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UPPER BOUNDS FOR THE RENEWAL FUNCTION VIA FOURIER METHODS

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Stone has used Fourier analytic methods to show that the renewal function $U(x) = \sum_0^\infty F^{n*}(x)$ for a random variable X with distribution function F, finite second moment and positive mean $\lambda^{-1} = EX$, is bounded above by $\lambda x_+ + C\lambda^2 EX^2$ for a universal constant C, $1 \leqq C < 3$. This paper refines his method to prove that $C < 2.081$, and shows that within certain constraints the smallest upper bound on C that the method will yield is 1.809.

Various authors' work on the simpler case where $X \geqq 0$ is summarized: the best result is the earliest published one, due to Lorden, who showed that then $C = 1$.

1. Introduction. Stone (1972) has proved that for any random variable (rv) X with finite first moment $\lambda^{-1} = EX > 0$ and finite second moment $\beta/\lambda^2 = EX^2$ (so that $\beta \geqq 1$), there exists a constant C independent of the distribution F of X such that the renewal function

$$U(x) = \textstyle\sum_{n=0}^\infty F^{n*}(x) \tag{1.1}$$

is bounded above as follows:

$$U(x) \leqq \lambda x_+ + C\beta \quad (\text{all} \ \ x)\,. \tag{1.2}$$

He showed that $1 \leqq C < 3$, and remarked that the upper bound could conceivably be made significantly smaller.

Lorden (1970) showed via methods using sub-additivity and Wald's lemma that the first passage time rv $N(x) = \inf\{n: S_n > x\}$ satisfies

$$EN(x) \leqq \lambda x + C\lambda^2 EX_+{}^2 \quad (\text{all} \ \ x > 0)$$

with $C = 1$. In the case that $X \geqq 0$, we have equality throughout in

$$U(x) = E\#\{n = 0, 1, \cdots : S_n \leqq x\} \geqq E \inf\{n: S_n > x\} \quad (x \geqq 0)$$

so that, as Professor Lorden kindly pointed out in correspondence, the proof via real variable methods in my recent paper (Daley (1976)), showing that $C \leqq 1.3186\cdots$, yields a result weaker than one already extant. It may be added here that O'Brien (1976) showed that the basic principle of my argument yields $C \leqq (3(3^{\frac{1}{2}}))/4 = 1.299\cdots$; a further modification to the method yields a tighter bounding constant, $C(\beta)$ say, for which $C(\beta)\uparrow$ with $\beta\uparrow$, $C(\beta)\downarrow 1$ for $\beta \downarrow 1$ and $\uparrow 1.299\cdots$ for $\beta\uparrow\infty$. O'Brien's approach enabled some progress to be made on using real variable methods to establish (1.2) for general X, but nothing as

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general as that given below. It should also be noted that Borovkov (1976) has given (in an appendix on renewal theory) an upper bound similar to (1.2), namely

$$U(x) \leqq (1 - p)^{-1}\{1 + 2(\lambda m_p)^{-1}\beta\}$$

where $0 < p < 1$ and m_p is nonzero and a pth quantile of the distribution of X which is assumed to be nonnegative.

The purpose of the present note is to examine Stone's Fourier analytic technique in some detail. The main result, which is a consequence of recognizing the broader validity of an identity and using a sharper inequality in Stone's line of reasoning, is the tighter bound in Theorem 1.

THEOREM 1. $1 \leqq C < 2.119418$.

Using the subadditivity of the renewal function in place of its nondecreasing property leads to the sharper bound in Theorem 2.

THEOREM 2. *There exists a function* $C(y)$ $(0 \leqq y < \infty)$, *decreasing monotonically*, $2.119418 > C(0) \geqq C(y) \downarrow 1$ $(y \to \infty)$, *such that*

$$U(x) - \lambda x_+ \leqq C(\lambda x/\beta)\beta . \tag{1.3}$$

Finally we shall show that the use of various inequalities on both the characteristic function and the renewal function in conjunction with the Fourier technique, will not yield as an upper bound on C anything smaller than 1.80884. The proof of this result suggests how the bound of Theorem 1 might be sharpened, and we do in fact effect a marginal improvement to

$$1 \leqq C < 2.080642 . \tag{1.4}$$

Further improvement is probably possible but slight.

In view of Lorden's sharp result for renewal processes, this limitation appears to be one of technique rather than suggesting the falsity of the conjecture that $C = 1$ irrespective of whether or not X is confined to be nonnegative.

2. Preliminaries. Let $\phi(\theta) = Ee^{i\theta X} = \int_{-\infty}^{\infty} e^{i\theta x}\, dF(x)$ be the characteristic function of X. Then the elementary trigonometric inequality $\sin u \geqq u(1 - u/\pi)$ (all u) yields

$$\begin{aligned} |1 - \phi(\theta)| &\geqq |\mathrm{Im}\, \phi(\theta)| = |E \sin \theta X| \\ &\geqq E \sin \theta X \geqq \theta EX - \theta^2 EX^2/\pi , \end{aligned} \tag{2.1}$$

and hence, since $|\mathrm{Im}\, \phi(\theta)| = |\mathrm{Im}\, \phi(-\theta)|$, that

$$|\theta|/|1 - \phi(\theta)| \leqq \lambda/(1 - \beta|\theta|/\pi\lambda) \quad (|\theta| \leqq \pi\lambda/\beta) . \tag{2.2}$$

This inequality is sharper than Stone's equation (7). The constant π appearing in (2.2) cannot be increased without making the inequality more complex algebraically: see Daley (1975).

Introduce $I(x) = 0$ or 1 as $x <$ or $\geqq 0$,

(2.3) $$G(x) = \int_{-\infty}^{x} (I(u) - F(u))\,du \to EX \quad (x \to \infty)\,,$$

(2.4) $$S(x) = \int_{-\infty}^{x} (EXI(u) - G(u))\,du \to EX^2/2 \quad (x \to \infty)\,.$$

We now write Stone's equation (11) more generally (Stone has $M = \lambda/\beta$) as

(2.5) $$\begin{aligned} J_1(x) &\equiv \int_{-\infty}^{\infty} MK(M(x-y))(U(y) - \lambda y_+ - \lambda^2 S(y))\,dy \\ &= \lambda^2(2\pi i)^{-1} \int_{-M}^{M} e^{-ix\theta} k(\theta/M)(\phi(\theta) - 1 - i\theta EX)^2\theta^{-3}(1 - \phi(\theta))^{-1}\,d\theta\,, \end{aligned}$$

valid where $|1 - \phi(\theta)| > 0$, viz., certainly where $0 < M < 2\pi\lambda/\beta$ (see, e.g., Daley (1975)), in which K and k are the probability density function

(2.6) $$MK(Mx) = (1 - \cos Mx)/\pi M x^2$$

and its characteristic function

(2.7) $$k(\theta/M) = \int_{-\infty}^{\infty} e^{i\theta x}\, MK(Mx)\,dx = (1 - |\theta|/M)_+\,.$$

The identity (2.5), cited by Stone, is a special case of a series of identities in Dubman (1970), on whose argument the derivation of (2.5) given in Section 6 is based. Observe the interpretation of (2.5) as the equality of the convolution of $MK(Mx)$ and the bounded measurable function $W(x) \equiv U(x) - \lambda x_+ - \lambda^2 S(x)$, with the Fourier inverse of the product of $k(\theta/M)$ and "Fourier transform" of $W(\cdot)$ (this last transform may be only formally defined). The feature of (2.5) that ensures its actual validity, and not merely formally so, is that $k(\cdot)$ vanishes outside $(-1, 1)$, and the scale factor M is so chosen as to ensure that any singularities of the "transform" of $W(\cdot)$ lie outside the support of $k(\cdot)$.

3. Proof of Theorem 1. Substitute into the identity (2.5) the inequalities (2.2) and

(3.1) $$|\phi(\theta) - 1 - i\theta EX| \leqq \theta^2 EX^2/2 = \theta^2\beta/2\lambda^2\,,$$

yielding (since $k(\cdot)$ is nonnegative and symmetric)

(3.2) $$\begin{aligned} |J_1(x)| &\leqq (\lambda^2/2\pi) \int_{-M}^{M} k(\theta/M)(\beta^2/4\lambda^4)\lambda(1 - |\theta|\beta/\pi\lambda)^{-1}\,d\theta \\ &= (\beta\zeta/4) \int_0^1 k(u)(1 - \zeta u)^{-1}\,du\,, \end{aligned}$$

provided that $0 < \zeta \equiv \beta M/\pi\lambda < 1$. Since

(3.3) $$\int_{-\infty}^{\infty} MK(M(x-y))\lambda^2 S(y)\,dy \leqq \beta/2\,,$$

we thus have in place of Stone's equation (12) the inequality

(3.4) $$\begin{aligned} \beta/2 + (\beta\zeta/4) &\int_0^1 k(u)(1 - \zeta u)^{-1}\,du \\ &\geqq J_2(x) \\ &\equiv \int_{-\infty}^{\infty} MK(M(x-y)((U(y) - \lambda y_+)\,dy \\ &= \int_{-\infty}^{\infty} K(-v)[U(x + v/M) - \lambda(x + v/M)_+]\,dv\,. \end{aligned}$$

Now $U(x) - \lambda x_+$ is nonnegative, and U is a nondecreasing function, so for all

$x > 0$, and any constants $A < B$,

$$J_2(x) \geqq \int_A^B K(-v)[U(x + v/M) - \lambda(x + v/M)_+]\,dv \tag{3.5}$$
$$\geqq \int_A^B K(-v)[U(x) - \lambda x - \lambda v/M]\,dv\,.$$

Set $A = 0$ and $B = M(U(x) - \lambda x)/\lambda$, so that on combining (3.5) with (3.4),

$$\int_0^{M(U(x)-\lambda x)/\lambda} [M(U(x) - \lambda x)/\lambda - v]K(-v)\,dv \tag{3.6}$$
$$\leqq \pi\zeta/2 + (\pi\zeta^2/4) \int_0^1 k(u)(1 - \zeta u)^{-1}\,du\,.$$

The function appearing on the left-hand side here (and recall that $K(\cdot)$ is symmetric), namely

$$g(y) \equiv \int_0^y (y - v)K(v)\,dv\,, \tag{3.7}$$

is a strictly increasing function of y, $\to \infty$ for $y \to \infty$, and $g(0) = 0$, so there is a *unique* root ξ of

$$g(\xi) = \pi\zeta/2 + (\pi\zeta^2/4) \int_0^1 k(u)(1 - \zeta u)^{-1}\,du\,. \tag{3.8}$$

Consequently, $M(U(x) - \lambda x)/\lambda \leqq \xi$ for all $x > 0$, or, observing that ξ depends on ζ,

$$U(x) - \lambda x_+ \leqq \lambda\xi(\zeta)/M = (\xi(\zeta)/\pi\zeta)\beta\,. \tag{3.9}$$

Thus, it remains to evaluate the bound

$$C \leqq \inf_{0<\zeta<1} \xi(\zeta)/\pi\zeta\,. \tag{3.10}$$

Substitute for K and k at (3.6) and (3.7) from (2.6) and (2.7). Writing

$$\mathrm{Si}\,(x) = \int_0^x u^{-1} \sin u\,du\,, \qquad \mathrm{Cin}\,(x) = \int_0^x u^{-1}(1 - \cos u)\,du\,, \tag{3.11}$$

$$\xi\,\mathrm{Si}\,(\xi) - \mathrm{Cin}\,(\xi) - (1 - \cos\xi) = \pi^2(3\zeta + (1 - \zeta)\ln(1 - \zeta))/4\,. \tag{3.12}$$

Solution of (3.12) yields for (3.10) the bound asserted in Theorem 1; it is attained at $(\xi, \zeta) = (5.582059, .8383549)$.

4. Using subadditivity and Theorem 2. We shall use here the subadditive property of the renewal function

$$U(x + y) \leqq U(x) + U(y) \quad (\text{all } x, y)\,. \tag{4.1}$$

PROOF OF (4.1). If both x and y are $\leqq 0$, (4.1) follows from the nonnegativity and monotonicity of U. Otherwise, assume $y > 0$. Set $S_0 = 0$, $S_{n+1} = S_n + X_{n+1}$ where the X_n's are independent random variables distributed like X, so that $U(x) = E\#\{n \geqq 0\colon S_n \leqq x\}$. Either there is a least n, $= \nu(x, y)$ say, such that $x < S_{\nu(x,y)} \leqq x + y$, or else for every $n = 0, 1, \cdots$, either $S_n \leqq x$ or $S_n > x + y$, and we can set $\nu(x, y) = \infty$. Writing $A_{x,y}(u) = \Pr\{\nu(x, y) < \infty, S_{\nu(x,y)} - x \leqq u\}$,

$$U(x + y) - U(x) = \int_0^y [U(y - u) - U(-u)]\,dA_{x,y}(u) \tag{4.2}$$
$$\leqq U(y) - U(-y) \leqq U(y)\,,$$

proving (4.1).

It follows from (4.1) that the function

$$V_0(x) = U(x) - \lambda x \tag{4.3}$$

is also subadditive; note that the function

(4.4) $$V(x) = U(x) - \lambda x_+ = V_0(x) - \lambda x_-$$

appearing in (3.4) and (3.5) need not be subadditive. In the first inequality in (3.5), set $B = \eta > 0$ and $A = -\eta$, and recall that $K(\cdot)$ is symmetric. Then for $x \geqq 0$ we can write

(4.5) $$\begin{aligned} J_2(x) &\geqq \int_0^\eta K(v)[V_0(x + v/M) + V_0(x - v/M) - \lambda(x - v/M)_-]\, dv \\ &\geqq \int_0^\eta K(v)[V_0(2x) - \lambda(x - v/M)_-]\, dv\,. \end{aligned}$$

Combine (4.5) with (3.4); we obtain

(4.6) $$\begin{aligned} (U(2x) - 2\lambda x)/\beta &\leqq C(y, \eta, \zeta)|_{y=2\lambda x/\beta} \\ &\equiv \frac{1+\frac{1}{2}\zeta \int_0^1 k(u)(1-\zeta u)^{-1}\, du + (2/\pi\zeta) \int_{\min(\eta, Mx)}^\eta vK(v)\, dv}{2 \int_0^\eta K(v)\, dv} \end{aligned}$$

(recall that $M = \pi\lambda\zeta/\beta$). Inequality (4.6) can be rewritten in the form of Theorem 2 by defining

(4.7) $$C(y) = \inf_{0<\zeta<1, \eta>0} C(y, \eta, \zeta)\,.$$

$C(y)$ decreases with increasing $y > 0$ by inspection, and to check that $C(y) \downarrow 1$ $(y \to \infty)$, take ζ small to make the first integral in the numerator sufficiently small, take η large enough to make the denominator close to 1, and then take y large enough to make the second integral in the numerator vanish.

To evaluate $C(y)$, observe that the infimum at (4.7) is attained in the restricted range $\eta \geqq \pi\zeta y/2$, and the two possibilities, $\eta >$ and $= \pi\zeta y/2$, require separate consideration (note that the former must hold for $y = 0$). Take $\eta > \pi\zeta y/2$, and, keeping ζ fixed, the infimum of (4.7) with respect to η occurs (by differentiation) where

(4.8) $$\begin{aligned} \eta \int_0^\eta K(v)\, dv = \pi\zeta/2 &+ (\pi\zeta^2/4) \int_0^1 k(u)(1 - \zeta u)^{-1}\, du \\ &+ \int_{\min(\eta, \pi\zeta y/2)}^\eta vK(v)\, dv\,. \end{aligned}$$

If we take (4.8) as an implicit definition of $\eta_y(\zeta)$, it follows from (4.7) that

(4.9) $$C(y) = \inf_{0<\zeta<1} \eta_y(\zeta)/\pi\zeta\,,$$

provided this infimum exceeds $y/2$. Referring to equation (3.8) and (3.10) it follows that the bound of Theorem 1 equals $C(0)$. Theorem 2 is proved.

Suppose finally that in (4.7), $\eta = \pi\zeta y/2$. Then differentiation shows that the infimum occurs where ζ satisfies

(4.10) $$\begin{aligned} (1 + (\zeta/2) \int_0^1 k(u)(1 - \zeta u)^{-1}\, du)\pi y K(\pi\zeta y/2) \\ = \int_0^{\pi\zeta y/2} K(v)\, dv \int_0^1 k(u)(1 - \zeta u)^{-2}\, du\,; \end{aligned}$$

complications may arise, as is the case in the computations referred to below, from (4.10) having more than one root ζ in $0 < \zeta < 1$.

We have evaluated the bound $C(y)$ for the kernel function $k(\theta) = (1 - |\theta|)_+$

used in Section 3. The infimum occurs where $\eta = \pi\zeta y/2$ for $y \geqq 3.0687$, with $0 < \eta < 2\pi$ for $y < 19.7$, $2\pi < \eta < 4\pi$ for $19.7 < y < 58$, and $\eta > 4\pi$ for larger y ($2n\pi < \eta < 2(n+1)\pi$ for an interval of values of y). For large y, $C(y) \cong 1 + 3/\pi(2y)^{\frac{1}{2}}$.

5. Possible limitations of the technique. It is shown in Section 6 that equation (2.5) holds on the range $0 < M < 2\pi\lambda/\beta$ for certain $K(\cdot)$ and $k(\cdot)$ more general than at (2.6) and (2.7) with $k(\cdot)$ real and vanishing outside $(-1, 1)$. In Sections 3 and 4 it was convenient to take $k(\cdot)$ nonnegative, and throughout this section we assume that $k(\cdot)$ is as in Section 6 and also that it is nonnegative.

From the inversion formula for probability density functions,

$$K(v) = \pi^{-1} \int_0^1 \cos \theta v k(\theta)\, d\theta\,, \tag{5.1}$$

so that equation (3.7) can be expressed as

$$g(y) = \pi^{-1} \int_0^1 \theta^{-2}(1 - \cos \theta y) k(\theta)\, d\theta\,, \tag{5.2}$$

enabling the equation (3.8) to be rewritten in the form

$$\int_0^1 [(1 - \cos \theta\xi)/\pi^2\theta^2 - \zeta^2/4(1 - \zeta\theta)] k(\theta)\, d\theta = \zeta/2\,. \tag{5.3}$$

For fixed $\xi < 2\pi$, $\theta^{-2}(1 - \cos \theta\xi)$ decreases monotonically on $0 < \theta < 1$, while $\zeta^2/4(1 - \zeta\theta)$ increases monotonically on the same interval. So if $2\xi^2 > \pi^2\zeta^2$ and $4(1 - \cos \xi) < \pi^2\zeta^2/(1 - \zeta)$, there exists θ_0 in (0, 1) such that the integrand in (5.3) is positive or negative as $\theta <$ or $> \theta_0$. Since $0 \leqq k(\theta) \leqq 1$ (all θ), it follows that

$$\zeta < 2 \int_0^{\theta_0} [(1 - \cos \theta\xi)/\pi^2\theta^2 - \zeta^2/4(1 - \zeta\theta)]\, d\theta\,,$$

i.e.,

$$2\pi^2\zeta\theta_0 < 4\xi\theta_0 \operatorname{Si}(\xi\theta_0) - 4(1 - \cos \xi\theta_0) + \pi^2\zeta\theta_0 \ln (1 - \zeta\theta_0)\,. \tag{5.4}$$

Now the equation

$$(1 - \zeta\theta_0)(1 - \cos \xi\theta_0) = (\pi\zeta\theta_0/2)^2 \tag{5.5}$$

and the inequality (5.4) are consistent only provided that $\xi\theta_0$ exceeds the unique positive root of (5.5) and (5.4) with equality. Hence, *there is a smallest possible upper bound obtainable by using this Fourier transform technique in conjunction with the inequalities* (2.2), (3.1), *the nondecreasing nature of* $U(x)$, *and the nonnegativity of* $U(x) - \lambda x_+$. This least possible upper bound on C equals 1.80884, obtained at $(\xi\theta_0, \zeta\theta_0) = (3.30133, .580952)$.

The bound could in principle be tightened by replacing (2.2) by an inequality based not on (2.1) but on the relation

$$\begin{aligned} |1 - \phi(\theta)| &= \{(E(1 - \cos \theta X))^2 + (E \sin \theta X)^2\}^{\frac{1}{2}} \\ &\geqq \{(|\theta EX| - \theta^2 EX^2/2\pi)^4/4 + (|\theta EX| - \theta^2 EX^2/\pi)^2\chi\}^{\frac{1}{2}} \\ &\qquad (|\theta| < 2\pi|EX|/EX^2) \end{aligned} \tag{5.6}$$

where $\chi = 1$ or 0 as $|\theta| <$ or $\geqq \pi|EX|/EX^2$. The additional algebraic complexity is readily apparent.

The argument concerning (5.3) suggests seeking an improvement to Theorem 1 via a ch.f. $k(\theta)$ that has a broader peak around $\theta = 0$ and higher order contact with the axis at $\theta = 1$ than the function $(1 - |\theta|)_+$. An example of such a ch.f. is the following (obtained by convolving $(1 - |\theta|)_+$ with itself and rescaling: see Esseen (1945)):

$$(5.7)\qquad \begin{aligned} k_2(\theta) &= 1 - 6\theta^2 + 6|\theta|^3, \quad & |\theta| < .5\,, \\ &= (1 - |\theta|)_+^3, & |\theta| \geqq .5\,. \end{aligned}$$

(The corresponding density $K_2(x) = (3/8\pi)\,(\sin(x/4)/(x/4))^4$.) Our calculations showed that the infimum at (3.10) occurs for $\zeta = 1$, a happy chance since the integral at (3.8) is then simpler to compute, and leads to the slight improvement on Theorem 1 asserted at (1.4).

6. Proof of the identity (2.5). Throughout this section which is largely based on part of Section 3 of Dubman (1970), $K(\cdot)$ denotes a continuous symmetric probability density function whose characteristic function $k(\cdot)$ vanishes outside $(-1, 1)$. We take $M > 0$ to be such that $|\phi(\theta)| < 1$ for $-M \leqq \theta \leqq M$ excluding the point $\theta = 0$. Then the inversion theorem for characteristic functions of continuous probability density functions ensures that

$$(6.1)\qquad \begin{aligned} MK(Mx) &= (2\pi)^{-1} \textstyle\int_{-M}^{M} e^{-ix\theta} k(\theta/M)\, d\theta \\ &= (2\pi)^{-1} \textstyle\int_{-M}^{M} \cos x\theta k(\theta/M)\, d\theta\,. \end{aligned}$$

Similarly, the probability density function

$$(6.2)\qquad \begin{aligned} M \textstyle\int_{-\infty}^{\infty} K(M(x - y))\, dF^{n*}(y) &= (2\pi)^{-1} \textstyle\int_{-M}^{M} e^{ix\theta} k(\theta/M)\phi^n(\theta)\, d\theta \\ &= (2\pi)^{-1} \textstyle\int_{-M}^{M} \operatorname{Re}\,(e^{-ix\theta} k(\theta/M)\phi^n(\theta))\, d\theta\,. \end{aligned}$$

For any $0 \leqq r \leqq 1$, define $U_r(x) = \sum_0^\infty r^n F^{n*}(x)$, so that it follows from (6.2) for $0 \leqq r \leqq 1$ that

$$(6.3)\qquad M \textstyle\int_{-\infty}^{\infty} K(M(x-y))\, dU_r(x) = (2\pi)^{-1} \textstyle\int_{-M}^{M} \operatorname{Re}\,(e^{-ix\theta} k(\theta/M)/(1 - r\phi(\theta)))\, d\theta\,.$$

Feller and Orey (1961) show that the measures on $[-M, M]$ with density $\operatorname{Re}\,(1 - r\phi(\theta))^{-1}$ converge weakly as $r \uparrow 1$ to a measure with an atom of mass $\lambda\pi$ at $\theta = 0$ and an absolutely continuous component with density $\operatorname{Re}\,(1 - \phi(\theta))^{-1}$ $(\theta \neq 0)$. Since $dU_r(x) \uparrow dU(x)$ $(r \uparrow 1)$ and $\int_{-\infty}^{\infty} K(M(x - y))\, dU(y) < \infty$, the two sides of (6.3) converge as $r \uparrow 1$ by monotone convergence and weak convergence respectively to yield the identity

$$(6.4)\qquad \begin{aligned} &M \textstyle\int_{-\infty}^{\infty} K(M(x - y))\, dU(y) \\ &\qquad = \lambda/2 + (2\pi)^{-1} \textstyle\int_{-\infty}^{\infty} \operatorname{Re}\,(e^{-ix\theta} k(\theta/M)/(1 - \phi(\theta)))\, d\theta\,. \end{aligned}$$

Now from (6.1),

$$(6.5)\qquad \begin{aligned} \textstyle\int_0^\infty MK(M(x - y))\, dy &= \textstyle\int_{-\infty}^{x} MK(My)\, dy = \frac{1}{2} + \textstyle\int_0^x MK(My)\, dy \\ &= \tfrac{1}{2} + (2\pi)^{-1} \textstyle\int_{-M}^{M} \theta^{-1} \sin x\theta k(\theta/M)\, d\theta \\ &= \tfrac{1}{2} + (2\pi)^{-1} \textstyle\int_{-M}^{M} \operatorname{Re}\,(e^{-ix\theta} k(\theta/M)/(-i\theta))\, d\theta\,. \end{aligned}$$

Combining (6.5) with (6.4), we have with $\mu = EX = \lambda^{-1}$,

$$\begin{aligned}(6.6)\qquad & M \int_{-\infty}^{\infty} K(M(x-y))d(U(y) - \lambda y_+) \\ &= (2\pi)^{-1} \int_{-M}^{M} \mathrm{Re}\,(e^{-ix\theta}k(\theta/M)\{(1-\phi(\theta))^{-1} + (i\theta u)^{-1}\})\,d\theta \\ &= (2\pi)^{-1} \int_{-M}^{M} \mathrm{Re}\,(e^{-ix\theta}k(\theta/M)\{(1+i\theta\mu-\phi(\theta))/(1-\phi(\theta))(i\theta\mu)\})\,d\theta\,.\end{aligned}$$

Now the term in braces is continuous and bounded on $(-M, M)$, hence integrable there, so the restriction of the integrand to the real part, introduced at (6.2) for integrability considerations, can be relaxed.

The function $\lambda^2 S(\cdot)$ (see (2.4) above) is a distribution function which, it may be verified, has characteristic function $(\phi(\theta) - 1 - i\theta\mu)/(i\theta\mu)^2$. Thus we may write down an expression similar to (6.2) involving S in place of F^{n*} and this then yields in conjunction with (6.6) (without the restriction to the real part)

$$\begin{aligned}(6.7)\qquad & M \int_{-\infty}^{\infty} K(M(x-y))\,dW(y) \\ &= \frac{1}{2\pi} \int_{-M}^{M} e^{-ix\theta}k(\theta/M) \left\{\frac{1}{1-\phi(\theta)} + \frac{1}{i\theta\mu} - \frac{\phi(\theta) - 1 - i\theta\mu}{(i\theta\mu)^2}\right\} d\theta\end{aligned}$$

where $W(y) = U(y) - \lambda y_+ - \lambda^2 S(y)$. The expression in braces equals

$$(1 + i\theta\mu - \phi(\theta))^2/(1 - \phi(\theta))(i\theta\mu)^2\,,$$

and when divided by θ, it is still continuous and bounded on $(-M, M)$, hence integrable there. Since $k(\theta/M)$ vinishes for $|\theta| > M$, this property of integrability will enable the Riemann–Lebesgue lemma to be applied later.

Let $K_1(\cdot)$ be the distribution function with density $MK(\cdot)$. Formal integration of the left side of (6.7) followed by formal integration by parts yields for fixed x and $-A < x$

$$\begin{aligned}(6.8)\qquad & \int_{-\infty}^{\infty} W(y)M(K(M(x-y)) - K(M(-A-y)))\,dy \\ &= \int_{-\infty}^{\infty} (K_1(M(x-y)) - K_1(M(-A-y)))\,dW(y) \\ &= \int_{-A}^{x} du \int_{-\infty}^{\infty} MK(M(x-y))\,dW(y) \\ &= \frac{1}{2\pi} \int_{-M}^{M} \frac{(e^{-ix\theta} - e^{iA\theta})k(\theta/M)(1 + i\theta\mu - \phi(\theta))^2}{(-i\theta)(1-\phi(\theta))(i\theta\mu)^2}\,d\theta\,.\end{aligned}$$

Replacing the infinite range of integration in the first three expressions here by $\int_{-B}^{B}$, for some B sufficiently large that $W(-B)$, $K_1(-MB)$, and

$$|(\int_{-\infty}^{\infty} - \int_{-B}^{B})MK(M(u-y))\,dW(y)|$$

are sufficiently small for $-A < u < x$, the first two equalities asserted at (6.8) can be verified, and then (6.7) used to deduce the last equality. Using the Riemann–Lebesgue lemma on the right hand side of (6.8) establishes (2.5) since, for the left-hand side of (6.8) $MK(M(\cdot))$ is a probability density.

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