The Annals of Probability
1978, Vol. 6, No. 5, 847–869

EXTREMES OF MOVING AVERAGES OF STABLE PROCESSES

BY HOLGER ROOTZÉN[1]

University of Lund and University of North Carolina at Chapel Hill

In this paper we study extremes of non-normal stable moving average processes, i.e., of stochastic processes of the form $X(t) = \sum a(\lambda - t)Z(\lambda)$ or $X(t) = \int a(\lambda - t)\, dZ(\lambda)$, where $Z(\lambda)$ is stable with index $\alpha < 2$. The extremes are described as a marked point process, consisting of the point process of (separated) exceedances of a level together with marks associated with the points, a mark being the normalized sample path of $X(t)$ around an exceedance. It is proved that this marked point process converges in distribution as the level increases to infinity. The limiting distribution is that of a Poisson process with independent marks which have random heights but otherwise are deterministic. As a byproduct of the proof for the continuous-time case, a result on sample path continuity of stable processes is obtained.

1. Introduction. A distribution is stable (γ, α, β) if it has the characteristic function

$$\phi(u) = \exp\{-\gamma^\alpha |u|^\alpha (1 - i\beta h(u, \alpha)u/|u|)\}, \tag{1.1}$$

with $0 \leqq \gamma$, $0 < \alpha \leqq 2$, $|\beta| \leqq 1$ and with $h(u, \alpha) = \tan(\pi\alpha/2)$ for $\alpha \neq 1$, $h(u, 1) = 2\pi^{-1} \log |u|$. Here γ is a scale parameter, α is the *index*, and β is the *symmetry parameter* of the stable distribution. If $\beta = 0$, then the distribution is symmetric, while the distribution is said to be *completely asymmetric* if $|\beta| = 1$ and $\alpha < 2$. A stochastic process $\{X(t);\ t \in T\}$ is stable with index α if for $n = 1, 2, \cdots$ and for arbitrary real numbers $a_1, \cdots, a_n$ and $t_1, \cdots, t_n \in T$, the random variable $a_1 X(t_1) + \cdots + a_n X(t_n)$ is stable with index α. In particular, as can be seen from (1.1), a collection of independent stable random variables with index α is a stable process.

In this paper, further use of the linear structure is made by restricting attention to the subclass consisting of moving averages, i.e., to stationary processes of the form $X(t) = \sum_\lambda a(\lambda - t)Z(\lambda)$, where $\{Z(\lambda)\}_{\lambda=-\infty}^{\infty}$ is an independent, stationary stable sequence, or of the form $X(t) = \int a(\lambda - t)\, dZ(\lambda)$ where $\{Z(\lambda);\ -\infty < \lambda < \infty\}$ has independent, stationary stable increments. If $\alpha = 2$ the process is normal, and it can be represented as a moving average iff its spectral distribution is absolutely continuous. Presently no simple characterization of the class of moving averages of stable processes is known for $\alpha < 2$. Of course normal

Received December 22, 1976.

[1] This research was sponsored in part by the Office of Naval Research, Contract No. N00014-75-C-0809.

AMS 1970 *subject classifications*. Primary 60F05; Secondary 60G17.

Key words and phrases. Extreme values, stable processes, moving average, sample path continuity.

processes are extensively analyzed but, partly because of the common linear structure, also stable processes with $\alpha < 2$ constitute a class of probability models that is amenable to analysis.

The subject of the present study is the asymptotic distribution of extremes of moving averages of stable processes with $\alpha < 2$. As can be seen from, e.g., Leadbetter et al. (1976), a suitable framework for dealing with extremes of stationary processes is the theory of point processes as given in, e.g., Kallenberg (1975) used to study the process of exceedances of a high level. Here we will go one step further and adjoin a *mark* to each point in the process of exceedances, the mark being the entire sample path of the process, normalized and centered at (a point close to) the upcrossing. The main results are that both when $X(t)$ has discrete parameter (or "time") and when $X(t)$ has continuous parameter, the marked point process converges in distribution. The limiting distribution is that of a Poisson process (possibly with multiple points) with independent marks that are distributed as $a(-t)$ multiplied by a certain random variable.

One of the main differences between the normal distribution (stable with $\alpha = 2$) and stable distributions with $\alpha < 2$ is that the tails of the latter decrease much slower. This leads to a radically different behavior of extremes. For the normal distribution the tails are of the order $e^{-x^2/2}/x$, while for stable distributions with $\alpha < 2$ the tails decrease as $x^{-\alpha}$. This affects extremes of moving averages in two different ways. To fix ideas, consider, e.g., maxima of a process $X(t) = \sum a(\lambda - t)Z(\lambda)$ with discrete parameter. First, extremes increase much slower when $\alpha = 2$ than when $\alpha < 2$, viz. as $(\log n)^{\frac{1}{2}}$ compared with $n^{1/\alpha}$. Secondly, when the independent sequence $\{Z(\lambda)\}$ is normal, many of the $Z(\lambda)$'s, $0 \leqq \lambda \leqq n$, will be almost as large as the largest one, and $X(t)$ will be large when many rather large $Z(\lambda)$'s are added. This entails that the limiting distribution of $M_n = \max_{1 \leqq t \leqq n} X(t)$ only depends on $\sum a(\lambda)^2$ and that it is the same as if $\{X(t)\}$ were an independent sequence with the same marginal distributions. On the other hand, when $\alpha < 2$ the maximum of $Z(\lambda)$ will be much larger than the typical values, and $X(t)$ will be large when one very large $Z(\lambda)$ is multiplied by a large $a(\lambda)$. In this case the limiting distribution of M_n depends on $\max_{-\infty<\lambda<\infty} a(\lambda)$ and on $\min_{-\infty<\lambda<\infty} a(\lambda)$ and is in general not the same as if $X(t)$ were an independent sequence with the same marginals.

In an earlier paper (Rootzén (1974)), the limiting distribution of maxima of moving averages of symmetric stable sequences with $\alpha < 2$ was obtained, but apart from that there do not seem to be any results published on extremes of stationary stable processes with $\alpha < 2$.

The plan of this paper is as follows. Section 2 deals with convergence in distribution of marked point processes. In Section 3 rather complete asymptotic results on extremes of moving averages of stable sequences are obtained. Section 4 contains preliminaries concerning moving averages of continuous parameter stable processes. In particular some conditions that ensure sample path continuity are found, which may be of independent interest. Finally, in Section 5 results for continuous parameter processes corresponding to those of Section 3 are established for $\alpha \neq 1$, under some restrictions on $a(\lambda)$.

2. Convergence in distribution of marked point processes. We are interested in the times $0 \leqq t_1 < t_2 < \cdots$ of occurrence of extreme values of a stochastic process $\{X(t)\}$ and in the behavior of the sample paths of $\{X(t)\}$ near the t_i's, and will describe them as a marked point process. In this section we introduce some notation and develop techniques needed in the remaining sections to prove convergence in distribution of marked point processes. Unfortunately the notation is somewhat cumbersome, but nevertheless we think it is well warranted, considering the completeness of results it makes possible.

Write $\mathscr{N}$ for the space of integer-valued and locally finite Borel measures on R^+ and define a metric on $\mathscr{N}$ in the following way: let $\mathscr{F} = \{f_i\}_{i=1}^{\infty}$ be a sequence of functions in $C_c = \{f : R^+ \to R^+; f \text{ is continuous with compact support}\}$ such that any $f \in C_c$ can be uniformly approximated by functions in $\mathscr{F}$. For $\mu, \nu \in \mathscr{N}$ put $\rho(\mu, \nu) = \sum_{i=1}^{\infty} 2^{-i} \rho_i(\mu, \nu)$, where $\rho_i(\mu, \nu) = \min(1, |\int f_i \, d\mu - \int f_i \, d\nu|)$. Then ρ is a metric on $\mathscr{N}$ that generates the topology of vague convergence ($\mu_n \in \mathscr{N}$ converges vaguely to $\mu \in \mathscr{N}$ if $\int f \, d\mu_n \to \int f \, d\mu$ for all $f \in C_c$); see, e.g., Bauer (1972), page 241. A point process in R^+ is defined to be a (Borel measurable) random variable with values in $(\mathscr{N}, \rho)$. As soon as we have (Borel measurable) random variables in a metric space we may of course consider convergence in distribution, using the theory of convergence in distribution in metric spaces as given in, e.g., Billingsley (1968). For further information on convergence in distribution of point processes see [7]. We regard the times $0 \leqq t_1 < t_2 < \cdots$ of occurrence of extremes of $\{X(t)\}$ as a point process N by putting $N(B) = \#\{t_i \in B\}$ for any Borel set $B \subset R^+$.

With each of the t_i's we associate a mark Y_i, where Y_i is the entire sample path of $\{X(t)\}$, normalized and centered to show the behavior close to t_i. If $\{X(t)\}$ is a discrete parameter process, Y_i is a random variable in a space $R^\infty = \{(\cdots, x_{-1}, x_0, x_1, \cdots); x_i \in R, i = 0, \pm 1, \cdots\}$. We consider R^∞ as a metric space with the metric $\delta(x, y) = \sum_{i=-\infty}^{\infty} 2^{-|i|} \delta_i(x, y)$ that generates the product topology, where $\delta_i(x, y) = \min(\frac{1}{3}, |x_i - y_i|)$. If $X(t)$ has continuous parameter we will impose conditions that make Y_i a random variable in $D(-\infty, \infty)$, the space of functions on $(-\infty, \infty)$ which are right continuous and have left-hand limits. On $D(-\infty, \infty)$ we use a slight modification of the metric given by Lindvall (1973). Since there is no risk of confusion we will use the same notation as for the metric on R^∞. Thus for $x, y \in D(-\infty, \infty)$ we let $\delta(x, y) = \sum_{i=-\infty}^{\infty} 2^{|i|} \delta_i(x, y)$ where $\delta_i(x, y) = \min(\frac{1}{3}, h(c_i x, c_i y))$ and $h(c_i x, c_i y)$ is the quantity given on pages 113–115 of Lindvall's paper, modified to $D(-\infty, \infty)$ instead of $D(0, \infty)$ in the way proposed on page 121. Thus $h(c_i x, c_i y) = d_0'(g_i x \circ \phi, g_i y \circ \phi)$, where $\phi(t) = \log t(1 - t)^{-1}$, where g_i is the function which is one on $[-i, i]$, zero on $[-i - 1, i + 1]^*$ and linear in between, and where d_0' is the metric defined on page 110 of the cited reference.

The marked point process is the vector $\eta = (N, Y_1, Y_2, \cdots)$ with values in $S = \mathscr{N} \times R^\infty \times R^\infty \times \cdots$ (or in $S = \mathscr{N} \times D(-\infty, \infty) \times D(-\infty, \infty) \times \cdots$) which we again consider as a metric space given a product metric $d(x, y) = \sum_{i=0}^\infty 2^{-i} d_i(x, y)$, where for $x = (\nu, x_1, x_2, \cdots)$ and $y = (\nu, y_1, y_2, \cdots)$ we put $d_0(x, y) = \rho(\nu, \mu)$ and $d_i(x, y) = \delta(x_i, y_i)$, $i \geqq 1$. Our aim is to prove convergence in distribution of marked point processes, and to this end we need the following simple criteria, which we state as lemmas for easy reference.

LEMMA 2.1.[2] *Let $\eta_n = (N_n, Y_{n1}, Y_{n2}, \cdots)$ and $\eta = (N, Y_1, Y_2, \cdots)$ be random variables in the product space (S, d). Suppose that $N_n, Y_{n1}, Y_{n2}, \cdots$ are independent for each n and that $N, Y_1, Y_2, \cdots$ are independent. Then $\eta_n \rightarrow_d \eta$ if (and only if) $N_n \rightarrow_d N$ and $Y_{ni} \rightarrow_d Y_i$, $i \geqq 1$.*

PROOF. Since S is a product of separable spaces this follows as on page 21 of [3]. □

LEMMA 2.2. *Let $\eta_n^k = (N_n^k, Y_{n1}^k, Y_{n2}^k, \cdots)$, $n \geqq 1$, $k \geqq 1$, be random variables in (S, d). Suppose that for $k \geqq 1$, $\eta_n^k \rightarrow_d \eta^k$ as $n \rightarrow \infty$ and that $\eta^k \rightarrow_d \eta$ as $k \rightarrow \infty$. Suppose further that*

$$(2.1) \qquad \lim_{k\to\infty} \limsup_{n\to\infty} P(d_i(\eta_n^k, \eta_n) > \varepsilon) = 0\,, \qquad \forall\, \varepsilon > 0,\ i = 0, 1, \cdots.$$

Then $\eta_n \rightarrow_d \eta$ as $n \rightarrow \infty$.

PROOF. For given $\varepsilon > 0$ choose i to make $2^{-i} < \varepsilon/2$ and thus $\sum_{j=i+1}^\infty 2^{-j} d_j(\eta_n^k, \eta_n) < \varepsilon/2$. Then

$$P(d(\eta_n^k, \eta_n) > \varepsilon) \leqq \sum_{j=0}^i P(d_i(\eta_n^k, \eta_n) > \varepsilon/(2(i+1)))$$

and by (2.1) we thus have

$$\lim_{k\to\infty} \limsup_{n\to\infty} P(d(\eta_n^k, \eta_n) > \varepsilon) = 0\,, \qquad \forall\, \varepsilon > 0\,,$$

which by Theorem 4.2 of [3] proves the lemma. □

For $i \geqq 1$, $d_i(\eta_n^k, \eta_n) = \delta(Y_{ni}^k, Y_{ni})$ and repeating the above argument once more we see that (2.1) holds for $i \geqq 1$ *if*

$$(2.2) \qquad \lim_{k\to\infty} \limsup_{n\to\infty} P(\delta_j(Y_{ni}^k, Y_{ni}) > \varepsilon) = 0\,, \qquad \forall\, \varepsilon > 0,\ j \geqq 1\,.$$

In the discrete-parameter case (2.2) is easy to check, but when the parameter is continuous further simplification is needed.

LEMMA 2.3. *Suppose that for each $i \geqq 1$ there are random variables $\{\varepsilon_n^k\}$ with $\lim_{k\to\infty} \limsup_{n\to\infty} P(|\varepsilon_n^k| > x) = 0$, $\forall\, x > 0$, and such that*

$$(2.3) \qquad \lim_{k\to\infty} \limsup_{n\to\infty} P(\sup_{-l\leq t\leq l} |Y_{ni}(t) - Y_{ni}^k(t + \varepsilon_n^k)| > x) = 0\,, \qquad \forall\, x > 0,\ l > 0\,,$$

[2] The results of this section are formulated in terms of a discrete parameter, n, which tends to infinity. They of course remain valid if the parameter tends to infinity in a continuous manner, and they will be used accordingly in Section 5.

and that furthermore $\lim_{u\to\infty} \limsup_{n\to\infty} P(\sup_{-l\le t\le l} |Y_{ni}(t)| > u) = 0, \forall\, l > 0$. *Then* (2.2) *holds and thus* (2.1), *for* $i \geqq 1$.

PROOF. Let ε be a real number with $|\varepsilon| \leqq 1$ and define a time transformation $\lambda(t) = \{1 - e^{-\varepsilon} + e^{-\varepsilon}/t\}^{-1}$, $t \in [0, 1]$. Then $\lambda(0) = 0$, $\lambda(1) = 1$ and λ is strictly increasing, so

$$\begin{aligned}\delta_j(y, z) &= d_0'(g_j\, y \circ \phi, g_j\, z \circ \phi)\\ &\leqq \sup_{0\le t\le 1} |g_j(\phi(t))y(\phi(t)) - g_j(\phi(\lambda(t)))z(\phi(\lambda(t)))| + s(\lambda)\end{aligned}$$

where $s(\lambda) = \sup_{s\ne t} |\log\{(\lambda(t) - \lambda(s))/(t - s)\}|$. Straightforward computations show that $s(\lambda) \leqq |\varepsilon|$. Further $\phi(\lambda(\phi^{-1}(s))) = s + \varepsilon$, so

$$\begin{aligned}\sup_{0\le t\le 1} &|g_j(\phi(t))y(\phi(t)) - g_j(\phi(\lambda(t)))z(\phi(\lambda(t)))|\\ &= \sup_{-\infty<s<\infty} |g_j(s)y(s) - g_j(s + \varepsilon)z(s + \varepsilon)|\\ &\leqq \sup_{-j-1\le s\le j+1} |y(s) - z(s + \varepsilon)| + \sup_{-j-2\le s\le j+2} |\varepsilon y(s)|\end{aligned}$$

by the definition of g_j. Hence

$$\delta_j(y, z) \leqq \sup_{-j-2\le s\le j+2} \{|y(t) - z(s + \varepsilon)| + |\varepsilon y(s)|\} + |\varepsilon|$$

and the lemma follows by routine calculations. □

LEMMA 2.4. *Let* $0 \leqq t_{n1}^k \leqq t_{n2}^k \leqq \cdots$ *be the atoms (repeated according to their multiplicities) of* $N_n^{\,k}$ *and similarly let* $0 \leqq t_{n1} \leqq t_{n2} \leqq \cdots$ *be the atoms of* N_n. *Suppose that*

$$\lim_{k\to\infty} \limsup_{n\to\infty} P(|t_{ni}^k - t_{ni}| > \varepsilon) = 0\,, \qquad \forall\, \varepsilon > 0,\ i = 1, 2, \cdots \tag{2.4}$$

and that in addition $N_n^{\,k} \to_d N^k$ *as* $n \to \infty$ *and that* $N^k \to_d N$ *as* $n \to \infty$. *Then* (2.1) *holds for* $i = 0$.

PROOF. As above, it is enough to prove

$$P = \lim_{k\to\infty} \limsup_{n\to\infty} P(\rho_j(N_n^{\,k}, N_n) > \varepsilon) = 0\,, \qquad \forall\, \varepsilon > 0,\ j = 1, 2, \cdots \tag{2.5}$$

where $\rho_j(N_n^{\,k}, N_n) \leqq |\int f_j\, dN_n^{\,k} - \int f_j\, dN_n|$ with $f_j \in C_c$. Suppose that the support of f_j is contained in $[0, T]$. For $\delta > 0$ there is a K with $P(N([0, T + 1]) > K) < \delta$ and thus $\limsup_{k\to\infty} P(N^k([0, T]) > K) < \delta$ and $\limsup_{k\to\infty} \limsup_{n\to\infty} P(N_n^{\,k}([0, T]) > K) < \delta$. Furthermore take a step function $g(t) = \sum_{i=1}^{m-1} a_i I(\tau_i < t \leqq \tau_{i+1})$, with $0 \leqq \tau_1 < \cdots < \tau_m \leqq T$ that approximates f_j uniformly, with $\sup_{0\le t} |f_j(t) - g(t)| < \varepsilon/(2K)$. Letting $0 \leqq t_1 \leqq t_2 \leqq \cdots$ be the atoms of N, we assume without loss of generality that

$$P(t_i = \tau_j) = 0\,, \qquad j = 1, \cdots, m,\ i \geqq 1\,. \tag{2.6}$$

On the set $\{N_n^{\,k}([0, T]) \leqq K,\ N_n([0, T]) \leqq K\}$ we have

$$|\textstyle\int f_j\, dN_n^{\,k} - \int f_j\, dN_n| < |\int g\, dN_n^{\,k} - \int g\, dN_n| + 2K\varepsilon/(2K)\,.$$

Hence

$$P \leqq \limsup_{k\to\infty} \limsup_{n\to\infty} \{P(N_n{}^k([0,T]) > K) + P(N_n([0,T]) > K) + P(|\textstyle\int g\,dN_n{}^k - \int g\,dN_n| > 0)\} \leqq 2\delta\,,$$

since $P(|\int g\,dN_n{}^k - \int g\,dN_n| > 0) \to 0$ by (2.6) and the hypothesis of the lemma. However, $\delta > 0$ is arbitrary, so (2.5) follows. □

Finally, it should perhaps be stressed that the convergence $\eta_n \to_d \eta$ that is to be proved in the following sections only says something about the sample paths *near* extremes: namely, $Y_{ni} \to_d Y_i$ where Y_{ni}, Y_i are random variables in (R^∞, δ) if and only if $(Y_{ni}(-k), \cdots, Y_{ni}(k)) \to_d (Y_i(-k), \cdots, Y_i(k))$ as $n \to \infty$, for each $k \geqq 1$. Similarly, since the limits of $Y_{ni} \in D(-\infty, \infty)$ which we obtain are continuous, the convergence in $D(-\infty, \infty)$ is equivalent to convergence in $D(-T, T)$ for each $T > 0$.

3. Extremes in discrete time. Let $\{Z(\lambda)\}_{\lambda=-\infty}^{\infty}$ be a sequence of independent stable $(1, \alpha, \beta)$ random variables. It is immediate that $\sum_{-\infty}^{\infty} a(\lambda)Z(\lambda)$ converges in distribution if and only if

$$\text{(3.1)} \qquad \sum_{\lambda=-\infty}^{\infty} |a(\lambda)|^\alpha < \infty \quad \text{and in addition, for} \quad \alpha = 1\,, \quad \beta \neq 0\,, \qquad |\textstyle\sum_{\lambda=-\infty}^{\infty} a(\lambda) \log |a(\lambda)|| < \infty\,.$$

Moreover, if (3.1) is satisfied then, since the $Z(\lambda)$'s are independent, $\sum_{-\infty}^{\infty} a(\lambda)Z(\lambda)$ converges with probability one also. The limiting distribution is stable with index α, scale parameter $(\sum_{-\infty}^{\infty} |a(\lambda)|^\alpha)^{1/\alpha}$ and with symmetry parameter $\beta \sum_{-\infty}^{\infty} \{a^+(\lambda)^\alpha - a^-(\lambda)^\alpha\}/\sum_{-\infty}^{\infty} |a(\lambda)|^\alpha$, where $a^+(\lambda) = \max(0, a(\lambda))$ and $a^-(\lambda) = \max(0, -a(\lambda))$. Further, if $\alpha = 1$ the distribution is translated by an amount $-\beta 2\pi^{-1} \sum_{-\infty}^{\infty} a(\lambda) \log |a(\lambda)|$.

Given $\{a(\lambda)\}_{\lambda=-\infty}^{\infty}$ satisfying (3.1) a moving average process $\{X(t)\}_{t=-\infty}^{\infty}$ is obtained by putting $X(t) = \sum_{\lambda=-\infty}^{\infty} a(\lambda - t)Z(\lambda)$. Let $x > 0$ be fixed, take a sequence $\{h(n)\}_{n=1}^{\infty}$ with $h(n) \uparrow \infty$ and $h(n)/n \to 0$ but otherwise arbitrary, and define the separated exceedances of $xn^{1/\alpha}$ recursively by putting $t_{n1} = \inf\{t \geqq h(n); X(t) > xn^{1/\alpha}\}$ and $t_{ni} = \inf\{t \geqq t_{n,i-1} + h(n); X(t) > xn^{1/\alpha}\}$, for $i \geqq 2$. When using *separated exceedances* we count several exceedances which are "a fixed distance apart" as one event only. The reason for this is that for large n the exceedances of the level $xn^{1/\alpha}$ will come in small clusters. Each cluster is caused by one very large variable $Z(\lambda)$. Therefore extremes belonging to the same cluster will be strongly dependent, whereas extremes belonging to different clusters will be almost independent. Furthermore, the distance between different clusters is of the magnitude n, and the number of exceedances in each cluster is asymptotically independent of n. By considering separated exceedances we will asymptotically get precisely one representative from each cluster.

At the end of the section, also ordinary exceedances will be considered. The time-normalized point process N_n of separated exceedances is defined by $N_n(B) = \#\{t_{ni}/n \in B\}$ for Borel sets $B \subset R^+$. Further, for a given sequence $\{\tau_{ni}\}_{i=1}^{\infty}$ the mark Y_{ni} at exceedance no. i is defined as

$$Y_{ni}(t) = X(t + \tau_{ni})/n^{1/\alpha}, \qquad t = 0, \pm 1, \cdots,$$

and we then have a marked point process $\eta_n = (N_n, Y_{n1}, Y_{n2}, \cdots)$.

In order to find the limiting distribution of η_n it is convenient to consider completely asymmetric processes first. Let $c_\alpha = \pi^{-1}\Gamma(\alpha)\sin(\alpha\pi/2)$ and put $A = \max_{-\infty<\lambda<\infty} a^+(\lambda)$. Further let Y_i have the distribution of the vector $(\cdots, a(1)Z, a(0)Z, a(-1)Z, \cdots)$ in R^∞, where Z is a random variable with distribution function $F(z) = 1 - x^\alpha A^{-\alpha} z^{-\alpha}$, $z \geqq xA^{-1}$. Then the limiting distribution is that of

(3.2) $(N, Y_1, Y_2, \cdots)$ where the components are independent, N is a Poisson process with intensity $\mu = 2c_\alpha A^\alpha x^{-\alpha}$ and where the Y_i's have the distribution given above.

LEMMA 3.1. *Suppose that* $\{Z(\lambda)\}_{-\infty}^{\infty}$ *are independent and stable* $(1, \alpha, 1)$, *that* $\{a(\lambda)\}_{-\infty}^{\infty}$ *satisfies* (3.1) *with* $A = \max_{-\infty<\lambda<\infty} a(\lambda) > 0$, *and that* $X(t) = \sum_{-\infty}^{\infty} a(\lambda - t)Z(\lambda)$. *Then there are time points* $\{\tau_{ni}; n \geqq 1, i \geqq 1\}$ *with* $\{t_{ni} - \tau_{ni}\}_{n=1}^{\infty}$ *tight for each* $i \geqq 1$ (i.e., $\lim_{k\to\infty} \limsup_{n\to\infty} P(|t_{ni} - \tau_{ni}| > k) = 0, i \geqq 1$) *such that if* η_n *is the marked point process of separated exceedances of* $xn^{1/\alpha}$ *defined above, then* $\eta_n \to_d \eta$ *with the distribution of* η *given by* (3.2).

REMARK. It would seem more natural to center the marks at the t_{ni}'s instead of at some τ_{ni}'s which are not explicitly defined in terms of $X(t)$, but unfortunately the limiting distribution then becomes much more complicated. However, using the entire observed structure of the sample path near extremes it is possible to find the centering. For instance, if the maximum of $\{a(\lambda)\}$ is unique then we may take τ_{ni} as the time point when $\{X(t); t \in [t_{ni}, t_{ni} + h(n)]\}$ first assumes its maximum. (The validity of this claim is verified at the end of the proof of the lemma.)

PROOF. The essential facts we will use are that X is a moving average and the following simple estimates of the tails of $F_{\alpha\beta}$, the stable $(1, \alpha, \beta)$ distribution (see Bergström (1953)):

$$1 - F_{\alpha 1}(z) \sim 2c_\alpha z^{-\alpha} \quad \text{as} \quad z \to \infty \tag{3.3}$$
$$F_{\alpha 1}(z) = o(|z|^{-\alpha}) \quad \text{as} \quad z \to -\infty$$

(where $f \sim g$ means $f = g(1 + o(1))$) and

$$F_{\alpha\beta}(z) + (1 - F_{\alpha\beta}(z)) \leqq k_\alpha z^{-\alpha}, \qquad z > 0 \tag{3.4}$$

for some constant k_α.

Define $0 \leqq \tau_{n1} < \tau_{n2} < \cdots$ as the times when $Z(\lambda) > xA^{-1}n^{1/\alpha}$, put $N_n'(B) = \#\{\tau_{ni}/n \in B\}$ for Borel sets $B \subset R^+$, and let $Y'_{ni}(t) = Z(\tau_{ni})/n^{1/\alpha}$ for $t = 0$ and $Y_{ni}(t) = 0$ for $t \neq 0$. Further, let ζ have the distribution obtained from (3.2) by putting $a(0) = A$, $a(\lambda) = 0$, $\lambda \neq 0$ in the definition of Y_i. The first step is to prove that for $\zeta_n = (N_n', Y'_{n1}, Y'_{n2}, \cdots)$ we have

$$\zeta_n \to_d \zeta \quad \text{as} \quad n \to \infty\,. \tag{3.5}$$

Obviously ζ_n has independent components, so according to Lemma 2.1 it is sufficient to prove that each of the components converges. From (3.3) we have

$$P(Z(0) > xA^{-1}n^{1/\alpha}) \sim 2c_\alpha A^\alpha x^{-\alpha} n^{-1}$$

which by Theorem 3.2 of Leadbetter (1976) proves that $N_n' \to_d N$. Furthermore, for $z \geqq xA^{-1}$,

$$\begin{aligned} P(Y'_{ni}(0) \leqq z) &= 1 - P(Y'_{ni}(0) > z) \\ &= 1 - P(Z(0) > n^{1/\alpha}z \mid Z(0) > n^{1/\alpha}xA^{-1}) \\ &\sim 1 - \frac{nx^\alpha A^{-\alpha}}{nz^\alpha} = 1 - x^\alpha A^{-\alpha} z^{-\alpha} \end{aligned}$$

and it follows that $Y'_{ni} \to_d Y_i'$ and thus that (3.5) holds.

The next step is to prove that $\{t_{ni} - \tau_{ni}\}_{n=1}^\infty$ is tight for $i \geqq 1$, i.e.,

$$\lim_{k\to\infty} \limsup_{n\to\infty} P(|t_{ni} - \tau_{ni}| > k) = 0\,, \qquad i = 1, 2, \cdots\,. \tag{3.6}$$

Now, putting $A_{ni} = \{|t_{ni} - \tau_{ni}| > k\}$, we have $P(A_{ni}) \leqq P(A^*_{n,i-1}A_{ni}) + P(A_{n,i-1})$, (defining $A^*_{n0} = \Omega$), and by recursion (3.6) follows if we prove

$$\lim_{k\to\infty} \limsup_{n\to\infty} P(A^*_{n,i-1}A_{ni}) = 0\,, \qquad i = 1, 2, \cdots\,. \tag{3.7}$$

Let N be a positive integer, put $B_n = \{\tau_{ni} > nN\}$ and put $C_n = \{Z(t) \in n^{1/\alpha}A^{-1}(x - 2\varepsilon, x]$ for some $t \in (0, nN)\}$. Taking $x/3 > \varepsilon > 0$ we have, for n such that $h(n) > 2k$, that

$$\begin{aligned} &A^*_{n,i-1}\{t_{ni} < \tau_{ni} - k\} \\ &\quad \subset \{X(t_{ni}) > n^{1/\alpha}x,\ Z(t) \leqq n^{1/\alpha}xA^{-1} \ \text{for}\ |t - t_{ni}| \leqq k,\ t_{ni} < nN - k\} \cup B_n \\ &\quad \subset \{X(t_{ni}) > n^{1/\alpha}x,\ Z(t) \leqq n^{1/\alpha}A^{-1}(x - 2\varepsilon) \ \text{for}\ |t - t_{ni}| \leqq k,\ t_{ni} < nN - k\} \\ &\qquad \cup B_n \cup C_n\,. \end{aligned}$$

Let D_n be the event that $Z(t) < -n^{1/\alpha}\varepsilon(2k + 1)^{-1}(\max_\lambda a^-(\lambda))^{-1}$ for some $t \in (0, nN)$, and let E_n be the event that there are time points t', t'' $(0 < t', t'' < nN)$ with $|t' - t''| \leqq 2k + 1$ and $Z(t'), Z(t'') > n^{1/\alpha}\varepsilon(2k + 1)^{-1}A^{-1}$. Further introduce $X_k(t) = \sum_{\lambda=-k+t}^{k+t} a(\lambda - t)Z(\lambda)$ and write F_n for the event that $\sup\{|X(t) - X_k(t)|;\ 0 < t \leqq nN\}$ exceeds $n^{1/\alpha}\varepsilon$. Then

$$\begin{aligned} &\{X(t_{ni}) > n^{1/\alpha}x,\ Z(t) \leqq n^{1/\alpha}A^{-1}(x - 2\varepsilon) \ \text{for}\ |t - t_{ni}| \leqq k,\ t_{ni} < nN - k\} \\ &\qquad \subset \{X_k(t_{ni}) > n^{1/\alpha}(x - \varepsilon),\ Z(t) \leqq n^{1/\alpha}A^{-1}(x - 2\varepsilon) \ \text{for}\ |t - t_{ni}| \leqq k, \\ &\qquad\qquad t_{ni} < nN - k\} \cup F_n \\ &\qquad \subset D_n \cup E_n \cup F_n\,, \end{aligned}$$

where the last inclusion follows from the fact that if $D_n{}^*$ occurs, if $X_k(t_{ni}) = \sum_{\lambda=-k}^{k} a(\lambda)Z(\lambda + t_{ni}) > n^{1/\alpha}(x - \varepsilon)$, and if $Z(\lambda + t_{ni}) \leqq n^{1/\alpha}A^{-1}(x - 2\varepsilon)$ for $|\lambda| \leqq k$, then for at least two values of λ with $|\lambda| \leqq k$ the summands $a^+(\lambda)Z(\lambda + t_{ni})$, which are not larger than $AZ(\lambda + t_{ni})$, have to exceed $n^{1/\alpha}\varepsilon(2k+1)^{-1}$. It follows that

$$A^*_{n,i-1}\{t_{ni} < \tau_{ni} - k\} \subset B_n \cup C_n \cup D_n \cup E_n \cup F_n . \tag{3.8}$$

We proceed to estimate the probabilities of the events in the right-hand side of (3.8). From (3.5)

$$P(B_n) = P(N_n'(0, N) \leqq i-1) \to \sum_{j=0}^{i-1} \frac{(\mu N)^j}{j!} e^{-\mu N}$$

as $n \to \infty$ and, using Boole's inequality and (3.3),

$$\begin{aligned} P(C_n) &\leqq nNP(Z(0) \in n^{1/\alpha}A^{-1}(x-2\varepsilon, x]) \\ &\to N \cdot 2 \cdot c_\alpha A^\alpha((x-2\varepsilon)^{-\alpha} - x^{-\alpha}) \end{aligned}$$

as $n \to \infty$, and similarly

$$P(D_n) \leqq nNP(Z(0) < -n^{1/\alpha}\varepsilon(2k+1)^{-1}(\max_\lambda a^-(\lambda))^{-1}) \to 0 .$$

Again by Boole's inequality and by independence and (3.3) we have

$$\begin{aligned} P(E_n) &\leqq ([nN/(2k+1)]+1)P(\{\exists\, t', t''; 0 \leqq t' < t'' \leqq 4k+2, \\ &\qquad Z(t') \geqq n^{1/\alpha}\varepsilon(2k+1)^{-1}A^{-1}, Z(t'') > n^{1/\alpha}\varepsilon(2k+1)^{-1}A^{-1}\}) \\ &\leqq ([nN/(2k+1)]+1)(2k+1)(4k+3)\{P(Z(0) > n^{1/\alpha}\varepsilon(2k+1)^{-1}A^{-1})\}^2 \\ &\sim ([nN/(2k+1)]+1)(2k+1)(4k+3)\{2c_\alpha\varepsilon^{-\alpha}(2k+1)^\alpha A^\alpha n^{-1}\}^2 \\ &\to 0 \end{aligned}$$

as $n \to \infty$. Finally, $X(t) - X_k(t)$ is stable with index α and scale parameter $\{\sum_{|\lambda|>k} |a(\lambda)|^\alpha\}^{1/\alpha}$ so (3.4) gives

$$\begin{aligned} P(F_n) &\leqq nNP(|X(0) - X_k(0)| > n^{1/\alpha}\varepsilon) \\ &\leqq nNk_\alpha\varepsilon^{-\alpha} \sum_{|\lambda|>k} |a(\lambda)|^\alpha n^{-1} \\ &= k_\alpha N\varepsilon^{-\alpha} \sum_{|\lambda|>k} |a(\lambda)|^\alpha . \end{aligned}$$

Hence, by (3.8),

$$\begin{aligned} &\limsup_{n\to\infty} P(A^*_{n,i-1}\{t_{ni} < \tau_{ni} - k\}) \\ &\qquad \leqq \sum_{j=0}^{i-1} \frac{(\mu N)^j}{j!} e^{-\mu N} + 2c_\alpha A^\alpha N((x-2\varepsilon)^{-\alpha} - x^{-\alpha}) + k_\alpha N\varepsilon^{-\alpha} \sum_{|\lambda|>k} |a(\lambda)|^\alpha \end{aligned}$$

so

$$\begin{aligned} &\lim_{k\to\infty} \limsup_{n\to\infty} P(A^*_{n,i-1}\{t_{ni} < \tau_{ni} - k\}) \\ &\qquad \leqq \sum_{j=0}^{i-1} \frac{(\mu N)^j}{j!} e^{-\mu N} + 2c_\alpha A^\alpha N((x-2\varepsilon)^{-\alpha} - x^{-\alpha}) \end{aligned}$$

and since $\varepsilon > 0$ and N are arbitrary (subject to $x/3 > \varepsilon > 0$) we get

$$\lim_{k\to\infty} \limsup_{n\to\infty} P(A^*_{n,i-1}\{t_{ni} < \tau_{ni} - k\}) = 0 .$$

Similarly, we can show

$$\lim_{k\to\infty} \limsup_{n\to\infty} P(A^*_{n,i-1}\{\tau_{ni} < t_{ni} - k\}) = 0 ,$$

the main change of the proof being that in the definition of E_n we have to consider t', t'' satisfying $|t' - t''| \leqq h(n) + 2k$ (instead of $|t' - t''| \leqq k$) and thus have to use $h(n)/n \to 0$ to prove $P(E_n) \to 0$. Now (3.7) and thus (3.6) follows.

To prove the remainder of the theorem, introduce $Y_{ni}^k = (\cdots, 0, a(k)Z(\tau_{ni}), \cdots, a(-k)Z(\tau_{ni}), 0, \cdots)$, put $N_n{}^k = N_n{}'$, and let $\eta_n{}^k = (N_n{}^k, Y_{n1}^k, Y_{n2}^k, \cdots)$. Since the function that maps ζ_n into $\eta_n{}^k$ is continuous, (3.5) implies that $\eta_n{}^k \to_d \eta^k$, where η^k has the distribution that is obtained from (3.2) by putting $a(\lambda) = 0$ for $|\lambda| > k$. Furthermore it is immediate from Lemma 2.1 that $\eta^k \to_d \eta$. Thus, by Lemma 2.2, $\eta_n \to_d \eta$ follows if we prove that (2.1) holds. The atoms of $N_n{}^k$ are τ_{n1}/n, $\tau_{n2}/n, \cdots$ and the atoms of N_n are $t_{n1}/n, t_{n2}/n, \cdots$ and thus, since it follows from (3.6) that $P(|\tau_{ni}/n - t_{ni}/n| > \varepsilon) \to 0$, as $n \to \infty$ $\forall \varepsilon > 0$, the hypothesis of Lemma 2.4 is satisfied so (2.1) holds for $i = 0$. Next, by definition $\delta_j(Y_{ni}, Y_{ni}^k) \leqq |Y_{ni}(j) - Y_{ni}^k(j)| = |X(j+\tau_{ni}) - a(-j)Z(\tau_{ni})|n^{-1/\alpha} \leqq |X(j+\tau_{ni}) - X_k(j+\tau_{ni})|n^{-1/\alpha} + |X_k(j + \tau_{ni}) - a(-j)Z(\tau_{ni})|n^{-1/\alpha}$ if $j \leqq k$. Hence

$$\begin{aligned}\{\delta_j(Y_{ni}, Y_{ni}^k) > 2\varepsilon\} \subset \{|X_k(j + \tau_{ni}) - a(-j)Z(\tau_{ni})| > n^{1/\alpha}\varepsilon, \tau_{ni} < nN - j - k\} \\ \cup \{\tau_{ni} \geqq nN - j - k\} \cup F_n \\ \subset \{\tau_{ni} \geqq nN - j - k\} \cup E_n \cup F_n ,\end{aligned}$$

and as in the proof of (3.7) we obtain

$$\lim_{k\to\infty} \limsup_{n\to\infty} P(\delta_j(Y_{ni}, Y_{ni}^k) > 2\varepsilon) = 0 , \qquad \forall \varepsilon > 0, j \geqq 1 ,$$

i.e., that (2.2) holds for $i \geqq 1$. Since this implies that also (2.1) holds for $i \geqq 1$, it completes the proof that $\eta_n \to_d \eta$.

Finally, let τ_{ni}' be the first time when $\{X(t); t \in [t_{ni}, t_{ni} + h(n)]\}$ assumes its maximum and suppose that for λ_0 satisfying $a(\lambda_0) = A$ we have $\min_{\lambda \neq \lambda_0}(a(\lambda_0) - a(\lambda)) = 2\varepsilon > 0$. To verify the claim of the remark we show that it is then possible to replace $\{\tau_{ni}\}$ by $\{\tau_{ni}' - \lambda_0\}$ in the statement of the lemma. To do this, it is sufficient to prove

$$P(\tau_{ni}' - \lambda_0 \neq \tau_{ni}) \to 0 \quad \text{as} \quad n \to \infty . \tag{3.9}$$

Let τ_{ni}^k be defined from $X_k(t) = \sum_{\lambda=-k+t}^{k+t} a(\lambda - t)Z(\lambda)$ in the same way as τ_{ni}' is defined from $X(t)$. Now $\{\tau_{ni}' - \lambda_0 \neq \tau_{ni}\} \subset \{\tau_{ni}' \neq \tau_{ni}^k, \tau_{ni}^k - \lambda_0 = \tau_{ni}\} \cup \{\tau_{ni}^k - \lambda_0 \neq \tau_{ni}\}$ and for $k \geqq |\lambda_0|$ we have $\{\tau_{ni}' \neq \tau_{ni}^k, \tau_{ni}^k - \lambda_0 = \tau_{ni}\} \subset D_n \cup E_n{}' \cup F_n \cup \{\tau_{ni} > nN - k - \lambda_0 - h(n)\}$ where $E_n{}'$ is defined in the same way as E_n except that $|t' - t''| \leqq 2k + 1$ is replaced by $|t' - t''| \leqq h(n) + 2k$. Furthermore $\{\tau_{ni}^k - \lambda_0 \neq \tau_{ni}\} \subset E_n{}' \cup \{\tau_{ni} > nN - k - \lambda_0 - h(n)\}$ and thus (3.9) follows as in the proof of (3.8). □

The general result follows rather easily from Lemma 3.1. Recall the notation $A = \max_\lambda a^+(\lambda)$, put $a = \max_\lambda a^-(\lambda)$ and set $\mu' = c_\alpha A^\alpha(1 + \beta)x^{-\alpha}$, $\mu'' = c_\alpha a^\alpha(1 - \beta)x^{-\alpha}$. Further let Z' and Z'' be independent with distribution functions $F_1(z) = 1 - x^\alpha A^{-\alpha}(1+\beta)^{-1}z^{-\alpha}$, $z \geqq x^{-1}A(1+\beta)^{1/\alpha}$ and $F_2(z) = 1 - x^\alpha a^{-\alpha}(1-\beta)^{-1}z^{-\alpha}$, $z \geqq x^{-1}a(1-\beta)^{1/\alpha}$ respectively and let $Y_i = (\cdots, a(1)Z', a(0)Z', a(-1)Z', \cdots)$ with probability $\mu'/(\mu' + \mu'')$ and $Y_i = (\cdots, -a(1)Z'', -a(0)Z'', -a(-1)Z'', \cdots)$

otherwise. Then the limiting distribution is that of the marked point process

(3.10) $(N, Y_1, Y_2 \cdots)$ where the components are independent, N is a Poisson process with intensity $\mu = \mu' + \mu''$ and where the Y_i's have the distribution given above.

THEOREM 3.2. *Let* $\{a(\lambda)\}_{-\infty}^{\infty}$ *satisfy* (3.1) *and let* $\{Z(\lambda)\}_{-\infty}^{\infty}$ *be an independent, stable* $(1, \alpha, \beta)$ *sequence. Suppose that the moving average sequence* $\{X(t)\}_{-\infty}^{\infty}$ *is given by* $X(t) = \sum_{-\infty}^{\infty} a(\lambda - t)Z(\lambda)$. *Then there exist* $\{\tau_{ni}\}$ *with* $\{t_{ni} - \tau_{ni}\}_{n=1}^{\infty}$ *tight for each* $i \geqq 1$, *such that* $\eta_n \to_d \eta$, *where* η_n *is the marked point process of separated exceedances of* $n^{1/\alpha}x$ *by* $\{X(t); t \geqq 0\}$ *and where the distribution of* η *is given by* (3.10).

PROOF. It is immediate from (1.1) that if X and Y are independent and stable $(1, \alpha, 1)$ with $\alpha \neq 1$, then $((1 + \beta)/2)^{1/\alpha}X - ((1 - \beta)/2)^{1/\alpha}Y$ is stable $(1, \alpha, \beta)$. If $\alpha = 1$ a constant has to be added to this representation, but since this introduces only trivial complications we assume $\alpha \neq 1$ for the remainder of the proof. Let $\{Z'(\lambda)\}_{-\infty}^{\infty}$ and $\{Z''(\lambda)\}_{-\infty}^{\infty}$ be independent stable $(1, \alpha, 1)$ sequences and put $X'(t) = \sum a(\lambda - t)((1 + \beta)/2)^{1/\alpha}Z'(\lambda)$ and $X''(t) = \sum a(\lambda - t)((1 - \beta)/2)^{1/\alpha}Z''(\lambda)$. Then the stochastic process $\{X'(t) - X''(t)\}_{t=-\infty}^{\infty}$ has the same distribution as $\{X(t)\}_{t=-\infty}^{\infty}$ and since we are interested only in distributional properties, we may thus consider $X'(t) - X''(t)$ instead of $X(t)$.

Let $0 \leqq \tau'_{n1} < \tau'_{n2} < \cdots$ be the times when $\{Z'(t); t \geqq 0\}$ exceeds $n^{1/\alpha}A^{-1}((1 + \beta)/2)^{-1/\alpha}x$ and let $0 \leqq \tau''_{n1} < \tau''_{n2} < \cdots$ be the times when $\{Z''(t); t \geqq 0\}$ exceeds $n^{1/\alpha}a^{-1}((1 - \beta)/2)^{-1/\alpha}x$. Using $\{\tau'_{ni}\}$ and $\{\tau''_{ni}\}$, define marked point processes of separated exceedances of the level $n^{1/\alpha}x$, $\eta_n' = (N_n', Y'_{n1}, Y'_{n2}, \cdots)$ from $\{X'(t)\}$ and $\eta_n'' = (N_n'', Y''_{n1}, Y''_{n2}, \cdots)$ from $\{X''(t)\}$. Further let η' and η'' be independent and with the distributions obtained from (3.2) by replacing $a(\lambda)$ with $a(\lambda)((1 + \beta)/2)^{1/\alpha}$ and with $-a(\lambda)((1-\beta)/2)^{1/\alpha}$ respectively. From Lemma 3.1 we have $\eta_n' \to_d \eta'$ and $\eta_n'' \to_d \eta''$, and since η_n' and η_n'' are independent, $(\eta_n', \eta_n'') \to_d (\eta', \eta'')$ (using the product metric on $S \times S$). Let $N_n^0 = N_n' + N_n''$, let $0 \leqq \tau_{n1} \leqq \tau_{n2} \leqq \cdots$ be the atoms of N_n^0, and if $\tau_{ni} = \tau'_{nk}$ for some k put $Y^0_{ni} = Y'_{nk}$, otherwise put $Y^0_{ni} = Y''_{nk}$ for the k that satisfies $\tau_{ni} = \tau''_{nk}$. Then the function that maps (η_n', η_n'') into η_n^0 is continuous except on the set where N_n' and N_n'' have common atoms. Since this set has (η', η'') probability zero it follows that $\eta_n^0 \to_d \eta$, where the distribution of η is given by (3.10). Finally, using the independence of $\{X_n'(t)\}$ and $\{X_n''(t)\}$ and similar (but easier) arguments as in the proof of Lemma 3.1, it follows that $P(d(\eta_n, \eta_n^0) > \varepsilon) \to 0$ as $n \to \infty$, for all $\varepsilon > 0$, and the theorem is proven. □

Theorem 3.2 gives a rather complete description of the asymptotic behaviour of extremes of linear stable processes, but it is somewhat complicated, and we will spend the rest of this section on some (simpler) corollaries to it.

The point process N_n gives the separated exceedances of $xn^{1/\alpha}$, but also ordinary exceedances are interesting. For Borel sets $B \subset R^+$ put $E_n(B) = \#\{t/n \in B; X(t) > xn^{1/\alpha}\}$, let $\nu^+(z) = \#\{\lambda; za(\lambda) > x\}$ and $\nu^-(z) = \{\lambda; -za(\lambda) > x\}$, and let the point process E have the following distribution: atoms occur according to a Poisson process with intensity $\mu = \mu' + \mu''$, the multiplicities of different atoms are independent and with the distribution of ν, where $\nu = \nu^+(Z')$ with probability $\mu'/(\mu' + \mu'')$ and $\nu = \nu^-(Z'')$ otherwise. (μ', μ'' and the distributions of Z' and Z'' are given on page 856.)

COROLLARY 3.3. *Suppose that* $\{X(t)\}_{t=-\infty}^{\infty}$ *satisfies the assumptions of Theorem* 3.2. *Then* $E_n \to_d E$, *where* E_n *is the point process of exceedances of* $xn^{1/\alpha}$ *and where the distribution of* E *is given above.*

PROOF. We only sketch the proof. Let E_n^k have the same atoms as N_n, but with the multiplicity of the atom at t_{ni} equal to $\#\{t \in [t_{ni}, t_{ni} + k]; X(t) > xn^{1/\alpha}\}$. According to the theorem $\eta_n \to_d \eta$ and with probability one η is of the form $(\nu, y_1, y_2, \cdots)$ where ν is a locally finite measure and where $y_i \in R^\infty$ is of the form $y_i(t) = z_i a(-t)$, $t = 0, \pm 1, \cdots$. However, for vectors of this form, the function that maps η_n into E_n^k is continuous except if $z_i a(-t) = x$ for some $i \geqq 1$. As the set of such vectors has η probability zero, it follows that E_n^k converges in distribution to some point process, say E^k. It is easily seen that $E^k \to_d E$ as $k \to \infty$, and the proof can be finished, using similar methods as in the proof of Lemma 3.1, by approximating E_n by E_n^k. $\square$

COROLLARY 3.4. *Suppose that* $\{X(t)\}_{-\infty}^{\infty}$ *satisfies the hypothesis of Theorem* 3.2 *and that the Borel set* $B \subset R^+$ *has boundary with Lebesgue measure zero* ($|\partial B| = 0$). *Then*

$$P(E_n(B) = k) \to \sum_{j=0}^{k} \frac{(\mu|B|)^j}{j!} e^{-\mu|B|} \Pr\left(\sum_{i=1}^{j} \nu_i = k\right)$$

as $n \to \infty$, *where the* ν_i*'s are independent and with the same distributions as* ν. *Similarly, if* $B_1, \cdots, B_l \subset R^+$ *are disjoint and have boundaries with Lebesgue measure zero, then* $P(E_n(B_1) = k_1, \cdots, E_n(B_l) = k_l)$ *tends to the product of the corresponding probabilities.*

Also the joint limiting distribution of heights and locations of the exceedances can be obtained from Theorem 3.2, but since the limiting distributions are complicated we only give the simplest result.

COROLLARY 3.5. *Suppose that* $\{X(t)\}_{-\infty}^{\infty}$ *satisfies the hypothesis of Theorem* 3.2 *and let* $M_n = \max_{1 \leqq t \leqq n} X(t)$. *Then*

$$P(M_n/n^{1/\alpha} \leqq x) \to \exp\{-c_\alpha(A^\alpha(1 + \beta) + a^\alpha(1 - \beta))x^{-\alpha}\}$$

as $n \to \infty$.

PROOF. Obviously $P(M_n/n^{1/\alpha} \leqq x) = P(E_n([0, 1]) = 0)$, and the latter probability converges to $\exp\{-c_\alpha(A^\alpha(1 + \beta) + a^\alpha(1 - \beta))x^{-\alpha}\}$ by Corollary 3.4. $\square$

There is a loose end left from Theorem 3.2. Namely, if $a(\lambda) \geqq 0$ and $\beta = -1$ (or $a(\lambda) \leqq 0$ and $\beta = 1$) then $\mu_1 = \mu_2 = 0$ and the theorem only says that $M_n/n^{1/\alpha} \to_P 0$. However, there may still be some other normalization which gives a nondegenerate limit. If $0 < \alpha < 1$ then $X(t) \leqq 0$ for all t and $0 \geqq M_n \rightarrow_P 0$ as $n \rightarrow \infty$. If instead $1 < \alpha < 2$ then, as can be seen from Skorokhod (1961), $1 - F_{\alpha,-1}(x) \sim A_\alpha x^{-\alpha/(2(\alpha-1))} \exp\{-(B_\alpha x)^{\alpha/(\alpha-1)}\}$, as $x \rightarrow \infty$, where the constants A_α and B_α are given in the same reference. It is hence straightforward that if the sequence $\{X(t)\}$ is independent, with scale parameter one, (i.e., $a(0) = 1$ and $a(\lambda) = 0$ for $\lambda \neq 0$) then

$$P(a_n(M_n - b_n) \leqq x) \rightarrow e^{-e^{-x}} \tag{3.11}$$

as $n \rightarrow \infty$, for $a_n = B_\alpha \alpha(\alpha - 1)^{-1}(\log n)^{1/\alpha}$ and $b_n = B_\alpha^{-1}(\log n)^{1-1/\alpha} + B_\alpha^{-1}(1 - 1/\alpha)2^{-1}(\log n)^{1/\alpha}\{2 \log A_\alpha + \alpha(\alpha - 1)^{-1} \log B_\alpha - \log \log n\}$. Moreover it can be checked that if only finitely many of the $a(\lambda)$'s are nonzero then the conditions of Theorem 3.1 of Leadbetter (1976) are satisfied, so (3.11) holds also in this case (still supposing that the scale parameter $\sum a(\lambda)^\alpha = 1$). It also follows that the point process of (ordinary) exceedances of the level $x/a_n + b_n$ tends in distribution to a Poisson process with intensity e^{-x}. Thus extremes behave in the same way for the dependent sequence and for the independent sequence. Of course the same results hold in the case of infinitely many nonzero $a(\lambda)$'s, provided $a(\lambda)$ decreases to zero quickly enough when $\lambda \rightarrow \pm\infty$.

In the boundary case, $\alpha = 1$, we have $1 - F_{1,-1}(x) \sim \pi^{-1} \exp\{\frac{1}{2} - \pi 4^{-1}x - 2(\pi e)^{-1}e^{\pi 2^{-1}x}\}$ as $x \rightarrow \infty$, and the extremes behave in the same way as above, except that the normalizing constants now should be $a_n = 2^{-1}\pi \log n$ and $b_n = 2\pi^{-1}\{\log \log n + \log \pi e/2 - (2^{-1} \log \log n + \log (\pi^2/2))/\log n\}$.

It is interesting to note that this behaviour is quite different from that described in Theorem 3.2 and that it instead is similar to the behaviour of extremes of normal sequences. It seems to be a quite general phenomenon that extremes of moving averages $\sum a(\lambda)Z(t - \lambda)$ behave like those of an independent sequence with the same marginal distribution if the tails of the distribution of the Z's decrease exponentially or faster, while behaviour like that described in Theorem 3.2 occurs if the tails decrease polynomially or slower. One aspect of this was noted in connection with Lemma 3.1, namely that the essential property of the stable distribution used in the proof is that the tails decrease as $x^{-\alpha}$. Thus results similar to those of Theorem 3.2 hold for moving averages as soon as the tails of the distribution of the independent variables decrease as powers of x, e.g., if the independent variables belong to the domain of normal attraction of a stable law with exponent $\alpha < 2$.

4. Moving averages of stable processes in continuous time. Consider a stochastic process $\{Z(\lambda); \lambda \in R\}$ that has stationary independent increments with $Z(0) = 0$ and $Z(1)$ stable $(1, \alpha, \beta)$. In the sequel we will assume that $\alpha \neq 1$. The usual way of obtaining an integral $\int a(\lambda)\, dZ(\lambda)$ is to first define it for step functions of the form $a(\lambda) = \sum_{i=1}^k a_i I_{(b_i,c_i]}(\lambda)$ with $-\infty < b_1 < c_1 \leqq b_2 < \cdots < c_k < \infty$ by putting $\int a(\lambda)\, dZ(\lambda) = \sum_{i=1}^k a_i(Z(c_i) - Z(b_i))$. Then, as is easily seen from (1.1), $\int a(\lambda)\, dZ(\lambda)$ is stable with index α, scale parameter $\{\int |a(\lambda)|^\alpha\, d\lambda\}^{1/\alpha}$, and symmetry parameter $\{\int (a^+(\lambda)^\alpha - a^-(\lambda)^\alpha)\, d\lambda\}/\int |a(\lambda)|^\alpha\, d\lambda$. Next, if $a(\lambda)$ is

(Lebesgue) measurable and satisfies

$$\int |a(\lambda)|^\alpha \, d\lambda < \infty \qquad 0 < \alpha < 1 \text{ or } 1 < \alpha < 2, \tag{4.1}$$

we can find a sequence $\{a_n(\lambda)\}_{n=1}^\infty$ of step functions with $\int |a(\lambda) - a_n(\lambda)|^\alpha \, d\lambda \to 0$ as $n \to \infty$. Then, for $I_n = \int a_n(\lambda) \, dZ(\lambda)$, the scale parameter of $I_n - I_m$ is $\{\int |a_n(\lambda) - a_m(\lambda)|^\alpha \, d\lambda\}^{1/\alpha}$ which tends to zero as $\min(m, n) \to \infty$. Hence $\{I_n\}_{n=1}^\infty$ is a Cauchy sequence in the sense of convergence in probability and there is a random variable I with $I_n \to_P I$. The integral is then defined (uniquely a.s.) by $\int a(\lambda) \, dZ(\lambda) = I$.

The object of study is moving averages, i.e., processes of the form $X(t) = \int a(\lambda - t) \, dZ(\lambda)$ with $a(\lambda)$ satisfying (4.1). We always assume that a separable version has been chosen. Our approach is to approximate $a(\lambda)$ by step functions $a_k(\lambda) = \sum a_i I(i2^{-k} < \lambda \leqq (i+1)2^{-k})$ and thus to approximate $X(t)$ by $X_k(t) = \sum a_i\{Z((i+1)2^{-k} + t) - Z(i2^{-k} + t)\}$. The necessary estimates are given by the following two lemmas.

Lemma 4.1. *Suppose* $X(t) = \sum a_i\{Z((i+1)2^{-k} + t) - Z(i2^{-k} + t)\}$, *where* $\sum |a_i|^\alpha < \infty$ *and where* $\{Z(\lambda); \lambda \in R\}$ *has stationary independent increments with* $Z(0) = 0$ *and* $Z(1)$ *stable* $(1, \alpha, 1)$, *and put* $X_l = \sup_{0 \leqq t \leqq 1} |X((l+t)2^{-k}) - X(l2^{-k})|$. *If* $0 < \alpha < 1$ *then, for some constant* K_α,

$$P(\max_{0 \leqq l \leqq 2^k N - 1} X_l > x) \leqq K_\alpha \sum A_i^\alpha N x^{-\alpha} \tag{4.2}$$

where $A_i = \max_{0 \leqq j \leqq 2^k} |a_{i2^k + j}|$. *If* $1 < \alpha < 2$ *then*

$$P(\max_{0 \leqq l \leqq 2^k N - 1} X_l > x) \leqq K_\alpha \sum |a_i|^\alpha N x^{-\alpha}. \tag{4.3}$$

Proof. We have

$$\begin{aligned} X((l+t)2^{-k}) &- X(l2^{-k}) \\ &= \sum a_i\{Z((i+1+l+t)2^{-k}) - Z((i+l+t)2^{-k})\} \\ &\quad - \sum a_i\{Z((i+1+l)2^{-k}) - Z((i+l)2^{-k})\} \\ &= \sum (a_{i-1} - a_i)\{Z((i+l+t)2^{-k}) - Z((i+l)2^{-k})\}. \end{aligned} \tag{4.4}$$

Suppose $0 < \alpha < 1$. Then $\{Z(\lambda)\}$ has nondecreasing sample paths and hence for $0 \leqq t \leqq 1$,

$$\begin{aligned} &|X((l+t)2^{-k}) - X(l2^{-k})| \\ &\quad = |\textstyle\sum_{i=-\infty}^\infty \sum_{j=1}^{2^k} (a_{i2^k+j-1} - a_{i2^k+j})\{Z(i + (j+l+t)2^{-k}) - Z(i + (j+l)2^{-k})\}| \\ &\quad \leqq 2 \textstyle\sum_{i=-\infty}^\infty A_i \sum_{j=1}^{2^k} \{Z(i + (j+l+t)2^{-k}) - Z(i + (j+l)2^{-k})\} \\ &\quad \leqq 2 \textstyle\sum_{i=-\infty}^\infty A_i \sum_{j=1}^{2^k} \{Z(i + (j+l+1)2^{-k}\} - Z\{i + (j+l)2^{-k})\} \\ &\quad = 2 \textstyle\sum A_i\{Z(i+1+(l+1)2^{-k}) - Z(i + l2^{-k})\}. \end{aligned}$$

It follows that

$$\begin{aligned} \max_{0 \leqq l \leqq 2^k - 1} X_l &\leqq 2 \textstyle\sum A_i\{Z(i+2) - Z(i)\} \\ &\leqq 2 \textstyle\sum (A_i + A_{i-1})\{Z(i+1) - Z(i)\}. \end{aligned}$$

The sequence $\{Z(i+1) - Z(i)\}_{i=-\infty}^{\infty}$ is independent and stable $(1, \alpha. 1)$ and thus, by (3.4),

$$P(\max_{0 \leqq l \leqq 2^k - 1} X_l > x) \leqq 2^\alpha k_\alpha \sum (A_i + A_{i-1})^\alpha x^{-\alpha}$$
$$\leqq 2 \cdot 2^\alpha k_\alpha \sum A_i^\alpha x^{-\alpha} .$$

Hence

$$P(\max_{0 \leqq l \leqq N2^k - 1} X_l > x) \leqq NP(\max_{0 \leqq l \leqq 2^k - 1} X_l > x)$$
$$\leqq 2 \cdot 2^\alpha k_\alpha \sum A_i^\alpha N x^{-\alpha}$$

and (4.2) holds with $K_\alpha = 2 \cdot 2^\alpha k_\alpha$.

Next suppose that $1 < \alpha < 2$. Put $Y(t) = X(t2^{-k}) - X(0)$ and set $b_i = a_i - a_{i-1}$. It follows from (4.4) that $\{Y(t); 0 \leqq t \leqq 1\}$ has stationary independent increments with $Y(0) = 0$ and $Y(1)$ stable with index α and scale parameter $(2^{-k} \sum |b_i|^\alpha)^{1/\alpha}$. Thus, according to Fristedt (1974, page 353), there is a constant c_α such that

$$P(\sup_{0 \leqq t \leqq 1} |Y(t)| > x) \leqq c_\alpha 2^{-k} \sum |b_i|^\alpha x^{-\alpha} .$$

Using Boole's inequality, stationarity and that $b_i = a_{i-1} - a_i$ we have

$$P(\max_{0 \leqq l \leqq 2^k N - 1} X_l > x) \leqq 2^k NP(X_0 > x)$$
$$\leqq 2^k N c_\alpha 2^{-k} \sum |b_i|^\alpha x^{-\alpha}$$
$$\leqq c_\alpha 2^\alpha \sum |a_i|^\alpha N x^{-\alpha} ,$$

and, taking $K_\alpha = c_\alpha 2^\alpha$, (4.3) follows. (This argument works also for $0 < \alpha < 1$, but the inequality (4.2) is better.) □

In order to apply Lemma 4.1 to $X(t) = \int a(\lambda - t)\, dZ(\lambda)$ some conditions are needed. Assume $a(\lambda) \geqq 0$, let $B_{ki} = \sup_{\lambda 2^k \in (i, i+1]} a(\lambda)$, $b_{ki} = \inf_{\lambda 2^k \in (i, i+1]} a(\lambda)$ and put $a_{ki} = a(i2^{-k})$. One possibility is to require

(4.5) $a(\lambda)$ is uniformly continuous, $\sum_{i=-\infty}^{\infty} B_{1i}^\alpha < \infty$ and $0 < \alpha < 1$.

The second part of this condition is of course equivalent to $\sum_{i=-\infty}^{\infty} B_{ki}^\alpha < \infty$, for all $k \geqq 1$. Another possibility is to require

(4.6) $a(\lambda)$ is uniformly continuous, $\sum_{i=-\infty}^{\infty} |a_{1i}|^\alpha < \infty$, there exist $\delta > 0$ and K such that $2^{k\delta} \sum |a_{ki} - a_{k-1,[i/2]}|^\alpha \leqq K$, and $1 < \alpha < 2$.

Obviously, this condition implies that $\sum |a_{ki}|^\alpha < \infty$ for all $k \geqq 1$. The latter part of the condition perhaps needs some motivation. Suppose that $a(\lambda)$ is continuously differentiable, except possibly at the points $\{i2^{-k+1}\}_{i=-\infty}^{\infty}$, and put $f_{ki} = \sup_{\lambda 2^k \in (i-1, i]} |a'(\lambda)|$. Then $|a_{ki} - a_{k-1,[i/2]}| \leqq f_{ki} 2^{-k}$ and hence $\sum |a_{ki} - a_{k-1,[i/2]}|^\alpha \leqq 2^{-k(\alpha-1)} \sum f_{ki}^\alpha 2^{-k}$. Thus the latter part of (4.6) holds with $\delta = \alpha - 1$ if, e.g., $\sum f_{ki}^\alpha 2^{-k}$ converges as $k \to \infty$, and to require that this holds is rather close to requiring $\int |a'(\lambda)|^\alpha\, d\lambda < \infty$.

LEMMA 4.2. *Suppose that* $X(t) = \int a(\lambda - t)\, dZ(\lambda)$, *where* $a(\lambda)$ *is nonnegative and satisfies* (4.1) *and* $\{Z(\lambda);\, \lambda \in R\}$ *is as in Lemma* 4.1. *Furthermore put* $X_k(t) = \sum a_{ki}\{Z((i+1)2^{-k}+t) - Z(i2^{-k}+t)\}$ (*the sum converges, by* (4.5) *or by* (4.6)). *If* $a(\lambda)$ *satisfies* (4.5) *then, for some constant* K_α'

$$P(\sup_{0 \leqq t \leqq N} |X(t) - X_k(t)| > x) \leqq K_\alpha' \sum_{i=-\infty}^{\infty} A_{ki}^\alpha N x^{-\alpha} \tag{4.7}$$

where $A_{ki} = \max_{0 \leqq l \leqq 2^k} |B_{k,i2^k+j} - b_{k,i2^k+j}|$. (*Note that* $A_{1i} \geqq A_{2i} \geqq \cdots$ *for* i *fixed.*) *If* $a(\lambda)$ *satisfies* (4.6) *then*

$$P(\sup_{0 \leqq t \leqq N} |X(t) - X_k(t)| > x) \leqq K_\alpha' 2^{-k\delta} N x^{-\alpha}\,. \tag{4.8}$$

PROOF. Suppose $0 < \alpha < 1$ and let $\bar{X}_k(t) = \sum B_{ki}\{Z((i+1)2^{-k}+t) - Z(i2^{-k}+t)\}$, $\underline{X}_k(t) = \sum b_{ki}\{Z((i+1)2^{-k}+t) - Z(i2^{-k}+t)\}$. Since $Z(\lambda)$ has nondecreasing sample paths, $D_k(t) = \bar{X}_k(t) - \underline{X}_k(t) \geqq |X(t) - X_k(t)|$. Using the same methods as in the first part of Lemma 4.1 it is easily seen that $P(\sup_{0 \leqq t \leqq N} D_k(t) > x) \leqq K_\alpha \sum_{i=-\infty}^{\infty} A_{ki} N x^{-\alpha}$ and thus (4.7) follows with $K_\alpha' = K_\alpha$.

Now consider $1 < \alpha < 2$. In this case let $D_k(t) = X_k(t) - X_{k-1}(t)$ and put $d_{ki} = a_{ki} - a_{k-1,[i/2]}$, making $D_k(t) = \sum d_{ki}\{Z((i+1)2^{-k}+t) - Z(i2^{-k}+t)\}$. Further let $D_l = \sup_{0 \leqq t \leqq 1} |D_k((l+t)2^{-k}) - D_k(l2^{-k})|$ and use (3.4) and Lemma 4.1 to obtain

$$\begin{aligned} P(\sup_{0 \leqq t \leqq N} D_k(t) > x) &\leqq P(\max_{0 \leqq l \leqq 2^n N - 1} |D_k(l2^{-k})| > x/2) \\ &\quad + P(\max_{0 \leqq l \leqq 2^n N-1} D_l > x/2) \\ &\leqq k_\alpha N \sum |d_{ki}|^\alpha x^{-\alpha} + K_\alpha N \sum |d_{ki}|^\alpha x^{-\alpha}\,. \end{aligned}$$

Hence by (4.6)

$$P(\sup_{0 \leqq t \leqq N} D_k(t) > x) \leqq KN\{(k_\alpha + K_\alpha) 2^{-k\delta} x^{-\alpha}\}\,. \tag{4.9}$$

Let $x_i = 2^{-(i-1)\delta/(2\alpha)}(1 - 2^{-\delta/(2\alpha)})x$, so that $\sum_{i=1}^{\infty} x_i = x$ and $\sum_{i=1}^{\infty} 2^{-(k+i)\delta} x_i^{-\alpha} \leqq$ constant $\times\, 2^{-k\delta} x^{-\alpha}$. Since $X_k(t) \to_p X(t)$ we have

$$P(\sup_{0 \leqq t \leqq N} |X(t) - X_k(t)| > x) \leqq \sum_{i=1}^{\infty} P(\sup_{0 \leqq t \leqq N} D_{k+i}(t) > x_i)\,, \tag{4.10}$$

and (4.8) follows from (4.9) and (4.10). □

The conditions used above imply that $X(t)$ has continuous sample paths, and although we do not need this result for the sequel, it is interesting in its own right.

THEOREM 4.3. *Let* $\{Z(\lambda);\, \lambda \in R\}$ *have stationary independent increments, which are stable with index* α. *Further suppose that* $a(\lambda)$ *satisfies* (4.1) *and that both* $a^+(\lambda)$ *and* $a^-(\lambda)$ *satisfy either* (4.5) *or* (4.6). *Then the moving average* $X(t) = \int a(\lambda - t)\, dZ(\lambda)$ *has continuous sample paths.*

REMARK 4.4. There is of course no claim of necessity of the conditions for path continuity. For $0 < \alpha < 1$ it seems probable that the necessary and sufficient condition is that $a(\lambda)$ is continuous and $\int |a(\lambda)|^\alpha\, d\lambda < \infty$, which is not too far from Condition (4.5).

PROOF. Obviously it is no restriction to assume that $Z(0) = 0$ and $Z(1)$ is stable$(1, \alpha, \beta)$, and since $\int a(\lambda - t)\,dZ(\lambda) = \int a^+(\lambda - t)\,dZ(\lambda) + \int a^-(\lambda - t)\,dZ(\lambda)$, it is enough to show that each of the terms is continuous, i.e., we may also assume $a(\lambda) \geqq 0$.

Let $\{Z'(\lambda); \lambda \in R\}$ and $\{Z''(\lambda); \lambda \in R\}$ be independent and have stationary independent increments with $Z'(0) = Z''(0) = 0$ and $Z'(1)$, $Z''(1)$ stable $(1, \alpha, 1)$. Then $\int a(\lambda - t)\,dZ(\lambda)$ has the same distribution as $((1 + \beta)/2)^{1/\alpha} \int a(\lambda - t)\,dZ'(\lambda) - ((1 - \beta)/2)^{1/\alpha} \int a(\lambda - t)\,dZ''(\lambda)$, and thus we may further assume $\beta = 1$.

The proof proceeds by approximating $X(t)$ by $V_k(t) = \int a_k(\lambda - t)\,dZ(\lambda)$, where $a_k(t)$ is defined by the requirement that $a_k(t) = 0$, $|t| \geqq k'$, for $k' = k'(k)$ to be specified later, that $a_k(l2^{-k}) = a(l2^{-k})$, $l = 0, \pm 1, \cdots, \pm k'2^k - 1$, and that $a_k(t)$ is linear between these points. Using the definition of the integral as a limit of sums and Abelian summation, it is seen that ("partial integration")

$$\begin{aligned}\int a_k(\lambda - t)\,dZ(\lambda) &= \int a_k(\lambda)\,dZ(\lambda + t)\\ &= -\int a_k'(\lambda) Z(\lambda + t)\,d\lambda\\ &= -\sum_{i=-k'2^k}^{k'2^k-1} \left\{\frac{a((i+1)2^{-k}) - a(i2^{-k})}{2^{-k}}\right\} \int_{i2^{-k}}^{(i+1)2^{-k}} Z(\lambda + t)\,d\lambda\,,\end{aligned}$$

$$\text{(here } a(k') = a(-k') = 0)$$

where the integrals are defined as limits in probability of sums. However, $Z(\lambda) \in D(-\infty, \infty)$ (see, e.g., Breiman (1968), page 306), and is thus locally Riemann integrable and hence $\int_{i2^{-k}}^{(i+1)2^{-k}} Z(\lambda + t)\,d\lambda$ is a.s. a Rieman integral and is thus a.s. continuous in t, and it follows that also $V_k(t) = \int a_k(\lambda - t)\,dZ(\lambda)$ is continuous in t a.s.

Hence, if we prove, e.g.,

$$\sup_{0 \leqq t \leqq 1} |X(t) - V_k(t)| \to_p 0\,, \tag{4.11}$$

then the desired result follows, since there is then a sequence $\{k_n\}$ of integers with $P(\sup_{0 \leqq t \leqq 1} |X(t) - V_{k_n}(t)| \to 0 \text{ as } n \to \infty) = 1$, i.e., $X(t)$ is a.s. a uniform (in $[0, 1]$) limit of continuous functions and is thus continuous in $[0, 1]$ and hence, by stationarity, in all of R. Now, let $X_k(t)$ be as in Lemma 4.2 and put $X_k'(t) = \sum_{|i2^{-k}| < k'} a_{ki}\{Z((i + 1)2^{-k} + t) - Z(i2^{-k} + t)\}$. We have

$$\begin{aligned}&P(\sup_{0 \leqq t \leqq 1} |X(t) - V_k(t)| > x)\\ &\quad \leqq P(\sup_{0 \leqq t \leqq 1} |X(t) - X_k(t)| > x/3)\\ &\quad\quad + P(\sup_{0 \leqq t \leqq 1} |X_k(t) - X_k'(t)| > x/3)\\ &\quad\quad + P(\sup_{0 \leqq t \leqq 1} |X_k'(t) - V_k(t)| > x/3)\,.\end{aligned} \tag{4.12}$$

Thus, if $0 < \alpha < 1$, Lemmas 4.1 and 4.2 give that

$$\begin{aligned}&P(\sup_{0 \leqq t \leqq 1} |X(t) - V_k(t)| > x)\\ &\quad \leqq K_\alpha'\{\textstyle\sum_{i=-\infty}^{\infty} A_{ki}^\alpha + \sum_{|i| \geqq k'} B_{1i}^\alpha + \sum_{i=-k'+1}^{k'-1} A_{ki}^\alpha\}3^\alpha x^{-\alpha}\,.\end{aligned} \tag{4.13}$$

Choosing, e.g., $k'(k) \equiv k$ it follows from (4.5) and the dominated convergence theorem that the right-hand side of (4.13) tends to zero as $k \to \infty$, and thus (4.11) holds for $0 < \alpha < 1$.

It is no loss of generality to assume $\delta \leqq \alpha$ in (4.6), and then it can be seen that, regardless of the value of k', $a_k(\lambda)$ satisfies (4.6) with K not depending on k and with the same δ as $a(\lambda)$, and thus if $1 < \alpha < 2$ it follows from Lemma 4.2 that the first and the third terms of (4.12) are bounded by $K_\alpha' 2^{-k\delta} 3^\alpha x^{-\alpha}$. Furthermore, by (3.4) the second term is bounded by $K_\alpha \sum_{|i2-k|>k'} |a_{ki}|^\alpha 3^\alpha x^{-\alpha}$, and since $\sum |a_{ki}|^\alpha < \infty$ by (4.6), k' can be chosen large enough to make $\sum_{|i2-k|>k'} |a_{ki}|^\alpha \to 0$ as $k \to \infty$, and it follows that (4.11) holds also for $1 < \alpha < 2$. □

5. Extremes in continuous time. Let $\{X(t);\ t \in R\}$ be a moving average, and in analogy with Section 3 define recursively $t_{T1} = \inf\{t \geqq h(T);\ X(t) > xT^{1/\alpha}\}$, $t_{Ti} = \inf\{t \geqq t_{T,i-1} + h(T);\ X(t) > xT^{1/\alpha}\}$ for $i \geqq 2$. For a given sequence $\{\tau_{Ti}\}_{i=1}^\infty$ put $Y_{Ti}(t) = X(t + \tau_{Ti})/T^{1/\alpha}$, let $N_T(B) = \#\{t_{Ti}/T \in B\}$ for Borel sets $B \subset R^+$ and consider the marked point process $\eta_T = (N_T, Y_{T1}, Y_{T2}, \cdots)$ of separated exceedances of $xT^{1/\alpha}$. Furthermore, put $A = \sup_{\lambda \in R} a^+(\lambda)$, let μ and Z be as defined on page 856, and let Y_i have the distribution of the random variable $\{Za(-t);\ t \in R\}$ in $D(-\infty, \infty)$. If $Z(\lambda)$ is completely asymmetric the limiting distribution will be that of

(5.1) $(N, Y_1, Y_2, \cdots)$ where the components are independent, N is a Poisson process with intensity μ, and where the Y_i's have the distribution given above.

LEMMA 5.1. *Suppose that $\{Z(\lambda);\ \lambda \in R\}$ has stationary independent increments, with $Z(0) = 0$ and $Z(1)$ stable $(1, \alpha, 1)$. Further suppose that $a(\lambda)$ satisfies (4.1), that both $a^+(\lambda)$ and $a^-(\lambda)$ satisfy either (4.5) or (4.6), and that $A > 0$. Then there exists $\{\tau_{Ti}\}$ such that $\{t_{Ti} - \tau_{Ti};\ T \geqq 1\}$ is tight for each $i \geqq 0$ and such that the marked point process η_T of separated exceedances of $xT^{1/\alpha}$ by $X(t) = \int a(\lambda - t)\,dZ(\lambda)$ converges in distribution to η, where the distribution of η is given by (5.1).*

PROOF. Without loss of generality we assume that $a(0) = A$. First suppose that $0 < \alpha < 1$. Recall the definitions of $X_k(t)$ and a_{ki} from Lemma 4.2 and put $\bar{X}_k(t) = X_k(2^{-k}[2^k t])$ so that $\bar{X}_k(i2^{-k}) = X_n(i2^{-k})$ and $\bar{X}_k(t)$ is constant for $t \in [i2^{-k}, (i+1)2^{-k})$. We have

$$
\begin{aligned}
P(\sup_{0\leqq t\leqq T} |X(t) - \bar{X}_k(t)| > T^{1/\alpha}x) \\
&\leqq P(\sup_{0\leqq t\leqq T} |X(t) - X_k(t)| > T^{1/\alpha}x/2) \\
&\quad + P(\sup_{0\leqq t\leqq T} |X_k(t) - \bar{X}_k(t)| > T^{1/\alpha}x/2) \\
&\leqq 2K_\alpha' \textstyle\sum_{i=-\infty}^\infty A_{ki}^\alpha (T+1)T^{-1}x^{-\alpha}2^\alpha \\
&\quad + K_\alpha \textstyle\sum_{i=-\infty}^\infty A_{ki}^\alpha (T+1)T^{-1}x^{-\alpha}2^\alpha
\end{aligned}
\tag{5.2}
$$

by Lemma (4.2) applied to both $a^+(\lambda)$ and $a^-(\lambda)$ and by Lemma (4.1). From (4.5) it follows that $\sum_{i=-\infty}^\infty A_{ki}^\alpha \to 0$ when $k \to \infty$ and hence

$$\lim_{k\to\infty} \limsup_{T\to\infty} P(\sup_{0\leqq t\leqq T} |X(t) - \bar{X}_k(t)| > T^{1/\alpha}x) = 0\,. \tag{5.3}$$

Now, let $\zeta_T^k = (N_T^k, Y_{T1}^k, Y_{T2}^k, \cdots)$ be the marked point process of $h(T)$-separated upcrossings of $xT^{1/\alpha}$ by the discrete process $\{\bar{X}_k(i2^{-k})\}_{i=-\infty}^{\infty}$, where $Y_{Ti}^k(t) = \bar{X}_k(t2^{-k} + \tau_{Ti}^k)$, $t = 0, \pm 1, \cdots$, with τ_{Ti}^k the time of the ith exceedance of $xT^{1/\alpha}/A$ by the sequence $\{Z(i2^{-k}) - Z((i-1)2^{-k})\}_{i=0}^{\infty}$. According to Lemma 3.1 ζ_T^k converges in distribution, to ζ^k say, as $T \to \infty$. Furthermore, if η_T^k is the marked point process of upcrossings of $xT^{1/\alpha}$ by the continuous-time process $\{\bar{X}_k(t);\, t \in R\}$, with the marks centered at the τ_{Ti}^k's, then the function $f\colon \mathscr{N} \times R^\infty \times R^\infty \times \cdots \to \mathscr{N} \times D(-\infty, \infty) \times D(-\infty, \infty) \times \cdots$ that maps ζ_T^k into η_T^k is continuous and hence $\eta_T^k \to_d f(\zeta^k) = \eta^k$, say. It is easily seen that the distribution of η^k is obtained from (5.1) by replacing $a(\lambda)$ with $a_k(\lambda) = \sum_{i=-k2^k}^{k2^k-1} a_{ki} I(i2^{-k} < \lambda \leqq (i+1)2^{-k})$. Thus $\eta^k = (N^k, Y_1^k, Y_2^k, \cdots)$ has independent components, and if $\eta = (N, Y_1, Y_2, \cdots)$ has the distribution given by (5.1) then N^k has the same distribution as N and $Y_i^k \to_d Y_i$, since $\sup_{\lambda \in R} |a(\lambda) - a_k(\lambda)| \to 0$ by (4.5). By Lemma 2.1 it follows that $\eta^k \to_d \eta$ as $n \to \infty$.

The appropriate centering for the ith mark, τ_{Ti}, of η_T is the time of the ith jump larger than $T^{1/\alpha}x/A$ in the process $\{Z(t);\, t \geqq 0\}$. To show this we first prove

$$(5.4) \qquad \lim_{T\to\infty} P(|\tau_{Ti} - \tau_{Ti}^k| > 2^{-k}) = 0\,.$$

To do this, let $\varepsilon \in (0, x/(2A))$ and write $Z(t) = Z^1(t) + Z^2(t) + Z^3(t) + Z^4(t)$, where $Z^1(t)$ is the sum of jumps by $Z(t)$ in $[0, t]$ of size larger than $T^{1/\alpha}(x/A + \varepsilon)$, where $Z^2(t)$ is the sum of jumps of size belonging to $T^{1/\alpha}(x/A - \varepsilon, x/A + \varepsilon]$, and where $Z^3(t)$ is the sum of jumps of size belonging to $T^{1/\alpha}(\varepsilon/n, x/A - \varepsilon]$, $(n > 0)$. Further, for $l = 1, 2, 3$, let E^l be the point process which has its atoms at the times of jumps of $Z^l(t)$. We recall that, putting $Z_j = Z(j2^{-k}) - Z((j-1)2^{-k})$, τ_{Ti}^k is equal to 2^{-k} times the location of the ith exceedance of $T^{1/\alpha}x/A$ by the sequence $\{Z_j\}_{j=1}^{\infty}$. Put $Z_j^l = Z^l(j2^{-k}) - Z^l((j-1)2^{-k})$ so that $Z_j = \sum_{l=1}^4 Z_j^l$, let N be a positive number and denote the event that $E^1([0, TN]) < i$ by A_T, the event that $E^2([0, TN]) > 0$ by B_T, the event that $E^2(2^{-k}(j-1, j]) > 1$, for some $j \in [1, 2^kNT]$, by C_T, and the event that $E^3(2^{-k}(j-1, j]) > 1$, for some $j \in [1, 2^kNT]$, by D_T. If $|\tau_{Ti} - \tau_{Ti}^k| > 2^{-k}$ and $A_T^* \cap B_T^*$ occurs, then at least one of the following three events must happen: either $Z_j > T^{1/\alpha}x/A$ and $E^1(2^{-k}(j-1, j]) = 0$, for some $j \in [1, 2^kNT]$, or $Z_j \leqq T^{1/\alpha}x/A$ and $E^1(2^{-k}(j-1, j]) > 0$, for some $j \in [1, 2^kNT]$, or else C_T occurs. Moreover, if $B_T^* \cap C_T^* \cap D_T^*$ occurs, then the first two events both imply that $E_T = \{|Z_j^4| > T^{1/\alpha}\varepsilon$, some $j \in [1, 2^kNT]\}$ happens. Thus we have proved

$$\{|\tau_{Ti} - \tau_{Ti}^k| > 2^{-k}\} \subset A_T \cup B_T \cup C_T \cup D_T \cup E_T\,.$$

Let $\mu_1 = c_\alpha \alpha^{-1}(x/A + \varepsilon)^{-\alpha}$, $\mu_2 = c_\alpha \alpha^{-1}\{(x/A - \varepsilon)^{-\alpha} - (x/A + \varepsilon)^{-\alpha}\}$ and $\mu_3 = c_\alpha \alpha^{-1}\{(\varepsilon/n)^{-\alpha} - (x/A - \varepsilon)^{-\alpha}\}$. Then, for $l = 1, 2, 3$, E^l is a Poisson process with intensity μ_l/T (see, e.g., Breiman (1968)) and thus

$$P(A_T) = P(E^1([0, TN]) < i) = \sum_{j=0}^{i-1} e^{-N\mu_1} \frac{(N\mu_1)^j}{j!}$$

and

$$P(B_T) = P(E^2([0, TN]) \geqq 1) = 1 - e^{-N\mu_2}\,.$$

Furthermore,

$$\begin{aligned} P(C_T) &\leqq 2^k NTP(E^1([0, 2^{-k}]) > 1) \\ &= 2^k NT(1 - e^{-2^{-k}\mu_1 T^{-1}} - 2^{-k}\mu_1 T^{-1} e^{-2^{-k}\mu_1 T^{-1}}) \\ &\to 0 , \end{aligned}$$

as $T \to \infty$, and similarly $P(D_T) \to 0$. Further, by differentiating the Lévy representation of the characteristic function of Z_1^4 (cf. the proof of Lemma 4.1) it is seen that $E((Z_1^4)^2) \leqq K2^{-k}(\varepsilon/n)^{2-\alpha} T^{2/\alpha - 1}$, for some constant K, and then Chebychev's inequality gives

$$\begin{aligned} P(E_T) &\leqq 2^k NTP(|z_1^4| > T^{1/\alpha}\varepsilon) \\ &\leqq 2^k NTE(|z_1^4|^2) T^{-2/\alpha}\varepsilon^{-2} \\ &\leqq KN\varepsilon^{-\alpha} n^{2-\alpha} , \end{aligned}$$

as $T \to \infty$. Hence

$$\begin{aligned} &\limsup_{T\to\infty} P(|\tau_{Ti} - \tau_{Ti}^k| > 2^{-k}) \\ &\qquad \leqq \sum_{j=0}^{i-1} e^{-N\mu_1} \frac{(N\mu_1)^j}{j!} + 1 - e^{-N\mu_2} + KN\varepsilon^{-\alpha} n^{2-\alpha} , \end{aligned}$$

and inserting the values of μ_1 and μ_2 and letting first $n \to \infty$, then $\varepsilon \to 0$, and then $N \to \infty$, this proves (5.4).

Now we are in a position to show that $\{t_{Ti} - \tau_{Ti}; T \geqq 1\}$ is tight, or equivalently to prove

$$(5.5) \qquad \lim_{y\to\infty} \limsup_{T\to\infty} P(|t_{Ti} - \tau_{Ti}| > y) = 0 .$$

Let $\{\underline{t}_{Ti}^k\}$ be the $h(T)$-separated upcrossings of $T^{1/\alpha}(x - A\varepsilon)$ by $\bar{X}_k(t)$ and let $\{\bar{t}_{Ti}^k\}$ be the $h(T)$-separated upcrossings of $T^{1/\alpha}(x + A\varepsilon)$ by $\bar{X}_k(t)$. Further, let $\underline{\tau}_{Ti}^k$ be the location of the ith exceedance of $T^{1/\alpha}(x/A - \varepsilon)$ and $\bar{\tau}_{Ti}^k$ the location of the ith exceedance of $T^{1/\alpha}(x/A + \varepsilon)$ by the discrete process $\{Z(j2^{-k}) - Z((j-1)2^{-k})\}_{j=1}^{\infty}$, write F_T for the event that $\sup_{0 \leqq t \leqq NT} |X(t) - X_k(t)| > T^{1/\alpha} A\varepsilon$ and, changing the notation slightly, let $A_T = \{\bar{t}_{Ti}^k < NT - y\}$. On the event $A_T^* \cap F_T^*$ we have $\underline{t}_{Ti}^k \leqq t_{Ti} \leqq \bar{t}_{Ti}^k$ and thus $\{|t_{Ti} - \tau_{Ti}| > y\} \subset \{\bar{t}_{Ti}^k - \tau_{Ti} > y\} \cup \{\bar{\tau}_{Ti} - \underline{t}_{Ti}^k > y\} \cup A_T \cup F_T$. Let G_T be the event that $z_j \in T^{1/\alpha}(x/A - \varepsilon, x/A + \varepsilon]$ for some $j \in [1, 2^k NT]$. If $A_T^* \cap G_T^* \cap \{|\bar{\tau}_{Ti}^k - \bar{\tau}_{Ti}| \leqq y\}$ occurs, then $\underline{\tau}_{Ti}^k = \tau_{Ti}^k = \bar{\tau}_{Ti}^k$ and thus

$$\begin{aligned} &\{|t_{Ti} - \tau_{Ti}| > y\} \\ &\quad \subset \{\bar{t}_{Ti}^k - \tau_{Ti}^k > y - 2^{-k}\} \cup \{\tau_{Ti}^k - \underline{t}_{Ti}^k > y - 2^{-k}\} \cup \{|\tau_{Ti} - \tau_{Ti}^k| > 2^{-k}\} \\ &\qquad \cup A_T \cup F_T \\ &\quad \subset \{\bar{t}_{Ti} - \bar{\tau}_{Ti}^k > y - 2^{-k}\} \cup \{\underline{\tau}_{Ti}^k - \underline{t}_{Ti}^k > y - 2^{-k}\} \cup \{|\tau_{Ti} - \tau_{Ti}^k| > 2^{-k}\} \\ &\qquad \cup A_T \cup F_T \cup G_T . \end{aligned}$$

Here $P(A_T) \to \sum_{j=0}^{i-1} e^{-N\mu_1}((N\mu_1)^j/j!)$, $P(|\tau_{Ti} - \tau_{Ti}^k| > 2^{-k}) \to 0$ as $T \to \infty$, and $P(G_T) \leqq 2^k NTP(Z_1 \in T^{1/\alpha}(x/A - \varepsilon, x/A + \varepsilon]) \sim 2Nc_\alpha\{(x/A - \varepsilon)^{-\alpha} - (x/A + \varepsilon)^{-\alpha}\}$.

Hence

$$\begin{aligned}\lim_{y\to\infty} \limsup_{T\to\infty} P(|t_{Ti} - \tau_{Ti}| > y) &\leqq \lim_{y\to\infty} \limsup_{T\to\infty} \{P(\bar{t}^k_{Ti} - \bar{\tau}^k_{Ti} > y - 2^{-k}) + P(\underline{\tau}^k_{Ti} - \underline{t}^k_{Ti} > y - 2^{-k})\} \\ &\quad + \sum_{j=0}^{i-1} e^{-N\mu_1} \frac{(N\mu_1)^j}{j!} + \lim_{y\to\infty} \limsup_{T\to\infty} P(F_T) \\ &\quad + 2Nc_\alpha\{(x/A - \varepsilon)^{-\alpha} - (x/A + \varepsilon)^{-\alpha}\},\end{aligned}$$

and the first term is zero by Lemma 4.1, the third term tends to zero as $k \to \infty$ by (5.3) and the remaining two terms tend to zero as first $\varepsilon \to 0$ and then $N \to \infty$, and thus (5.5) follows.

To complete the proof that $\eta_T \to_d \eta$ it is enough to show that the conditions of Lemmas 2.3 and 2.4 are satisfied. However, the atoms of N_T are $t_{T1}/T, t_{T2}/T, \cdots$ and the atoms of $N_T{}^k$ are $t^k_{T1}/T, t^k_{T2}/T, \cdots$, and $(t_{Ti} - t^k_{Ti})/T \to_p 0$ follows from (5.4) and the facts that $\{t_{Ti} - \tau_{Ti}; T \geqq 1\}$ and $\{t^k_{Ti} - \tau^k_{Ti}; T \geqq 1\}$ are tight, so the hypothesis of Lemma 2.4 is satisfied. Further, let $\varepsilon_T{}^k = \tau_{Ti} - \tau^k_{Ti}$. Then, by (5.4), $\lim_{k\to\infty} \limsup_{T\to\infty} P(|\varepsilon_T{}^k| > x) = 0$ for $x > 0$. Since $Y^k_{Ti}(t) = \underline{X}_k(t + \tau^k_{Ti})/T^{1/\alpha}$ we have $\{\sup_{-l\leq t\leq l} |Y_{Ti}(t) - Y^k_{Ti}(t + \varepsilon_T{}^k)| > x\} = \{\sup_{-l\leq t\leq l} |X(t + \tau_{Ti}) - \underline{X}_k(t + \tau_{Ti})| > T^{1/\alpha}x\} \subset \{\sup_{-l\leq t\leq NT+l} |X(t) - \underline{X}_k(t)| > T^{1/\alpha}x\} \cup A_T$ and from (5.3) it then follows that

$$\begin{aligned}\lim_{k\to\infty} \limsup_{T\to\infty} P(\sup_{-l\leq t\leq l} |Y_{Ti}(t) - Y^k_{Ti}(t + \varepsilon_{Tk})| > x) &\leqq \lim_{k\to\infty} \limsup_{T\to\infty} P(A_T) \\ &= \sum_{j=0}^{i-1} e^{-N\mu_1} \frac{(N\mu_1)^j}{j!},\end{aligned}$$

and since N is arbitrary it follows that (2.3) holds. Further,

$$\lim_{u\to\infty} \limsup_{T\to\infty} P(\sup_{-l\leq t\leq l} |Y_{ni}(t)| > u) = 0$$

follows easily from Lemma 4.2, and thus the hypothesis of Lemma 2.3 is satisfied. This concludes the proof of the lemma for the case $0 < \alpha < 1$.

If instead $1 < \alpha < 2$ we have to use the second parts of Lemmas 4.1 and 4.2 instead of the first ones to prove (5.3), but apart from that, the lemma follows in precisely the same way as above also in this case. □

Since the restriction that $Z(\lambda)$ is completely asymmetric can be removed in precisely the same way as Theorem 3.2 is obtained from Lemma 3.1, we omit the details of this proof and only state the result.

Recall the notation $A = \sup_{\lambda\in R} a(\lambda)$, put $a = \sup_{\lambda\in R} a^-(\lambda)$, let $\mu' = c_\alpha A^\alpha(1 + \beta)x^{-\alpha}$, $\mu'' = c_\alpha a^\alpha(1 - \beta)x^{-\alpha}$, let Z' and Z'' have the distributions given on page 856 and let $Y_i(t) = Z'a(-t)$, $t \in R$, with probability $\mu'/(\mu' + \mu'')$ and $Y_i(t) = -Z''a(-t)$, $t \in R$, otherwise. Then the limiting distribution of η_T is that of the marked point process

(5.6) $(N, Y_1, Y_2, \cdots)$ where the components are independent, N is a Poisson process with intensity $\mu = \mu_1 + \mu_2$ and the Y_i's have the distribution given above.

THEOREM 5.2. *Let* $a(\lambda)$ *satisfy* (4.1) *and let both* $a^+(\lambda)$ *and* $a^-(\lambda)$ *satisfy either* (4.5) *or else* (4.6). *Further let* $\{Z(\lambda); \lambda \in R\}$ *have stationary independent increments with* $Z(0) = 0$ *and* $Z(1)$ *stable* $(1, \alpha, \beta)$ *and let* $X(t) = \int a(\lambda - t)\, dZ(\lambda)$. *Then there exist* $\{\tau_{Ti}; i = 1, 2, \cdots, T \geqq 1\}$ *with* $\{t_{Ti} - \tau_{Ti}; T \geqq 1\}$ *tight for each* $i \geqq 1$ *such that* $\eta_T \to_d \eta$ *as* $T \to \infty$, *where* η_T *is the marked point process of separated exceedances of* $T^{1/\alpha}x$ *by* $\{X(t); t \geqq 0\}$ *and where the distribution of* η *is given by* (5.6).

Similarly as for the discrete time case, various corollaries concerning the behavior of extremes can be deduced from Theorem 5.2. Here we only give the very simplest result, concerning $M_T = \sup_{0 \leqq t \leqq T} X(t)$.

COROLLARY 5.3. *Suppose that* $X(t)$ *satisfies the hypothesis of Theorem* 5.2. *Then*

$$P(M_T/T^{1/\alpha} \leqq x) \to \exp\{c_\alpha(A^\alpha(1 + \beta) + a^\alpha(1 - \beta))x^{-\alpha}\}$$

as $T \to \infty$.

We have not treated the case $\alpha = 1$ above. However, using methods rather similar to those for $1 < \alpha < 2$, conditions for the result of Theorem 5.2 to hold can be obtained also for $\alpha = 1$.

Finally we note that it is easy to see that all of the limit theorems of this paper are mixing in the sense of Rényi (the first result in this direction is proved in [12]). Hence they can be extended to cases where the level is random, and possibly depending on the process $X(t)$.

Acknowledgment. I am grateful to Stamatis Cambanis, Ross Leadbetter, and Georg Lindgren for some stimulating conversations on the subject of this paper. Further I want to thank the referee for some helpful comments.

REFERENCES

[1] BAUER, H. (1972). *Probability Theory*. De Gruyter, Berlin.
[2] BERGSTRÖM, H. (1953). Theorie der stabilen verteilungsfunktionen. *Arch. Math.* **4** 380–391.
[3] BILLINGSLEY, P. (1968). *Convergence of Probability Measures*. Wiley, New York.
[4] BREIMAN, L. (1968). *Probability*. Addison-Wesley, Reading, Mass.
[5] FELLER, W. (1971). *An Introduction to Probability Theory and Its Applications, Vol. II*. Wiley, New York.
[6] FRISTEDT, B. (1974). Sample functions of stochastic processes with stationary independent increments. In *Advances in Probability, Vol. III* (P. Ney, S. Port, eds.) 241–410. Dekker, New York.
[7] KALLENBERG, O. (1975). *Random measures*. Akademische Verlag, Berlin.
[8] LEADBETTER, M. R. (1976). Weak convergence of high level exceedances by a stationary sequence. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **34** 11–15.
[9] LEADBETTER, M. R., LINDGREN, G. and ROOTZÉN, H. (1978). Extremal and related properties of stationary processes. Unpublished.
[10] LÉVY, P. (1954). *Théorie de l'addition des variables aléatoires, 2nd ed.* Gauthier-Villars, Paris.
[11] LINDVALL, T. (1973). Weak convergence of probability measures and random functions in the function space $D[0, \infty)$. *J. Appl. Probability* **10** 109–121.
[12] ROOTZÉN, H. (1974). Some properties of convergence in distribution of sums and maxima of dependent random variables. Report 7, Dept. of Math. Statist., Univ. of Lund.

[13] SKOROKHOD, A. (1961). Asymptotic formulas for stable distribution laws. *In Selected Translations in Mathematical Statistics and Probability, Vol. I.* 157–161.

DEPARTMENT OF MATHEMATICAL STATISTICS
UNIVERSITY OF LUND AND LUND INSTITUTE OF TECHNOLOGY
BOX 725
S-220 07 LUND
SWEDEN