The Annals of Probability
1978, Vol. 6, No. 5, 771-787

ASYMMETRIC CAUCHY PROCESSES: SAMPLE FUNCTIONS AT LAST ZERO

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For asymmetric Cauchy processes an integral test is given for the sample function growth immediately after the process has been zero for the last time.

1. Introduction. $\{X_t, t \geqq 0\}$ be a one dimensional stable process of index α. We shall assume that X has right continuous paths with left limits. For each real number x, let $T_x = \inf\{t > 0 : X_t = x\}$ denote the first hitting time of $\{x\}$. Assume that 0 is regular for $\{0\}$, i.e., that $P^0\{T_0 = 0\} = 1$. Let us consider a time interval during which the path $t \to X_t(\omega)$ is not zero. (The interval covering the point $\{t_0\}$ for example.) We can then ask how the process behaves at the endpoints of this interval. By time reversal (see [16]) it follows that the way the process returns to zero is symmetric to the way it leaves zero. We need therefore only study the latter.

In [7] Itô and McKean describe the initial sample function growth of the Brownian motion (the case $\alpha = 2$) at the left-hand endpoints of its zero-free intervals. The case $1 < \alpha < 2$ has been analyzed by Millar in [10] and by the author in [12]. In this paper the remaining case: the asymmetric Cauchy processes ($\alpha = 1$) is studied. The approach is that of [10] and [12]. But the analysis is complicated by the fact that the asymmetric Cauchy processes are not strictly stable. We shall study the initial behavior of the process $Z_t = X_{L+t}$, $t \geqq 0$, where L denotes the last time that the asymmetric Cauchy process X is zero. (This process, unlike the strictly stable processes of index $\alpha > 1$, is transient.) A consequence of the zero-one law in Section 3, and the stationary independent increments of X is that X leaves zero in exactly the same way at each of the left-hand endpoints of its zero-free intervals.

In this paper it is shown that if X has both positive and negative jumps and f is a nonnegative decreasing function, then with probability 1

$$\limsup_{t\to 0} Z(t)/tf(t) = 0 \quad \text{or} \quad \infty$$

and

$$\liminf_{t\to 0} |Z(t)|/\exp(-f(t)) = \infty \quad \text{or} \quad 0$$

according as $\int_0^1 (tf(t))^{-1}\,dt < \infty$ or $= \infty$. If X has no positive jumps (say), then the process $\{Z(t), t > 0\}$ is positive for an initial period of time. Furthermore

$$\limsup_{t\to 0} Z(t)/t|\log(t)| = 2/\pi \quad \text{a.s.}$$

Received May 28, 1976; revised June 16, 1977.

AMS 1970 *subject classifications.* Primary 60G17, 60J30; Secondary 60J25, 60J55, 60J40.

Key words and phrases. Stochastic process, stationary independent increments, stable process, local time, zero-free intervals, sample function growth, last zero, rate of escape.

And if f is a nonnegative decreasing function, then with probability 1

$$\liminf_{t\to 0} Z(t)/tf(t) = \infty \quad \text{or} \quad 0$$

according as $\int_0^1 f(t)(t\log^2(t))^{-1}\,dt < \infty$ or $= \infty$.

2. The asymmetric Cauchy process. From now on let $\{X_t, t \geqq 0\}$ be a one dimensional Cauchy process, i.e., a stable process with stationary independent increments and

$$E^0\{\exp(i\theta X_t)\} = \exp\{-t\psi(\theta)\},$$

where $\psi(\theta) = |\theta| + ih\theta\log|\theta|$. The skew parameter h satisfies $|h| \leqq 2/\pi$. If $h = 0$, then the process is the usual symmetric Cauchy process for which one point sets are polar. We will henceforth assume that $h \neq 0$. If $h = 2/\pi$, then X takes only positive jumps, and if $h = -2/\pi$ only negative jumps. The transition density of the Cauchy process with parameter h is

$$p_t(x) = p(t, x, h) = (2\pi)^{-1}\int_{-\infty}^{\infty} e^{-i\theta x}e^{-t(|\theta|+ih\theta\log|\theta|)}\,d\theta .$$

We have the scaling property

$$p(t, x, h) = t^{-1}p(1, xt^{-1} - h\log(t), h) .$$

In the following asymptotic formulas it will simplify notation if we introduce the parameter $\beta = h\pi/2$. Note that $-1 \leqq \beta \leqq 1$. If $h > 0$, we have the representation

$$p(1, x, h) = \pi^{-1}\int_0^{\infty}\exp\{-ux - hu\log(u)\}\sin\{(1+\beta)u\}\,du$$

(see [14]). It therefore follows from a theorem on Laplace transforms that

$$p(1, x, h) \sim \pi^{-1}(1+\beta)x^{-2}$$
$$p_x'(1, x, h) \sim -2\pi^{-1}(1+\beta)x^{-3}$$

as $x \to \infty$. In [14] Skorokhod shows that as $x \to -\infty$,

$$p(1, x, 2/\pi) \sim (\pi\xi/8)^{\frac{1}{2}}e^{-\xi},$$

where $\xi = (2/\pi)\exp(-\pi x/2 - 1)$. It can also be shown that

$$p_x'(1, x, 2/\pi) \sim (\pi/2)\xi p(1, x, 2/\pi)$$

as $x \to -\infty$. For $0 < h < 2/\pi$,

$$\begin{aligned} p(1, x, h) &= \int p(1-\beta, x-y, 0)p(\beta, y, 2/\pi)\,dy \\ &= \pi^{-1}(1-\beta)\int((1-\beta)^2 + (x-y)^2)^{-1}p(\beta, y, 2/\pi)\,dy . \end{aligned}$$

It follows that

$$p(1, x, h) \sim \pi^{-1}(1-\beta)x^{-2}$$
$$p_x'(1, x, h) \sim -2\pi^{-1}(1-\beta)x^{-3}$$

as $x \to -\infty$. We finally note that since

$$p(t, -x, -h) = p(t, x, h) ,$$

this also takes care of the asymptotic behavior in the case $h < 0$. From now on we will let $p_t(x)$ denote the transition density. For $\lambda \geqq 0$ the λ-potential kernel is $u^\lambda(x) = \int_0^\infty e^{-\lambda t} p_t(x)\, dt$. Let L_t^x be a local time at x with $E^0 \int_0^\infty e^{-\lambda t}\, d_t L_t^x = u^\lambda(x)$. Special properties of the local times of the asymmetric Cauchy processes are in [5] and [11].

It is well known that $u(x) = u^0(x)$ is continuous. (See [13].) In fact, $u(x)$ is differentiable everywhere except at 0. We shall need estimates of $u(0) - u(x)$ and of $u'(x)$ for small x. In the sequel, whenever f is a complex valued function we shall let Re $\{f\}$ denote the real part of f and Im $\{f\}$ the imaginary part of f.

LEMMA 2.1.

$$u(0) - u(x) = (2\pi)^{-1} \int_{-\infty}^{\infty} \mathrm{Re}\, \{(1 - e^{-i\theta x})\psi(\theta)^{-1}\}\, d\theta\,.$$

PROOF. Since for any x

$$u(x) = \int_0^\infty p_t(x)\, dt = (2\pi)^{-1} \int_0^\infty dt \int_{-\infty}^{\infty} e^{-i\theta x} e^{-t\psi(\theta)}\, d\theta$$

we derive the identity in question simply by changing the order of integration. Justifying this step is not trivial, however. Note that $\mathrm{Im}\, \{\psi(\theta)^{-1}\} = -h \log |\theta|(\theta + \theta h^2 \log^2 |\theta|)^{-1}$ is not integrable. Therefore the integral

$$\int \mathrm{Im}\, (1 - e^{-i\theta x})\, \mathrm{Im}\, \{\psi(\theta)^{-1}\}\, d\theta$$

is defined as

$$\lim_{N\to\infty} \int_{-N}^{N} \sin\, (\theta x)\, \mathrm{Im}\, \{\psi(\theta)^{-1}\}\, d\theta\,.$$

The convergence is ensured by the fact that $|\mathrm{Im}\, \{\psi(\theta)^{-1}\}|$ is decreasing for $\theta > 3$ and goes to zero for $\theta \to \infty$.

For $0 < \varepsilon$

$$\int_\varepsilon^\infty dt \int_{-\infty}^{\infty} |(1 - e^{i\theta x}) e^{-t\psi(\theta)}|\, d\theta \leqq \int_\varepsilon^\infty dt \int_{-\infty}^{\infty} |\theta x| e^{-t|\theta|}\, d\theta < \infty\,.$$

Hence by Fubini

$$\int_\varepsilon^\infty (p_t(0) - p_t(x))\, dt = (2\pi)^{-1} \int_{-\infty}^{\infty} (1 - e^{-i\theta x})\psi(\theta)^{-1} e^{-\varepsilon\psi(\theta)}\, d\theta\,.$$

Since $\mathrm{Re}\, \{\psi(\theta)^{-1}\} = |\theta|^{-1}(1 + h^2 \log^2 |\theta|)^{-1}$ is integrable

$$\lim_{\varepsilon\to 0} \int_{-\infty}^{\infty} \mathrm{Re}\, \{(1 - e^{-i\theta x}) e^{-\varepsilon\psi(\theta)}\}\, \mathrm{Re}\, \{\psi(\theta)^{-1}\}\, d\theta$$
$$= \int_{-\infty}^{\infty} \mathrm{Re}\, (1 - e^{-i\theta x})\, \mathrm{Re}\, \{\psi(\theta)^{-1}\}\, d\theta$$

by dominated convergence. Unfortunately,

$$\mathrm{Im}\, \{\psi(\theta)^{-1}\} = -h \log |\theta|(\theta + \theta h^2 \log^2 |\theta|)^{-1}$$

is not integrable. So we can only conclude that

$$\lim_{\varepsilon\to 0} \int_{-N}^{N} \mathrm{Im}\, \{(1 - e^{-i\theta x}) e^{-\varepsilon\psi(\theta)}\}\, \mathrm{Im}\, \{\psi(\theta)^{-1}\}\, d\theta$$
$$= \int_{-N}^{N} \mathrm{Im}\, (1 - e^{-i\theta x})\, \mathrm{Im}\, \{\psi(\theta)^{-1}\}\, d\theta$$

for any large positive constant N. Fix N. To finish the proof we will show that

$$\limsup_{\varepsilon\to 0} \left|\int_{|\theta|>N} \mathrm{Im}\, \{(1 - e^{-i\theta x}) e^{-\varepsilon\psi(\theta)}\}\, \mathrm{Im}\, \{\psi(\theta)^{-1}\}\, d\theta\right| = O(N^{-1})\,.$$

First consider

$$\int_N^\infty \operatorname{Im}\{e^{-\varepsilon\psi(\theta)}\} \operatorname{Im}\{\psi(\theta)^{-1}\}\, d\theta = -\int_N^\infty e^{-\varepsilon\theta} \sin(\varepsilon \operatorname{Im}\psi(\theta)) \operatorname{Im}\{\psi(\theta)^{-1}\}\, d\theta .$$

Using the change of variable $z = \operatorname{Im}\psi(\theta) = h\theta\log(\theta)$ and arguing as in the proof of Corollary 2.2 below, we find that as $\varepsilon \to 0$, this integral is of the same order of magnitude as $\log^{-1}(\varepsilon^{-1})$. To evaluate

$$\int_N^\infty \operatorname{Im}\{e^{-i\theta x}e^{-\varepsilon\psi(\theta)}\} \operatorname{Im}\{\psi(\theta)^{-1}\}\, d\theta$$
$$= -\int_N^\infty e^{-\varepsilon\theta} \sin(x\theta + \varepsilon \operatorname{Im}\psi(\theta)) \operatorname{Im}\{\psi(\theta)^{-1}\}\, d\theta ,$$

we write

$$\int_N^\infty = \int_N^{\varepsilon^{-1}} + \int_{\varepsilon^{-1}}^\infty = I_1 + I_2 .$$

By dominating the integrand it is easy to see that $I_2 \to 0$ as $\varepsilon \to 0$. Finally, using the substitution $z = x\theta + \varepsilon \operatorname{Im}\{\psi(\theta)\}$ we see that $\limsup_{\varepsilon\to 0} |I_1| = O(N^{-1})$. This completes the proof.

COROLLARY 2.2. *As* $x \to 0$

$$\log(|x|^{-1})(u(0) - u(x)) = (\pi h^2)^{-1} + \operatorname{sgn}(x)(2h)^{-1} + o(1)$$
$$|x| \log^2(|x|)u'(x) = -\operatorname{sgn}(x)(\pi h^2)^{-1} - (2h)^{-1} + o(1) .$$

PROOF.

$$u(0) - u(x) = \frac{1}{2\pi} \int \frac{1 - \cos(\theta x)}{|\theta|(1 + h^2\log^2|\theta|)}\, d\theta + \frac{h}{2\pi} \int \frac{\sin(\theta x)\log|\theta|}{\theta + \theta h^2 \log^2|\theta|}\, d\theta .$$

As $x \to 0$ the first term equals $(\pi h^2 \log(|x|^{-1}))^{-1}(1 + o(1))$ and the second term $\operatorname{sgn}(x)(2h\log(|x|^{-1}))^{-1}(1 + o(1))$. Let us prove the second assertion. Assume that $x > 0$. Let $0 < \varepsilon < N$ and write

$$\int_0^\infty \frac{\sin(\theta x)\log(\theta)}{\theta + \theta h^2\log^2(\theta)}\, d\theta = \int_0^{\varepsilon x^{-1}} + \int_{\varepsilon x^{-1}}^{Nx^{-1}} + \int_{Nx^{-1}}^\infty = I_1 + I_2 + I_3 .$$

As $x \to 0+$, $|I_1| = O(\varepsilon \log^{-1}(x^{-1}))$ and $|I_3| = O(N^{-1}\log^{-1}(x^{-1}))$. Finally, by dominated convergence,

$$\lim_{x\to 0+} \log(x^{-1}) I_2 = \lim_{x\to 0+} \log(x^{-1}) \int_\varepsilon^N \frac{\sin(\theta)\log(\theta x^{-1})}{\theta + \theta h^2 \log^2(\theta x^{-1})}\, d\theta$$
$$= h^{-2} \int_\varepsilon^N \frac{\sin(\theta)}{\theta}\, d\theta .$$

The estimate for $u'(x)$ is obtained in the same manner once we have convinced ourselves that we can compute $u'(x)$ by differentiating the integrands. We can now get estimates for the probability of hitting the point $\{y\}$ before the point $\{0\}$ in the case $|h| \neq 2/\pi$. The estimates for the case $|h| = 2/\pi$ can be found in Corollary 6.3.

COROLLARY 2.3. *Assume that* $|h| < 2/\pi$. *Then there exist constants* $0 < c < C$ *such that for all small enough* $y \neq 0$,

$$c\log|y|/\log|x| \leq P^x\{T_y < T_0\} \leq C\log|y|/\log|x| \qquad \text{if} \quad |x| \leq |y|$$
$$c/\log(|x| + 2) \leq P^x\{T_y < T_0\} \leq C/\log(|x| + 2) \qquad \text{if} \quad |y| < |x| .$$

PROOF. According to Getoor [4]

$$P^x\{T_y < T_0\} = \frac{u(0)u(y-x) - u(-x)u(y)}{u(0)^2 - u(y)u(-y)} .$$

If we write $G(x) = u(0) - u(x)$, then

$$P^x\{T_y < T_0\} \sim \frac{G(y) + G(-x) - G(y-x)}{G(y) + G(-y)}$$

as $x, y \to 0$. For $|y| < |x| < \varepsilon$,

$$G(-x) - G(y-x) \geqq -2|y|\,|u'(-x)| \geqq -\tfrac{1}{2}G(y)$$

by virtue of Corollary 2.2. This implies that

$$0 < c \leqq P^x\{T_y < T_0\} \leqq 1$$

for a suitable positive constant c. For $y^2 < |x| \leqq |y| < \varepsilon$,

$$-2(h \log|x|)^{-2} \leqq -2|x|\,|u'(y)| \leqq G(y) - G(y-x) \leqq G(y) .$$

For $|x| \leqq y^2 < \varepsilon^2$,

$$|G(y) - G(y-x)| \leqq 2|x|\,|u'(y)| \leqq (h \log|x|)^{-2} .$$

The estimates for $P^x\{T_y < T_0\}$ therefore follow from Corollary 2.2. Finally, if $|y| < \varepsilon \leqq |x|$, then $P^x\{T_y < T_0\}$ is proportional to $u(-x)$. And according to Proposition 2 of [13], $u(-x)$ is proportional to $\log^{-1}|x|$ as $|x| \to \infty$. This completes the proof.

According to Blumenthal and Getoor [1]

$$P^x\{T_0 \leqq t\} = \int_0^t H(t-s)p_s(-x)\,ds ,$$

where H is a positive, differentiable, and decreasing function on $(0, \infty)$ with Laplace transform $(\lambda u^\lambda(0))^{-\lambda}$. It follows by a Tauberian theorem that

$$\lim_{t\to 0} H(t)/\log(t^{-1}) = 4\beta^2/\pi(1 + |\beta|) ,$$

where $\beta = h\pi/2$. Furthermore, $H(t) \to u(0)^{-1}$ as $t \to \infty$.

LEMMA 2.4.

$$P^x\{T_0 > t\} = (u(0) - u(-x))H(t)(1 + e(x, t)) .$$

If $|h| < 2/\pi$, then $e(x, t)$ is a bounded function that goes to zero uniformly in x and t as $x/t \to 0$. If $|h| = 2/\pi$, then the same is true provided we only consider x of the opposite sign of h.

PROOF.

$$P^x\{T_0 > t\} = \int_0^t H(t-s)(p_s(0) - p_s(-x))\,ds .$$

We will first show that for small x,

$$\int_0^\infty |p_s(0) - p_s(-x)|\,ds = O(\log^{-1}(|x|^{-1})) .$$

Since $|p_s(0) - p_s(-x)| = 2(p_s(0) - p_s(-x))^+ - (p_s(0) - p_s(-x))$, it is sufficient to show that

$$\int_0^\infty (p_s(0) - p_s(-x))^+ \, ds = O(\log^{-1}(|x|^{-1}))$$

by virtue of Corollary 2.2.

$$\int_0^\infty (p_s(0) - p_s(-x))^+ \, ds \leqq \int_0^{|x|} p_s(0) \, ds + \int_{|x|}^\infty |p_s(0) - p_s(-x)| \, ds$$
$$= I_1 + I_2 \, .$$

From the scaling relation $p_s(y) = s^{-1}p_1(ys^{-1} - h \log(s))$ it follows that

$$I_1 \leqq C \int_0^{|x|} s^{-1} \log^{-2}(s) \, ds = C \log^{-1}(|x|^{-1}) \, .$$

Furthermore, for $s > |x|$

$$|p_s(0) - p_s(-x)| = |xs^{-2}p_1'(-\tilde{x}s^{-1} - h \log(s))|$$
$$= O(|x|s^{-2}(1 + |\log^3(s)|)^{-1}) \, .$$

Hence for small x, $I_2 = O(\log^{-3}(|x|^{-1}))$. This proves the assertion. We now proceed with the proof of Lemma 2.4.

$$P^x\{T_0 > t\} = H(t)(u(0) - u(-x)) - H(t) \int_t^\infty (p_s(0) - p_s(-x)) \, ds$$
$$+ \int_0^t (H(t - s) - H(t))(p_s(0) - p_s(-x)) \, ds \, .$$

We have to show that the two error terms get smaller and smaller compared to the leading term $H(t)(u(0) - u(-x))$ as $x/t \to 0$. This is true for the first error term which is $O(H(t)|x|/t(1 + |\log^3(t)|))$ for $|x| < t$. To evaluate the second error term we write

$$\int_0^t = \int_0^{|x|} + \int_{|x|}^{t/2} + \int_{t/2}^t = I_1 + I_2 + I_3 \, .$$

$$|I_1| \leqq (H(t - |x|) - H(t)) \int_0^\infty |p_s(0) - p_s(-x)| \, ds \, .$$

Here, $(H(t - |x|) - H(t))/H(t)$ is bounded for $2|x| < t$ and goes to zero uniformly in x and t as $x/t \to 0$. This, together with the fact that

$$\int_0^\infty |p_s(0) - p_s(-x)| \, ds \leqq C(u(0) - u(-x)) \, ,$$

shows that $I_1/H(t)(u(0) - u(-x)) \to 0$ as $x/t \to 0$.

$$|I_2| \leqq (H(t/2) - H(t)) \int_{|x|}^\infty |p_s(0) - p_s(-x)| \, ds$$
$$= O((H(t/2) - H(t))/(1 + |\log^3(|x|)|)) \, .$$

$(H(t/2) - H(t))/H(t)$ is a bounded function that goes to zero as $t \to \infty$. It follows that $I_2/H(t)(u(0) - u(-x)) \to 0$ as $x/t \to 0$. Finally,

$$|I_3| \leqq \int_{t/2}^t H(t - s)|p_s(0) - p_s(-x)| \, ds$$
$$= O(|x|H(t)/t(1 + |\log^3(t)|)) \, .$$

This completes the proof of Lemma 2.4.

REMARKS. This proof does not work for small positive x in the case $h = 2/\pi$, or for small negative x in the case $h = -2/\pi$. The reason is that $u(0) - u(-x)$

is of the magnitude $|x|$ (see Lemma 6.2) whereas $\int |p_s(0) - p_s(-x)|\, ds$ is much larger.

Since we do not have a scaling property for $P^x\{T_0 \leqq t\}$ we shall need

LEMMA 2.5. *For any $\varepsilon > 0$ there exists a constant $C > 0$ such that for t small enough and $x(ht \log(t))^{-1} > \varepsilon > 0$ we have*

$$P^x\{T_0 \leqq t\} \leqq C \log^{-1}(t^{-1}) .$$

PROOF. For $s \leqq t$

$$p_s(-x) = s^{-1}p_1(-xs^{-1} - h \log(s)) \leqq s^{-1}p_1(-xs^{-1}) \leqq Csx^{-2} .$$

So $P^x\{T_0 \leqq t\} \leqq Cx^{-2}t^2 \log(t^{-1})$. This completes the proof.

From now on let Q_t denote the probability distribution with density

$$q_t(y) = p_t(y) - H(t)^{-1} \int_0^t (p_t(y) - p_{t-s}(y))H'(s)\, ds .$$

To see that this is indeed a probability density we will show that

$$-\int_0^t H'(s)\, ds \int_{-\infty}^{\infty} |p_t(y) - p_{t-s}(y)|\, dy < \infty .$$

(Remember that $H(s)$ is a decreasing function.)

$$-\int_0^t = -\int_0^{t/2} - \int_{t/2}^t = I_1 + I_2 .$$

From the scaling property $p_t(y) = t^{-1}p_1(yt^{-1} - h \log(t))$ and the asymptotic formulas for p_1 and p_1' in the beginning of this section it follows that if $0 \leqq s \leqq t/2$, then

$$|p_t(y) - p_{t-s}(y)| \leqq Cs(1 + y^2)^{-1} ,$$

where the constant C depends on t. Hence

$$I_1 \leqq -\pi C \int_0^{t/2} sH'(s)\, ds \leqq \pi C \int_0^{t/2} H(s)\, ds < \infty .$$

Finally, since $|p_t(y) - p_{t-s}(y)| \leqq p_t(y) + p_{t-s}(y)$,

$$I_2 \leqq -2 \int_{t/2}^t H'(s)\, ds = 2(H(t/2) - H(t)) < \infty .$$

LEMMA 2.6. *Let f be a bounded Borel function and fix $t_0 > 0$. If $|h| < 2/\pi$, then*

$$\lim_{t\to t_0, x\to 0} E^x f(X_t)I\{T_0 > t\}/P^x\{T_0 > t\} = Q_{t_0}(f) .$$

The same is true if $|h| = 2/\pi$, provided we only consider x of the opposite sign of h.

PROOF. For every fixed $s > 0$,

$$\lim_{x\to 0} P^x\{T_0 > s\}/P^x\{T_0 > t_0\} = H(s)/H(t_0)$$

by virtue of Lemma 2.4. It follows that if $g(s, t)$ is a continuous function on R_+^2 that vanishes for $0 < s \leqq \varepsilon$, then

$$\lim_{t\to t_0, x\to 0} \int_0^t g(s, t)P^x\{T_0 \in ds\}/P^x\{T_0 > t\} = -H(t_0)^{-1} \int_0^{t_0} g(s, t_0)H'(s)\, ds . \tag{2.7}$$

Furthermore, if $0 < \delta < \frac{1}{2}$ and x is small, then

$$\int_0^\delta sP^x\{T_0 \in ds\} \leqq \int_0^\delta P^x\{T_0 > s\}\, ds \leqq CP^x\{T_0 > t\} \int_0^\delta \log(s^{-1})\, ds ,$$

where C does not depend on δ, t, or x. It follows that (2.7) continues to hold if $g(s, t)$ is any continuous function on R_+^2 for which

$$\limsup_{t \to t_0} \sup_{0<s<t} |g(s, t)|/s < \infty .$$

In particular, if $y \neq 0$, then (2.7) holds for

$$g(s, t) = p_t(y) - p_{t-s}(y) .$$

Therefore the density of the probability measure

$$P^x\{X_t \in dy, T_0 > t\}/P^x\{T_0 > t\}$$

which for small x is approximately

$$p_t(y) + \int_0^t (p_t(y) - p_{t-s}(y))P^x\{T_0 \in ds\}/P^x\{T_0 > t\}$$

converges to $q_{t_0}(y)$ as $t \to t_0$ and $x \to 0$. The proof is identical to Millar's proof of Lemma 4.5 in [10]. We will omit the details.

3. The last exit process. We shall now describe the process Z given by the path of X from its last zero. Let

$$g(x) = P^x\{T_0 = \infty\} = 1 - u(-x)/u(0)$$
$$L = \sup\{s > 0 : X(s) = 0\} .$$

Then $P^0\{0 < L < \infty\} = 1$. For $t \geqq 0$ put $Z(t) = X(L + t)$, and consider the σ-fields

$$\mathscr{F}_t = \bigcap_{s>t} \sigma\{Z(u) : 0 \leqq u \leqq s\} , \qquad t \geqq 0 .$$

According to [8], $\{Z_t, \mathscr{F}_t, t > 0\}$ is a strong Markov process with transition functions

$$H_t(x, f) = E^x[(fg)(X_t)I\{t < T_0\}]/g(x) .$$

Put $H_t(0, f) = E^0 f(Z_t)$ and let $\{P_L{}^x\}$ denote the usual family of measures associated with the transition functions $\{H_t\}$. The potential operator has the form

$$U_L f(x) = \int_0^\infty H_t(x, f)\, dt = \int f(y) u_L(x, y)\, dy ,$$

where $u_L(x, y) = E^x L_{T_0}^y g(y)/g(x)$. It is well known that

$$\lim_{t \to 0} Z(t) = 0 ,$$

because a Lévy process that is not compound Poisson never leaves zero in one big jump. (See Section 2 of [10].)

By time reversal and Corollary 3.5 of [9] it follows that if X has both positive and negative jumps, then Z immediately assumes both positive and negative values, jumping across zero an infinite number of times. If X has no positive jumps (say), then Z is nonnegative for an initial period of time.

The following result identifies the entrance law of Z.

LEMMA 3.1. *For each bounded continuous function f and each $t > 0$*

$$E^0 f(Z_t) = u(0)H(t)Q_t(fg) .$$

PROOF. For $x \neq 0$

$$H_s(x, f) = \frac{E^x[(fg)(X(s))I\{s < T_0\}]}{P^x\{s < T_0\}} \frac{P^x\{s < T_0\}}{P^x\{T_0 = \infty\}} .$$

Under the restrictions of Lemma 2.6, it follows from Lemma 2.4 and Lemma 2.6 that

$$\lim_{s\to t, x\to 0} H_s(x, f) = u(0)H(t)Q_t(fg) .$$

If $|h| = 2/\pi$, then Z_s has the opposite sign of h for all sufficiently small s. So by dominated convergence and the Markov property

$$E^0 f(Z_t) = \lim_{s\to 0} E^0\{H_{t-s}(Z_s, f)\} = \lim_{s\to 0} H_{t-s}(Z_s, f)$$
$$= u(0)H(t)Q_t(fg)$$

for all asymmetric Cauchy processes.

COROLLARY 3.2. *For $A \in \mathscr{F}_0$ we either have $P^0(A) = 0$ or $P^0(A) = 1$.*

PROOF. The zero-one law will follow from Proposition 5.17 of Chapter I in [2] once we have shown that $\{Z_t, \mathscr{F}_t\}$ is strong Markov not only for $t > 0$ (which we know from [8]) but for $t \geqq 0$. One way of proving this is to apply Theorem 8.11 of Chapter I in [2]. Because, as we have just seen, for each fixed $s > 0$ and every bounded continuous function f, the map $t \to H_s(Z_t, f)$ is right continuous not only for $t > 0$ (which we know from [8]) but also at $t = 0$.

4. Probability estimates. The estimates obtained in Lemmas 4.1 to 4.4 will be used to determine the upper envelope of the process $\{Z_t\}$ at 0. The estimates for the potential operator will be used to determine the lower envelope of the process $\{|Z_t|\}$ at 0. We shall need the following asymptotic formulas for the densities of the asymmetric Cauchy processes. (See Section 2.) If $h \neq -2/\pi$, then $p_1(y) \sim Ay^{-2}$ as $y \to \infty$. If $h = -2/\pi$, then $p_1(y) \sim A\exp(By - Ce^{Dy})$ as $y \to \infty$. In the following estimates the letters c and C will denote positive constants whose values are unimportant. We may change their values from line to line, even on the same line. We will assume that t is small.

LEMMA 4.1. *For $N > t$ we have*

$$\int_N^\infty p_t(y)\,dy < \int_N^\infty q_t(y)\,dy .$$

If $h \neq -2/\pi$, $\varepsilon_0 > 0$ is fixed, and $N > (1 + \varepsilon_0)t|h \log(t)|$, then

$$ctN^{-1} < \int_N^\infty q_t(y)\,dy < CtN^{-1} .$$

If $h = -2/\pi$ and $\varepsilon > \varepsilon_0 > 0$, then

$$\int_{(2/\pi+\varepsilon)t|\log(t)|}^\infty q_t(y)\,dy < \exp(-t^{-c\varepsilon}) .$$

PROOF. The first assertion follows from Fubini and the fact that for $s < t$

$$\int_N^\infty (p_t(y) - p_{t-s}(y))\,dy$$
$$= P^0\{Nt^{-1} - h\log(t) < X_1 < N(t-s)^{-1} - h\log(t-s)\} > 0 .$$

To get the upper bound in the case $h \neq -2/\pi$ we note that

$$\int_N^\infty (p_t(y) - p_{t-s}(y))\,dy < CsN^{-1}.$$

Hence

$$\int_0^t -H'(s)\,ds \int_N^\infty (p_t(y) - p_{t-s}(y))\,dy$$
$$< CN^{-1} \int_0^t -sH'(s)\,ds < CN^{-1} \int_0^t H(s)\,ds < CtN^{-1}H(t).$$

The case $h = -2/\pi$ is proved the same way.

Combining Lemma 3.1, Lemma 4.1, and the estimates for $H(t)$ and $P^s\{T_0 = \infty\}$ we get

LEMMA 4.2. *If* $|h| < 2/\pi$ *and* $t|\log(t)| < N$, *then*

$$c < P_L{}^0\{Z_t > N\}N \log(2 + N^{-1})/t|\log(t)| < C.$$

If $h = -2/\pi$, *then for fixed* $\varepsilon > 0$

$$P_L{}^0\{Z_t > (2/\pi + \varepsilon)t|\log(t)|\} < \exp(-t^{-\varepsilon}).$$

For small t the distribution of X_t is concentrated around $ht \log(t)$. The same is true for Z_t. If $h < 0$, then the first assertion in Lemma 4.3 follows from Lemma 4.1. By symmetry, the assertion is also true for $h > 0$.

LEMMA 4.3. *For* $k > 1$

$$\lim_{t\to 0} P_L{}^0\{k^{-1} < Z_t(ht \log(t))^{-1} < k\} = 1.$$

Fix $k > 1$. *If* $|h| < 2/\pi$, $k^{-1} < x(ht \log(t))^{-1} < k$, *and* $2kt|\log(t)| < N$, *then*

$$c < P_L{}^x\{Z_t > N\}N \log(2 + N^{-1})/t|\log(t)| < C.$$

PROOF. The last assertion is proved the same way that Lemma 3.4 in [12] is proved. By definition

$$P_L{}^x\{Z_t > N\} = \int_N^\infty P^x\{X_t \in dy,\, t < T_0\}P^y\{T_0 = \infty\}/P^x\{T_0 = \infty\}.$$

To evaluate this integral we replace $P^x\{X_t \in dy, t < T_0\}$ with $P^x\{X_t \in dy\}$. We can do this because $P^x\{X_t > y, T_0 \leqq t\} = O(|\log(t)|^{-1}P^x\{X_t > y\})$ by virtue of Lemma 2.5 and the first passage relation.

Put $Z_t{}^* = \sup\{Z_s: 0 < s \leqq t\}$.

LEMMA 4.4. *If* $h < 0$, *then*

$$P_L{}^0\{Z_t{}^* > N\} \leqq CP_L{}^0\{Z_t > N\}$$

for all N *and small* t. *If* $0 < h < 2/\pi$, *then the same conclusion holds for* $N > 2t|\log(t)|$.

PROOF. If $h < 0$ and $N < \frac{1}{2}ht \log(t)$, then $P_L{}^0\{Z_t > N\}$ is almost 1 by virtue of Lemma 4.3. Furthermore, arguing as in the proof of Lemma 4.3, we see that if $h < 0$, there exists a constant $C > 0$ such that

$$P_L{}^x\{Z_{t-s} > x\} \geqq C$$

for all $s < t$ and all $x \geqq \frac{1}{2}ht \log(t)$. By the first passage relation, this implies

$$P_L^0\{Z_t^* > N\} \leqq C^{-1}P_L^0\{Z_t > N\}$$

for all $N \geqq \frac{1}{2}ht \log(t)$. Next, if $0 < h < 2/\pi$ and $x \geqq N \geqq 2t|\log(t)|$, then

$$P_L^x\{Z_{t-s} \leqq \tfrac{1}{2}N\} \leqq CP_L^0\{Z_t > N\}$$

by Lemma 4.2 and an argument similar to the proof of Lemma 4.3. Hence

$$\begin{aligned} P_L^0\{Z_t^* > N\} &\leqq P_L^0\{Z_t > \tfrac{1}{2}N\} + P_L^0\{Z_t^* > N, Z_t \leqq \tfrac{1}{2}N\} \\ &\leqq CP_L^0\{Z_t > N\}. \end{aligned}$$

This completes the proof of Lemma 4.4.

REMARK. If $h = 2/\pi$, then $P_L^0\{Z_t^* > 0\} \to 0$ as $t \to 0$.

We shall now turn our attention to the potential operator. The Green's function $u_L(x, y)$ was introduced in Section 3. In this section we shall assume that $|h| < 2/\pi$. The case $|h| = 2/\pi$ is analyzed in Section 6.

LEMMA 4.5. *Assume that* $|h| < 2/\pi$. *If* $|x| < |y| < \varepsilon$, *then*

$$c \log^{-1}(|y|^{-1}) < u_L(x, y) < C \log^{-1}(|y|^{-1}).$$

If $|y| \leqq |x| < \varepsilon$, *then*

$$c \log(|x|^{-1}) \log^{-2}|y| < u_L(x, y) < C \log(|x|^{-1}) \log^{-2}|y|.$$

If $|y| < \varepsilon \leqq |x|$, *then*

$$c \log^{-1}(|x| + 2) \log^{-2}|y| < u_L(x, y) < C \log^{-1}(|x| + 2) \log^{-2}|y|.$$

PROOF. $u_L(x, y) = E^x L^y_{T_0} P^y\{T_0 = \infty\}/P^x\{T_0 = \infty\}$. Clearly, $E^x L^y_{T_0} = P^x\{T_y < T_0\} E^y L^y_{T_0}$.

$$\begin{aligned} E^y L^y_{T_0} &= E^y \textstyle\int_0^\infty dL_t^y - E^y \int_{T_0}^\infty dL_t^y \\ &= u(0) - P^y\{T_0 < \infty\} E^0 \textstyle\int_0^\infty dL_t^y \\ &= u(0) - (u(-y)/u(0))u(y) \\ &= G(y) + G(-y) - G(y)G(-y)/u(0), \end{aligned}$$

where $G(x) = u(0) - u(x)$. By Corollary 2.2,

$$c \log^{-1}(|y|^{-1}) \leqq E^y L^y_{T_0} \leqq C \log^{-1}(|y|^{-1}).$$

The estimates for $P^y\{T_0 = \infty\}$ and $P^x\{T_y < T_0\}$ follow from Corollaries 2.2 and 2.3.

For $b > 0$, put $B = [-b, b]$ and $T_B = \inf\{t > 0: |Z_t| \leqq b\}$.

LEMMA 4.6. *Assume that* $|h| < 2/\pi$. *For* $b < |x| < \varepsilon$,

$$c \log|x|/\log(b) \leqq P_L^x\{T_B < \infty\} \leqq C \log|x|/\log(b).$$

For $b < \varepsilon \leqq |x|$,

$$c/\log(b^{-1}) \log(|x| + 2) \leqq P_L^x\{T_B < \infty\} \leqq C/\log(b^{-1}) \log(|x| + 2).$$

PROOF. Write $T = T_B$. For all x

$$U_L(x, B) = E_L{}^x I\{T < \infty\} U_L(Z_T, B)$$

by the strong Markov property. Hence

$$\frac{U_L(x, B)}{\sup U_L(z, B)} \leqq P_L{}^x\{T < \infty\} \leqq \frac{U_L(x, B)}{\inf U_L(z, B)},$$

where the supremum and the infimum are taken over all $z \in B$. The result now follows from Lemma 4.5 and a couple of straightforward calculations.

LEMMA 4.7. *Assume that* $|h| < 2/\pi$. *For* $b < t < \varepsilon$

$$c \leqq P_L{}^0\{|Z_s| \leqq b \text{ for some } s > t\} \log(b)/\log(t) \leqq C.$$

PROOF.

$$\begin{aligned} P_L{}^0\{|Z_s| \leqq b \text{ for some } s > t\} &= E_L{}^0 P_L{}^{Z(t)}\{T_B < \infty\} \\ &= u(0)H(t) \textstyle\int P_L{}^y\{T_B < \infty\} P^y\{T_0 = \infty\} q_t(y)\, dy. \end{aligned}$$

The assertion therefore follows from Lemma 4.6 and the estimates for $H(t)$ and $P^y\{T_0 = \infty\}$.

COROLLARY 4.8. *For a suitable choice of* $a \in (0, 1)$

$$c \leqq P_L{}^0\{|Z_s| \leqq b \text{ for some } s \in (t, t^a)\} \log(b)/\log(t) \leqq C.$$

5. The case $|h| < 2/\pi$.

THEOREM 5.1. *Let* $f(t)$, $0 < t < 1$, *be a nonnegative decreasing function. If* $\int_0^1 (tf(t))^{-1} dt < \infty$, *then* $\lim_{t\to 0} Z_t/tf(t) = 0$ a.s. *If* $\int_0^1 (tf(t))^{-1} dt = \infty$, *then* $\limsup_{t\to 0} Z_t/tf(t) = \infty$ a.s.

PROOF. Assume that the integral is finite. Then $\sum f(2^{-n})^{-1}$ is finite, too. Put

$$A_n = \{Z^*(2^{-n}) > \varepsilon f(2^{-n})2^{-n}\}.$$

By virtue of Lemmas 4.2 and 4.4 we have $\sum P^0(A_n) < \infty$. By Borel–Cantelli, this implies

$$\limsup_{t\to 0} Z_t^*/tf(t) \leqq 2\varepsilon \quad \text{a.s.}$$

This proves the first part of the theorem. Next, assume that the integral is infinite. Then $\sum f(2^{-n})^{-1} = \infty$. We may assume that $f(t) > 2|\log(t)|$. Put $\varphi(t) = (ht\log(t))^{-1}$ and let $1 < k < K$. Put

$$B_n = \{k^{-1} < Z(2^{-n-1})\varphi(2^{-n-1}) < k, K2^{-n}f(2^{-n}) < Z(2^{-n})\}.$$

By the Markov property and Lemma 4.3, $\sum P^0(B_n) = \infty$. The events B_n are not independent. But for $m \neq n$ we have

$$P^0(B_m \cap B_n) \leqq CP^0(B_m)P^0(B_n)$$

by the Markov property and Lemma 4.3. By a generalization of the Borel–Cantelli lemma for dependent events (see page 317 of [15])

$$P^0(\limsup B_n) > 0 .$$

By Corollary 3.2, this implies $P^0(\limsup B_n) = 1$. Thus

$$\limsup_{t\to 0} Z_t/tf(t) \geqq K \quad \text{a.s.}$$

This completes the proof.

THEOREM 5.2. *Let $f(t)$, $0 < t < 1$, be a nonnegative decreasing function. If $\int_0^1 (tf(t))^{-1}\,dt < \infty$, then* $\liminf_{t\to 0} |Z_t|/\exp(-f(t)) = \infty$ a.s. *If $\int_0^1 (tf(t))^{-1}\,dt = \infty$, then* $\liminf_{t\to 0} |Z_t|/\exp(-f(t)) = 0$ a.s.

PROOF. Assume that the integral is finite. Then

$$\sum 2^n/f(\exp(-2^n)) < \infty .$$

Let K be a large positive constant and put

$$A_n = \{|Z_s| < K\exp(-f(\exp(-2^n))) \ \text{for some}\ s > \exp(-2^{n+1})\} .$$

By Lemma 4.7, $P^0(A_n) < C2^{n+1}/f(\exp(-2^n))$. Hence

$$\liminf_{t\to 0} |Z_t|/\exp(-f(t)) \geqq K \quad \text{a.s.}$$

This proves the first part of the theorem. Next, assume that the integral is infinite. We may assume that $f(t) > 2|\log(t)|$. Choose $a \in (0, 1)$ as in Corollary 4.8. Then

$$\sum a^{-n}/f(\exp(-a^{-n})) = \infty .$$

Let $K > 1$. To simplify notation, write $x_n = \exp(-a^{-n})$ and $I_n = \{x\colon |x| \leqq \exp(-Kf(x_n))\}$. Put

$$B_n = \{Z_s \in I_n \ \text{for some}\ s \in (x_n, x_{n-1})\} .$$

By Corollary 4.8,

$$ca^{-n}/f(\exp(-a^{-n})) < P^0(B_n) < Ca^{-n}/f(\exp(-a^{-n})) .$$

So $\sum P^0(B_n) = \infty$. The events B_n are not independent. For $m + 1 < n$,

$$\begin{aligned} &P^0(B_n \cap B_m) \\ &\qquad \leqq P^0(B_n) \sup_{y\in I_n} P_L{}^y\{Z_s \in I_m \ \text{for some}\ s > x_m - x_{n-1}\} \end{aligned}$$

by the strong Markov property. By virtue of Lemma 4.6 and Lemma 2.4 we have for t and b small and $|y| < t^2$

$$\begin{aligned} P_L{}^y\{&|Z_s| \leqq b \ \text{for some}\ s > t\} \\ &= E_L{}^y P_L{}^{Z(t)}\{T_B < \infty\} \\ &= E^y(I\{t < T_0\}P^{X(t)}\{T_0 = \infty\}P_L{}^{X(t)}\{T_B < \infty\})/P^y\{T_0 = \infty\} \\ &\leqq CP^y\{t < T_0\}/\log(b^{-1})P^y\{T_0 = \infty\} \leqq C\log(t^{-1})/\log(b^{-1}) . \end{aligned}$$

Hence

$$P^0(B_n \cap B_m) \leqq CP^0(B_n)a^{-m}/f(\exp(-a^{-m})) \leqq CP^0(B_n)P^0(B_m).$$

We may therefore apply the extended Borel–Cantelli lemma. So $P^0(\limsup B_n) > 0$. By Corollary 3.2, this implies

$$\liminf_{t\to 0} |Z_t|/\exp(-f(t)) = 0 \quad \text{a.s.}$$

6. The case $|h| = 2/\pi$. If $|h| = 2/\pi$, then $\{Z_t\}$ has the opposite sign of h for an initial period of time. If $\{X_t\}$ is a Cauchy process with parameter $h = -2/\pi$, then $\{-X_t\}$ is a Cauchy process with parameter $2/\pi$. So we need only consider the case $h = -2/\pi$.

THEOREM 6.1. *If* $h = -2/\pi$, *then*

$$\limsup_{t\to 0} Z_t/t|\log(t)| = 2/\pi \quad \text{a.s.}$$

PROOF. Given $\varepsilon > 0$, choose $b < 1$ and put

$$A_n = \{Z^*(b^n) > (2/\pi + \varepsilon)b^n|\log(b^n)|\}.$$

By virtue of Lemma 4.2 and Lemma 4.4 we have $\sum P^0(A_n) < \infty$. This implies that

$$\limsup_{t\to 0} Z_t/t|\log(t)| \leqq 2/\pi + 2\varepsilon \quad \text{a.s.}$$

provided we have chosen b close enough to 1. Next, put

$$B_n = \{2/\pi - \varepsilon < Z(n^{-1})n/\log(n)\}.$$

By virtue of Lemma 4.3, $P^0(B_n) \geqq c > 0$. Hence

$$\limsup_{t\to 0} Z_t/t|\log(t)| > 2/\pi - \varepsilon \quad \text{a.s.}$$

This completes the proof.

If $h = -2/\pi$, then Z is nonnegative for an initial period of time. We shall now determine the lower envelope of the process at 0.

LEMMA 6.2. *There exists a positive constant* c *such that* $u(x) = u(0)\exp(-cx)$ *for* $x > 0$.

PROOF. $u(x) = u(0)P^0\{T_x < \infty\}$. Let $x > 0$ and $y > 0$. By the strong Markov property,

$$\begin{aligned} P^0\{T_{x+y} < \infty\} &= P^0\{T_x < \infty\}P^x\{T_{x+y} < \infty\} \\ &= P^0\{T_x < \infty\}P^0\{T_y < \infty\}, \end{aligned}$$

since X has no upward jumps.

In the same manner that Corollary 2.3 was proved we get

COROLLARY 6.3. *Define*

$$\begin{aligned} B(x, y) &= \log(y)/\log(x) && \textit{if} \quad 0 < (1-x)x \leqq y \\ &= \log(y)\log(1 - y/x)/\log^2(x) && \textit{if} \quad 0 < y < (1-x)x. \end{aligned}$$

Then there exist constants $0 < c < C$ *such that for all small positive* x *and* y

$$cB(x, y) \leqq P^x\{T_y < T_0\} \leqq CB(x, y).$$

The next lemma fills the gap between Lemmas 2.4 and 2.5.

LEMMA 6.4. *For all small positive x and t*

$$c \log(t)/\log(xt) \leqq P^x\{T_0 > t\} \leqq C \log(t)/\log(xt).$$

LEMMA 6.5. *If $0 \leqq x < t \log(t^{-1})$ and $t < b < \frac{1}{2}t \log(t^{-1})$,*

$$P_L^x\{0 < Z_t < b\} < Cb/t \log^2(t).$$

PROOF. By weak convergence we need only consider the case $0 < x$. Then

$$P_L^x\{0 < Z_t < b\} = \frac{E^x[I\{t < T_0, 0 < X_t < b\}P^{X(t)}\{T_0 = \infty\}]}{P^x\{T_0 = \infty\}}.$$

The density of $P^x\{t < T_0, X_t \in dy\}$ is

$$\begin{aligned} &p_t(y - x) - \textstyle\int_0^t p_{t-s}(y)P^x\{T_0 \in ds\} \\ &\qquad = (p_t(y - x) - p_t(y))P^x\{T_0 \leqq t\} + p_t(y - x)P^x\{T_0 > t\} \\ &\qquad\quad + \textstyle\int_0^t (p_t(y) - p_{t-s}(y))P^x\{T_0 \in ds\}. \end{aligned}$$

Integrating each of the 3 terms we get

$$P^x\{t < T_0, 0 < X_t < b\} \leqq Cb/t \log(t) \log(x).$$

Hence

$$P_L^x\{0 < Z_t < b\} \leqq Cb/t \log(t) \log(b).$$

This proves the lemma.

COROLLARY 6.6. *Under the assumptions of Lemma 6.5,*

$$c < t \log(t^{-1}) \textstyle\int_b^{2t \log(t^{-1})} y^{-1}P_L^x\{Z(t) \in dy\} < C.$$

For $b > 0$, put $B = [0, b]$ and $T_B = \inf\{t > 0: Z_t \in B\}$. As usual, T_b denotes the first hitting time of $\{b\}$.

LEMMA 6.7. *If $0 < b < x < \varepsilon$, then*

$$P_L^x\{T_b < \infty\} \leqq P_L^x\{T_B < \infty\} \leqq CP_L^x\{T_b < \infty\}.$$

PROOF. Obviously, $P_L^x\{T_b < \infty\} = P^x\{T_b < T_0\}P^b\{T_0 = \infty\}/P^x\{T_0 = \infty\}$. So $P_L^x\{T_b < \infty\}$ is of the magnitude $B(x, b) \log(x)/\log(b)$. Arguing as in the proof of Lemma 4.6, we see that $b \log(b)/x \log(x)$ is an upper bound for $P_L^x\{T_B < \infty\}$. This is not the best possible upper bound. But it shows that

$$P_L^x\{\inf Z_t < b^2\} = o(P_L^x\{\inf Z_t < b\})$$

as $b \downarrow 0$. It follows that $P_L^x\{Z(T_B) > b^2\} \geqq cP_L^x\{T_B < \infty\}$. By the strong Markov property and Corollary 6.3

$$\begin{aligned} P_L^x\{T_b < \infty\} &\geqq P_L^x\{Z(T_B) > b^2, T_b < \infty\} \\ &= E_L^x\{Z(T_B) > b^2, P_L^{Z(T_B)}\{T_b < \infty\}\} \\ &\geqq cP_L^x\{Z(T_B) > b^2\} \geqq cP_L^x\{T_B < \infty\}. \end{aligned}$$

This completes the proof.

LEMMA 6.8. *If* $0 \leqq x < t \log(t^{-1})$ *and* $t < b < \frac{1}{4}t \log(t^{-1})$, *then* $c < P_L{}^x\{Z_s \in B$ *for some* $s > t\} t \log^2(t)/b < C$.

PROOF.

$$\begin{aligned} P_L{}^x\{T_B \circ \theta_t < \infty\} &= E_L{}^x P_L{}^{Z(t)}\{T_B < \infty\} \\ &\leqq P_L{}^x\{0 < Z_t < 2b\} + \int_{2b} P_L{}^y\{T_B < \infty\} P_L{}^x\{Z_t \in dy\} . \end{aligned}$$

By Lemmas 6.5, 6.7, and Corollary 6.6, these two terms are both less than $cb/t \log^2(t)$. On the other hand,

$$E_L{}^x P_L{}^{Z(t)}\{T_B < \infty\} \geqq \int_b P_L{}^y\{T_B < \infty\} P_L{}^x\{Z_t \in dy\} .$$

And by Lemma 6.7 and Corollary 6.6, this integral is greater than $cb/t \log^2(t)$.

We can now determine the lower envelope of the process $\{Z_t\}$ at 0. The proof is identical to the proof of Theorem 5.1 of [10] or Theorem 5.2 of the previous section.

THEOREM 6.9. *Assume that* $h = -2/\pi$. *If* f *is a nonnegative decreasing function, then with probability* 1

$$\liminf_{t \to 0} Z(t)/tf(t) = \infty \quad or \quad 0$$

according as $\int_0^1 f(t)(t \log^2(t))^{-1}\, dt < \infty$ *or* $= \infty$.

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