.2}$$

*then the diffusion is almost surely explosive*, i.e., $P_x(\zeta < \infty) = 1$ *for all* $x \in R^k$.

Secondly, the problem of studying diffusions (in our sense) on those open subsets of $R^k$ which are $C^2$-diffeomorphic to $R^k$ is easily reduced to the investigation on $R^k$ in view of Itô's lemma (see McKean (1969)) which enables one to compute "drift" and "diffusion" coefficients of the transformed process, at least when the corresponding process on $R^k$ has coefficients which are Lipschitzian on compacts.

Finally, suppose that the coefficients of $L$ satisfy (A) and are Hölder continuous on compacts. Using standard results from the theory of elliptic partial differential equations one can show that if $z$ is a positive recurrent point then the function $u(y) = E_y(\tau_{\overline{B(z:r_0)}^c})$ is continuous on $\overline{B(z:r_0)}^c$ for every $r_0 > 0$ (indeed, it is twice differentiable and satisfies $Lu = -1$). From the uniqueness (up to a constant multiple) of the invariant measure and Lemma 3.4 it then follows that all points in $R^k$ are positive recurrent. There is then a complete classification of diffusions into transient, null recurrent, and positive recurrent ones.

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF ARIZONA
TUCSON, ARIZONA 85721