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THE BIRTH AND DEATH CHAIN IN A RANDOM ENVIRONMENT: INSTABILITY AND EXTINCTION THEOREMS

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Let (Y_n) be a recurrent Markov chain with discrete or continuous state space. A model of a birth and death chain (Z_n) controlled by a random environment (Y_n) is formulated wherein the bivariate process (Y_n, Z_n) is taken to be Markovian and the marginal process (Z_n) is a birth and death chain on the nonnegative integers with absorbing state $z = 0$ when a fixed sequence of environmental states of (Y_n) is specified. In this paper, the property of uniform ϕ-recurrence of (Y_n) is used to prove that with probability one the sequence (Z_n) does not remain positive or bounded. An example is given to show that uniform ϕ-recurrence of (Y_n) is required to insure this instability property of (Z_n). Conditions are given for the extinction of the process (Z_n) when (i) (Z_n) possesses homogeneous transition probabilities and (Y_n) possesses an invariant measure on discrete state space, and (ii) (Z_n) possesses nonhomogeneous transition probabilities and (Y_n) has general state space.

0. Introduction. In this paper, we formulate a model of a discrete-time birth and death process (Z_n) evolving in a random environment controlled by a process (Y_n) as follows: we take the environmental process (Y_n) to be a recurrent, irreducible, discrete-time Markov chain on either discrete or continuous state space. The process (Z_n) moves on the nonnegative integers $\mathbb{Z}_0$ with transitions from $z_0 \in \mathbb{Z}_0$ permitted only to those states z for which $|z - z_0| \leqslant 1$. We assume all positive states of (Z_n) lead to 0 and we make 0 absorbing. The processes (Y_n) and (Z_n) are related through the definition of a transition probability function P for the bivariate process (Y_n, Z_n). This kernel P is defined in such a way that (Y_n, Z_n) is Markovian and its definition imposes the following two conditions: (i) (Y_n) is Markovian and time homogeneous and (ii) given a realization of (Y_n), the conditional distribution of (Z_n) is Markovian (but not time homogeneous, in general). Condition (i) means that there is no "feedback" from (Z_n) to (Y_n), while condition (ii) means that the evolution of (Z_n) is Markovian when a fixed sequence of environments (determined by the Y_n's) is specified. Nevertheless, since the sequence of environments is itself a stochastic process, the marginal (Z_n) process of (Y_n, Z_n) is not Markovian, in general. When no realization of the (Y_n) process is specified in advance, we will refer to (Z_n) as a birth and death chain in a random environment. In this paper, it is our intent to study the process (Z_n) using minimal information about the environment (Y_n). In terms of the bivariate process (Y_n, Z_n), this would reduce to the familiar model of a Markov chain. Given full information about the initial distribution and bivariate transition probability function P of

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(Y_n, Z_n), the classical theory of Markov chains would provide answers to the questions raised. The problem here is to study the behavior of the non-Markovian process (Z_n) without full knowledge of the bivariate stochastic structure. The two central questions which are of interest are the questions of instability and extinction of the process (Z_n). The results in Section 2 completely determine conditions on the environmental process (Y_n) which insure that the sequence (Z_n) does not remain positive and bounded with probability 1. Under these conditions, (Z_n) either goes to 0 or $+\infty$, thus exhibiting an "unstable" or "transient" quality. In Section 3, we give conditions for $P_{(y,z)}[Z_n \to 0] = 1$ for any initial state (y, z) of (Y_n, Z_n), examining both the cases when the "transition probabilities" of (Z_n) depend on the z-state of (Z_n) or not. In Section 1, we present the definitions and general results of discrete-time Markov chains which are pertinent to the results in this paper (see Chung (1960) and Orey (1971) for further reference). We also describe the exact structure of the processes (Y_n) and (Y_n, Z_n) as well as introduce the notation which will be used throughout the following discussion.

Recently there has been great interest in stochastic processes in random environments, notably the works of Griego and Hersh (1969), (1971) in which they introduce the concept of random evolutions, on which the present model is based; Smith (1968), Smith and Wilkinson (1969), (1971), and Athreya and Karlin (1971), (1971a) dealing with branching processes with random environments; and Solomon (1975), in his paper on random walks in a random environment. In each of these three models, there is some relationship between the present paper and the works cited above. The work of Griego and Hersh introduced random evolutions as an operator-theoretic version of the classical Feynman-Kac formula, and this has been their point of view throughout, that is, to use random evolutions as tools for solving differential equations. Our purpose in studying the birth and death chain in a random environment is more in consonance with the Smith-Wilkinson and Athreya-Karlin models; that is, determining the stochastic nature of (Z_n) with minimal knowledge of the environment (Y_n). The Smith-Wilkinson model assumes that the environment is chosen by i.i.d. random variables, a rather crucial assumption since it makes (Z_n) Markovian. Athreya and Karlin specify that the environment be controlled by a stationary ergodic process and they too are interested in instability and extinction characteristics of the branching process with random environments. Their results depend heavily on the iterates of the probability generating function of Z_n, a device which has no applicability in the analysis of the birth and death chain in a random environment. In Solomon's model of random walks in a random environment, he assumes that the transition probabilities of the walk are chosen randomly in space in contrast to the present model (as well as the branching process with random environments) where randomness of the environment occurs in time.

1. Preliminaries. Let (Y_n) be a discrete-time Markov chain with countable state space $\mathcal{Y}$ and transition probability function (tpf) K. Denote the n-step tpf of (Y_n) by $K^{(n)}$. For $y, y' \in \mathcal{Y}$, say y and y' *communicate* if there exists $n_1, n_2 \in \mathbb{N} =$

$\{1, 2, 3 \ldots\}$ such that $K^{(n_1)}(y, y') > 0$ and $K^{(n_2)}(y', y) > 0$. If y and y' communicate, we say y (and y') are *recurrent (transient)* if and only if $\sum_{n=1}^{\infty} K^{(n)}(y, y') = +\infty (< +\infty)$. A set of states $A \subset \mathcal{Y}$ is called (stochastically) *closed* if no state outside A can be reached from any state in A. A single state y forming a closed set will be called *absorbing*. A Markov chain (Y_n) is *irreducible* if there exists no closed set other than $\mathcal{Y}$. Of course, for countable state space, this is equivalent to saying that all states communicate. Define the *first return time* τ_y to a state $y \in \mathcal{Y}$ by $\tau_y = \inf\{n \geq 1, Y_n = y\}$ and let $\tau_y = +\infty$ if no $Y_n = y$, for $n \geq 1$. Call the recurrent state y *positive recurrent* if $E_y \tau_y < +\infty$ where $E_y \tau_y$ is the mean return time to y. We will use the following fact: if the Markov chain (Y_n) is positive recurrent, then it possesses a distribution $\{\pi_y\}$ such that $\pi K = \pi$, i.e.,

$$\sum_{y' \in \mathcal{Y}} \pi_{y'} K(y', y) = \pi_y.$$

We call such a distribution π an *invariant distribution*.

Consider now the case where (Y_n) is a Markov chain with continuous state space $(\mathcal{Y}, \mathcal{Q})$. Let K be the tpf for (Y_n). Then K satisfies

(i) $K(y, \cdot)$ is a probability measure on $\mathcal{Q}$ for all $y \in \mathcal{Y}$ and
(ii) $K(\cdot, A)$ is $\mathcal{Q}$-measurable for all $A \in \mathcal{Q}$.

Before introducing the concepts of indecomposability, irreducibility, and recurrence of Markov chains on general state space, we need the following notions: Let

$$Q(y, A) = P_y\Big(\bigcap_{m=1}^{\infty} \bigcup_{n=m}^{\infty} [Y_n \in A]\Big), \qquad y \in \mathcal{Y}, A \in \mathcal{Q},$$

$$L(y, A) = P_y\Big(\bigcup_{n=1}^{\infty} [Y_n \in A]\Big), \qquad y \in \mathcal{Y}, A \in \mathcal{Q},$$

$$A^0 = \{y \in \mathcal{Y} : L(y, A) = 0\}.$$

Thus $Q(y, A)$ is the probability that $Y_n \in A$ for infinitely many n, given $Y_0 = y$, and $L(y, A)$ is the probability that $Y_n \in A$ for some positive n given $Y_0 = y$. The set A^0 consists of those states y for which the process (Y_n) never reaches A given that $Y_0 = y$.

Let ϕ be a σ-finite, nontrivial measure on $(\mathcal{Y}, \mathcal{Q})$. A set $A \in \mathcal{Q}$ is (stochastically) *closed* if $A \neq \varnothing$ and $K(y, A) = 1$ for all $y \in A$. Note that the set A^0 is either closed or empty. A closed set which does not contain two disjoint closed sets is *indecomposable*. The Markov chain (Y_n) is called ϕ-*irreducible* if $L(y, A) > 0$ for all y whenever $\phi(A) > 0$. The Markov chain (Y_n) is ϕ-*recurrent* if $L(y, A) = 1$ for all $y \in \mathcal{Y}$ whenever $\phi(A) > 0$. The set $A \in \mathcal{Q}$ is *inessential* if $Q(y, A) = 0$, for all $y \in \mathcal{Y}$; otherwise A is *essential*. An essential set which is the union of countably many inessential sets is *improperly essential*; otherwise it is *properly essential*. We will use the fact that if y is indecomposable and A is closed, then $\mathcal{Y} - A$ is not properly essential. (See Orey (1971).)

The *first entrance time* τ_A to a set $A \in \mathcal{Q}$ is defined by $\tau_A = \inf\{n \geq 1 : Y_n \in A\}$ and $+\infty$ if no n exists satisfying the given condition. Similarly define $\tau_A^{(n)} = \inf\{k$

$> \tau_A^{(n-1)} : Y_k \in A\}$ and $+\infty$ if no such k exists, and call $\tau_A^{(n)}$ the nth *entrance time* to A. It is convenient to define $\tau_A^{(0)} = 0$ and set $\tau_A^{(1)} = \tau_A$. We now define the important notion of uniform ϕ-recurrence.

DEFINITION. The Markov chain (Y_n) on $(\mathcal{Y}, \mathcal{Q})$ is *uniformly ϕ-recurrent* if

$$\sup_{y \in \mathcal{Y}} P_y[\tau_A > n] \to 0 \qquad \text{as} \quad n \to \infty$$

whenever $\phi(A) > 0$.

Let $K_A^{(n)}(y, B) = P_y[\tau_A^{(n)} < +\infty, Y_{\tau_A^{(n)}} \in B]$ for all $B \in \mathcal{Q}$. *Suppose* $P_y[\tau_A^{(n)} < +\infty] = 1, \forall y \in \mathcal{Y}$. Let $\mathcal{Q}_A = \{B \in \mathcal{Q} : B \subset A\}$. Then $K_A^{(n)}(y, B)$, $B \in \mathcal{Q}_A$ is a tpf for the Markov chain $Y_0, Y_{\tau_A^{(1)}}, Y_{\tau_A^{(2)}}, \ldots$ with state space $(A, \mathcal{Q}_A)$. This chain is called the *process on A*. A set $D \in \mathcal{Q}$ is a *D-set* if the process on D is uniformly ϕ-recurrent. We assume that $\mathcal{Q}$ is *separable,* i.e., there exists a countable class of sets $A_0, A_1, \ldots$ that generate $\mathcal{Q}$. A probability distribution P_ϕ is determined on the product measurable space $(\mathcal{Y}^{\mathbb{N}}, \mathcal{Q}^{\mathbb{N}})$ by the transition kernel K through the relations

$$P_\phi[Y_0 \in A_0, \cdots, Y_n \in A_n]$$
$$= \int_{A_0} \phi(dy_0) \int_{A_1} K(y_0, dy_1) \cdots \int_{A_{n-1}} K(y_{n-2}, dy_{n-1}) K(y_{n-1}, A_n),$$

where $A_0, \cdots, A_n \in \mathcal{Q}$. The expectation operator corresponding to such a probability distribution will be denoted by E_ϕ. A σ-finite measure π on $(\mathcal{Y}, \mathcal{Q})$ is *invariant* for the tpf K if $\pi K = \pi$, i.e., if $\pi(A) = \int_{\mathcal{Y}} K(y, A)\pi(dy)$, $A \in \mathcal{Q}$. We know from the theorem of Harris (Orey (1971)) that every ϕ-recurrent chain on $(\mathcal{Y}, \mathcal{Q})$ has a nontrivial σ-finite measure π such that π is invariant and ϕ is absolutely continuous with respect to π.

Throughout the following discussion, (Y_n) will be a discrete-time Markov chain with tpf K. Surprisingly, however, it will be seen in Theorem 3.2 that a sufficient condition for extinction of the process (Z_n) is stated without any assumptions on (Y_n) so that it may be taken to be any stochastic process. When the state space $\mathcal{Y}$ of (Y_n) is countable, a minimal assumption we make is that (Y_n) be an irreducible, recurrent Markov chain. When $\mathcal{Y}$ is continuous, we will take (Y_n) to be ϕ-irreducible and ϕ-recurrent for an arbitrary but fixed nontrivial σ-finite measure ϕ on $(\mathcal{Y}, \mathcal{Q})$ with $\mathcal{Q}$ separable. The process (Y_n, Z_n) will be a discrete-time Markov chain on the state space $\mathbb{S} = \mathcal{Y} \times \mathbb{Z}_0$ with 0 made absorbing in $\mathbb{Z}_0$. Thus $F = \mathcal{Y} \times \{0\}$ is a closed set in $\mathbb{S}$. We assume that $\mathbb{S}$ is indecomposable. If λ is taken to be counting measure on $\mathbb{Z}_0$, then a further condition we impose on $\mathbb{S}$ is that $\mathbb{S} - F$ be irreducible in the countable case and $(\pi \times \lambda)$-irreducible when $\mathcal{Y}$ is continuous. When $\mathcal{Y}$ is countable, note that each state $(y, z) \in \mathbb{S} - F$ is transient, for we have specified that there is positive probability of (y, z) ever reaching F. When $\mathcal{Y}$ is continuous, $\mathbb{S} - F$ is not properly essential since F is closed (cf. Orey (1971)).

We define the tpf of (Y_n, Z_n), denoted by P, by the relation

$$P[(y, z), (y', z')] = K(y, y')P_y(z, z')$$

in the countable case and

$$P[(y, z), (A, z')] = K(y, A)P_y(z, z'), \qquad A \in \mathcal{Q},$$

in the continuous case where, in either case, for $z \geqslant 1$,

$$\begin{aligned} P_y(z, z') &= p_z^{(y)} \geqslant 0 && \text{if } z' = z + 1 \\ &= r_z^{(y)} \geqslant 0 && \text{if } z' = z \\ &= q_z^{(y)} \geqslant 0 && \text{if } z' = z - 1 \\ &= 0 && \text{otherwise,} \end{aligned}$$

and $r_0^{(y)} = 1 \forall y \in \mathcal{Y}$, with $p_z^{(y)} + r_z^{(y)} + q_z^{(y)} = 1$. Thus for each fixed y, P_y is the tpf for a birth and death chain on $\mathbb{Z}_0$ with 0 made absorbing. Note that (Z_n) is *not* Markovian in general since the sequence of environments is chosen according to a Markov process. In an abuse of language, we will refer to $\{p_z^{(y)}, q_z^{(y)}, r_z^{(y)}\}$ as "transition probabilities" for (Z_n) and 0 as an "absorbing state" for (Z_n).

2. Instability of the birth and death chain in a random environment. In this section, we obtain conditions on the control process (Y_n) for which

$$P_{(y_0, z_0)}[Z_n \to 0 \quad \text{or} \quad Z_n \to \infty] = 1 \tag{2.1}$$

for any initial state (y_0, z_0) of the bivariate process (Y_n, Z_n). The results in this section give a complete solution to this problem when the state space $\mathcal{Y}$ of (Y_n) is discrete or continuous and in so doing show that positive recurrence of (Y_n) does not insure the instability property (2.1). Indeed, we will show that the condition which insures instability of (Z_n) is the uniform ϕ-recurrence of (Y_n) and we present a counterexample of instability when (Y_n) is only positive recurrent. For $\mathcal{Y}$ discrete, we take $\mathcal{Q}$ to be the σ-algebra of all subsets of $\mathcal{Y}$, and ϕ to be counting measure on $(\mathcal{Y}, \mathcal{Q})$. Let $\mathbb{Z}^+ = \mathbb{Z}_0 - \{0\}$.

THEOREM 2.1. *Let (Y_n) be a uniformly ϕ-recurrent chain with discrete state space $(\mathcal{Y}, \mathcal{Q})$. Then property* (2.1) *holds.*

PROOF. Let $(y_0, z_0) \in \mathcal{Y} \times \mathbb{Z}^+$, $z \in \mathbb{Z}^+$. The uniform ϕ-recurrence of (Y_n) says that for each $y' \in \mathcal{Y}$ there exists $m \in \mathbb{N}$, and $\varepsilon > 0$ such that

$$P_{(y_0, z_0)}\Big(\bigcup_{j=1}^m [Y_{n+j} = y'] | (Y_n, Z_n) = (y, z)\Big) \geqslant \varepsilon \tag{2.5}$$

for all $y \in \mathcal{Y}$. Note that

$$\begin{aligned} &P_{(y_0, z_0)}\Big(\bigcup_{j=1}^m [(Y_{n+j}, Z_n) = (y', z)]\Big) \\ &\quad \doteq \Sigma_{y \in \mathcal{Y}} P_{(y_0, z_0)}\Big(\bigcup_{j=1}^m [Y_{n+j} = y'] \cap [(Y_n, Z_n) = (y, z)]\Big) \\ &\quad = \Sigma_{y \in \mathcal{Y}} \Big\{P_{(y_0, z_0)}\Big(\bigcup_{j=1}^m [Y_{n+j} = y'] | (Y_n, Z_n) = (y, z)\Big) \\ &\qquad \cdot P_{(y_0, z_0)}[(Y_n, Z_n) = (y, z)]\Big\}. \end{aligned}$$

Using inequality (2.5) gives

$$\begin{aligned} P_{(y_0, z_0)}\Big(\bigcup_{j=1}^m [(Y_{n+j}, Z_n) = (y', z)]\Big) &\geqslant \varepsilon \Sigma_{y \in \mathcal{Y}} P_{(y_0, z_0)}[(Y_n, Z_n) = (y, z)] \\ &= \varepsilon P_{(y_0, z_0)}[Z_n = z]. \end{aligned}$$

Summing over n gives

$$\Sigma_{n=1}^{\infty} P_{(y_0, z_0)}[Z_n = z] \leqslant \varepsilon^{-1} \Sigma_{n=1}^{\infty} P_{(y_0, z_0)}\Big(\bigcup_{j=1}^{m} [(Y_{n+j}, Z_n) = (y', z)] \Big).$$

We need only show that the right-hand side of the above inequality is finite, for then it follows as a consequence of the Borel-Cantelli lemma that $P_{(y_0, z_0)}[Z_n = z$ infintely often$] = 0$ which implies the desired conclusion. By Boole's inequality

$$P_{(y_0, z_0)}\Big(\bigcup_{j=1}^{m} [(Y_{n+j}, Z_n) = (y', z)] \Big) \leqslant \Sigma_{j=1}^{m} P_{(y_0, z_0)}[(Y_{n+j}, Z_n) = (y', z)]. \tag{2.6}$$

Now let $B = \{z \pm i, 0 \leqslant i \leqslant m\} \cap \mathbb{Z}^+$. We have

$$\begin{aligned} \Sigma_{j=1}^{m} P_{(y_0, z_0)}[(Y_{n+j}, Z_n) = (y', z)] &\leqslant \Sigma_{j=1}^{m} \{ P_{(y_0, z_0)}[(Y_{n+j}, Z_{n+j}) \in (y', B)] \\ &\quad + P_{(y_0, z_0)}[Z_n = z, Z_{n+j} = 0] \}. \end{aligned} \tag{2.7}$$

This follows because (Z_n) can move at most one step to the right or left. Thus after m steps its position is at most $z + m$ and at least $z - m$. The second term in the RHS of (2.7) adjusts for the event that (Z_n) may be absorbed at 0 before m steps have occurred.

Note that

$$\begin{aligned} \text{(i)}\ \Sigma_{j=1}^{m} \Sigma_{n=1}^{\infty} P^{(n+j)}[(y_0, z_0), (y', B)] &= \Sigma_{j=1}^{m} \Sigma_{n=j+1}^{\infty} P^{(n)}[(y_0, z_0), (y', B)] \\ &\leqslant \Sigma_{j=1}^{m} \Sigma_{n=1}^{\infty} P^{(n)}[(y_0, z_0), (y', B)] \\ &= m \Sigma_{n=1}^{\infty} P^{(n)}[(y_0, z_0), (y', B)] \end{aligned} \tag{2.8}$$

and

$$\begin{aligned} \text{(ii)}\ &\Sigma_{j=1}^{m} \Sigma_{n=1}^{\infty} P_{(y_0, z_0)}[Z_n = z, Z_{n+j} = 0] \\ &= E_{(y_0, z_0)}[\Sigma_{j=1}^{m} \Sigma_{n=1}^{\infty} \mathcal{X}_{[Z_n = z, Z_{n+j} = 0]}] \leqslant m^2 \end{aligned} \tag{2.9}$$

where $\mathcal{X}$ denotes the indicator function. Let $S_{n,j} = [Z_n = z, Z_{n+j} = 0]$. The inequality (2.9) is true because either a sample sequence contains no coordinates with a 0 entry, in which case for all such sequences ω, $\mathcal{X}_{S_{n,j}}(\omega) = 0$ for all n and all j, or the sample sequence has a coordinate with a 0 entry and all coordinates thereafter are 0. In the latter case, the maximum value of $\Sigma_{j=1}^{m} \Sigma_{n=1}^{\infty} \mathcal{X}_{S_{n,j}}$ occurs when we consider a sample sequence of the type $\omega = (z, \cdots, z, 0, 0, \cdots)$. It readily follows that $\Sigma_{n=1}^{\infty} \mathcal{X}_{S_{n,j}}(\omega) \leqslant j$. Thus $\Sigma_{j=1}^{m} \Sigma_{n=1}^{\infty} \mathcal{X}_{S_{n,j}}(\omega) \leqslant \Sigma_{j=1}^{m} j \leqslant m^2$. Now use (2.8) and (2.9) in (2.7) to obtain

$$\begin{aligned} &\Sigma_{n=1}^{\infty} \Sigma_{j=1}^{m} P_{(y_0, z_0)}[(Y_{n+j}, Z_n) = (y', z)] \\ &\qquad \leqslant m \Sigma_{n=1}^{\infty} P_{(y_0, z_0)}[(Y_n, Z_n) \in (y', B)] + m^2 \\ &\qquad = m \Sigma_{z' \in B} \Sigma_{n=1}^{\infty} P_{(y_0, z_0)}[(Y_n, Z_n) = (y', z')] + m^2. \end{aligned} \tag{2.10}$$

Since $(y', z') \in \mathcal{Y} \times \mathbb{Z}^+$, it is a transient state for (Y_n, Z_n) (since there is positive probability that (y', z') reaches the closed set $\mathcal{Y} \times \{0\}$) so that

$$\Sigma_{n=1}^{\infty} P_{(y_0, z_0)}[(Y_n, Z_n) = (y', z')] < +\infty.$$

Since B is a finite set the expression (2.10) is finite, and from (2.6) we can conclude that

$$\Sigma_{n=1}^{\infty} P_{(y_0, z_0)}\Big(\bigcup_{j=1}^{m} [(Y_{n+j}, Z_n) = (y', z)]\Big) < +\infty$$

which was to be shown. □

REMARK. Note that in Theorem 2.1, the assumption of uniform ϕ-recurrence of (Y_n) can be replaced by the following equivalent condition: there exists $y' \in \mathcal{Y}$, $m \in N$, and $\varepsilon > 0$ such that for all $y \in \mathcal{Y}$,

$$\sup_{1 \leqslant j \leqslant m} K^{(j)}(y, y') \geqslant \varepsilon,$$

where $K^{(j)}$ is the jth step transition probability function of (Y_n).

When $\mathcal{Y}$ is countable, there is a condition which guarantees property (2.1) in the case when (Y_n) is only recurrent. This condition is stated in the following proposition:

PROPOSITION 2.1. *Suppose (Y_n) is a recurrent Markov chain with discrete state space $\mathcal{Y}$. Assume that*

$$\liminf_n P_{(y_0, z_0)}[Y_n = y | Z_n = z] > 0$$

for $(y_0, z_0), (y, z) \in \mathcal{Y} \times \mathbb{Z}^+$. Then property (2.1) *holds.*

PROOF. Suppose that $P_{(y_0, z_0)}[Z_n = z$ infinitely often$] = 1$ for some state $z \in \mathbb{Z}^+$. By Borel-Cantelli, this implies

$$\Sigma_{n=1}^{\infty} P_{(y_0, z_0)}[Z_n = z] = +\infty. \tag{2.11}$$

From the identity

$$\Sigma_{n=1}^{\infty} P_{(y_0, z_0)}[(Y_n, Z_n) = (y, z)] = \Sigma_{n=1}^{\infty} P_{(y_0, z_0)}[Z_n = z] P_{(y_0, z_0)}[Y_n = y | Z_n = z]$$

and the hypothesis

$$\liminf_n P_{(y_0, z_0)}[Y_n = y | Z_n = z] = \varepsilon > 0,$$

we have

$$\Sigma_{n=1}^{\infty} P_{(y_0, z_0)}[(Y_n, Z_n) = (y, z)] \geqslant \varepsilon \Sigma_{n=1}^{\infty} P_{(y_0, z_0)}[Z_n = z].$$

Thus by (2.11) this forces

$$\Sigma_{n=1}^{\infty} P_{(y_0, z_0)}[(Y_n, Z_n) = (y, z)] = +\infty$$

contradicting the transience of the state (y, z). □

The fact that the positive states of (Y_n, Z_n) are transient played a crucial role in the proof of Theorem 2.1. Thus

$$\Sigma_{n=1}^{\infty} P_{(y_0, z_0)}[(Y_n, Z_n) = (y, z)] < \infty$$

for $(y, z) \in \mathcal{Y} \times \mathbb{Z}^+$ and consequently we were able to show

$$\Sigma_{n=1}^{\infty} P_{(y_0, z_0)}[Z_n = z] < +\infty,$$

which was the desired conclusion. When $\mathcal{Y}$ is continuous, there is no notion of transient states but there is a notion of transient sets, namely that of inessential sets. However, for continuous state space, A inessential in $\mathcal{Y}$ does not automatically imply $\Sigma_{n=1}^{\infty} P_{(y_0, z_0)}[(Y_n, Z_n) \in (A, z)] < +\infty$ (see Chung (1964) for a counter-example). Thus the extension of Theorem 2.1 to continuous state space $(\mathcal{Y}, \mathcal{Q})$ depends upon the existence of certain "small sets" $A \times B \subseteq \mathcal{Y} \times \mathbb{Z}^+$ where A is a set of positive ϕ-measure and B is a finite set such that

$$\Sigma_{n=1}^{\infty} P_{(y_0, z_0)}[(Y_n, Z_n) \in (A, z)] < +\infty$$

for all $z \in B$. Lemmas 2.1 and 2.2 establish the existence of such a set.

LEMMA 2.1. *Let (Y_n) be a Markov chain with continuous state space $(\mathcal{Y}, \mathcal{Q})$. Then for any $n_0 \in \mathbb{N}$, there exists $A \in \mathcal{Q}$ with $\phi(A) > 0$, $m \in \mathbb{N}$ and $\varepsilon > 0$ such that*

$$P_{(y,z)}\Big(\bigcup_{j=1}^{m}[Z_j = 0]\Big) \geqslant \varepsilon \qquad \text{for all} \quad (y, z) \in A \times \{1, \cdots, n_0\}.$$

PROOF. Let $F = \mathcal{Y} \times \{0\}$. Since $F^0 = \{(y, z) : L[(y, z), F] = 0\}$ is empty or closed, the indecomposability of $\mathcal{Y} \times \{0, 1, 2, \cdots\}$ implies F^0 is empty. Thus $L[(y, z), F] > 0$ for all $(y, z) \in \mathcal{Y} \times \{0, 1, 2, \cdots\}$. If we define

$$A_{m,\varepsilon} = \Big[(y, z) : P_{(y,z)}\Big(\bigcup_{j=1}^{m}[(Y_j, Z_j) \in F]\Big) \geqslant \varepsilon\Big]$$

then

$$\bigcup_{n}^{\infty} \bigcup_{m}^{\infty} A_{m, 1/n} = \mathcal{Y} \times \{0, 1, 2, \cdots\} \tag{2.12}$$

since $L[(y, z), F] > 0$ for every (y, z). Wlog assume $\phi(\mathcal{Y})$ is finite. For each fixed z, it follows from (2.12) that there exists finite $m(z)$, $n(z)$ such that letting $B_z = \{y : (y, z) \in A_{m(z), 1/n(z)}\}$, we have $\phi(B_z) > \phi(\mathcal{Y})(1 - 1/2^{z+1})$. Now given $n_0 \in \mathbb{N}$, let

$$m = \max_{1 \leqslant z \leqslant n_0}\{m(z)\}$$

and

$$\varepsilon = \min_{1 \leqslant z \leqslant n_0}\{1/n(z)\}.$$

Let $A = \bigcap_{z=1}^{n_0} B_z$. Then

$$\phi(A) \geqslant \phi(\mathcal{Y})\big(1 - \Sigma_{z=1}^{n_0} 1/2^{z+1}\big) > 0 \tag{2.13}$$

and the conditions of the lemma are satisfied for this A, m, and ε. □

LEMMA 2.2. *Let F and G be subsets of $\mathcal{Y} \times \{0, 1, 2, \cdots\}$ such that F is closed and $F \cap G = \varnothing$. Suppose there exists $m \in \mathbb{N}$ and $\varepsilon > 0$ such that*

$$P_{(y,z)}\Big(\bigcup_{j=1}^{m}[(Y_j, Z_j) \in F]\Big) \geqslant \varepsilon \qquad \text{for} \quad (y, z) \in G.$$

Then $\Sigma_{n=1}^{\infty} P_{(y_0, z_0)}[(Y_n, Z_n) \in G] < +\infty$ for any $(y_0, z_0) \in \mathcal{Y} \times \mathbb{Z}^+$.

PROOF. Let $\tau^{(n)}$ denote the time of the nth visit the (Y_n, Z_n) process pays to G. Then by hypothesis there exists $m \in \mathbb{N}$ such that for any $(y, z) \in G$, and using the fact that $F \cap G = \varnothing$ with F closed,

$$P_{(y,z)}[\tau^{(m+1)} < +\infty] \leqslant P_{(y,z)}\Big(\bigcup_{j>m}[(Y_j, Z_j) \in G]\Big) \leqslant 1 - \varepsilon. \tag{2.14}$$

Now

$$\begin{aligned} P_{(y_0, z_0)}[\tau^{(n+m+1)} < +\infty] &= P_{(y_0, z_0)}[\tau^{(n)} < +\infty] P_{(y_0, z_0)}[\tau^{(n+m+1)} < +\infty \mid \tau^{(n)} < +\infty] \\ &= E_{(y_0, z_0)}\{\mathfrak{X}_{[\tau^{(n)} < +\infty]} P_{(Y_{\tau^{(n)}}, Z_{\tau^{(n)}})}[\tau^{(m+1)} < +\infty]\} \\ &\leqslant (1 - \varepsilon) P_{(y_0, z_0)}[\tau^{(n)} < +\infty] \end{aligned}$$

by (2.14) and the strong Markov property. Iterating this relation gives

$$P_{(y_0, z_0)}[\tau^{(k(m+1))} < +\infty] \leqslant (1 - \varepsilon)^k. \tag{2.15}$$

If we let V be the number of times (Y_n, Z_n) visits G, then

$$\begin{aligned} \Sigma_{n=1}^{\infty} P_{(y_0, z_0)}[(Y_n, Z_n) \in G] \\ = E_{(y_0, z_0)}(V) = \Sigma_{n=1}^{\infty} P_{(y_0, z_0)}[\tau^{(n)} < +\infty] \end{aligned}$$

and by (2.15)

$$\leqslant \Sigma_{n=1}^{\infty} (1 - \varepsilon)^{n/(m+1)} < +\infty. \qquad \square$$

The following lemma (cf. Cogburn (1975)) is also needed for the proof of Theorem 2.2.

LEMMA 2.3. *Let (Y_n) be a uniformly ϕ-recurrent Markov chain on general state space $(\mathcal{Y}, \mathcal{Q})$. Let $\delta > 0$. Then there exist constants $a < +\infty$, $0 < b < 1$ (depending on δ) such that for all $A \in \mathcal{Q}$ with $\phi A \geqslant \delta$,*

$$\sup_{y \in \mathcal{Y}} P_y[\tau_A \geqslant m] \leqslant a b^m.$$

THEOREM 2.2. *Let (Y_n) be a uniformly ϕ-recurrent Markov chain with continuous state space $(\mathcal{Y}, \mathcal{Q})$. Then property* (2.1) *holds.*

PROOF. Let $(y_0, z_0) \in \mathcal{Y} \times \mathbb{Z}^+$, $z \in \mathbb{Z}^+$. Let $F = \mathcal{Y} \times \{0\}$. This is a closed set in $\mathcal{Y} \times \mathbb{Z}_0$. Wlog assume $\phi(\mathcal{Y}) < +\infty$ and let $\delta = \phi(\mathcal{Y})/2 > 0$. By Lemma 2.3, there exist constants $a < +\infty$, $0 < b < 1$ (depending on δ) such that whenever $\phi(A) \geqslant \delta$,

$$\sup_{y \in \mathcal{Y}} P_y[\tau_A \geqslant m] \leqslant a b^m.$$

Now choose m such that $ab^m \leqslant \frac{1}{2}$ and n_0 such that $n_0 \geqslant z + m$. From Lemma 2.1, there exists a set A with $\phi(A) \geqslant \phi(\mathcal{Y})/2$ (by 2.13), and there exist $\varepsilon > 0$ and $m' \in \mathbb{N}$ so that

$$P_{(y,z')}\Big(\bigcup_{j=1}^{m'}[(Y_j, Z_j) \in F]\Big) \geqslant \varepsilon \qquad \text{for all} \quad (y, z') \in A \times \{1, \cdots, n_0\}.$$

By Lemma 2.3

$$P_{(y_0, z_0)}\Big(\bigcup_{j=1}^{m}[Y_{n+j} \in A] | (Y_n, Z_n) = (y, z)\Big) \geq \tfrac{1}{2}$$

uniformly in y. Furthermore if we let $G = A \times \{1, \cdots, n_0\}$, then Lemma 2.2 gives

$$\Sigma_{n=1}^{\infty} P_{(y_0, z_0)}[(Y_n, Z_n) \in G] < +\infty. \quad (2.16)$$

Using identical methods as in the proof of Theorem 2.1 (see (2.5)–(2.9) with ε replaced by $\frac{1}{2}$ and y' replaced by A), we obtain

$$\begin{aligned}
&\tfrac{1}{2}\Sigma_{n=1}^{\infty} P_{(y_0, z_0)}[Z_n = z] \\
&\quad \leq \Sigma_{j=1}^{m}\Sigma_{n=1}^{\infty} P_{(y_0, z_0)}[(Y_{n+j}, Z_{n+j}) \in (A \times \{z \pm i, 0 \leq i \leq m\} \cap \mathbb{Z}^+)] \\
&\qquad + \Sigma_{j=1}^{m}\Sigma_{n=1}^{\infty} P_{(y_0, z_0)}[Z_n = z, Z_{n+j} = 0] \\
&\quad \leq m\{\Sigma_{n=1}^{\infty} P_{(y_0, z_0)}[(Y_n, Z_n) \in G]\} + m^2 < +\infty,
\end{aligned}$$

the last inequality following by (2.16). Thus $\Sigma_{n=1}^{\infty} P_{(y_0, z_0)}[Z_n = z] < +\infty$ and the theorem is proved. □

As Example 2.1 below shows, the assumption of uniform ϕ-recurrence of the control process (Y_n) is required to insure that (Z_n) satisfies the instability condition (2.1). However, by using the idea of a uniform set for $\mathcal{Y}$ (defined below), we can assume that (Y_n) is only ϕ-recurrent and show the result holds for a suitable subsequence (Z_n^*) of (Z_n). Let π be an invariant distribution for the ϕ-recurrent chain (Y_n) such that ϕ is absolutely continuous with respect to π.

DEFINITION 2.1. *A set $D \in \mathcal{Q}$ is uniform if for every $B \in \mathcal{Q}$ with $\pi B > 0$*

$$\sup_{y \in D} P_y[\tau_B \geq n] \to 0 \quad \text{as} \quad n \to \infty.$$

For this result, we will assume that D is a ϕ-positive uniform set for $\mathcal{Y}$ with $P_\phi(D) = 1$. (Since $\mathcal{Q}$ is separable, there exist uniform sets D with positive ϕ-measure and in fact there exist uniform sets D_n so that $\mathcal{Y} - \bigcup_{n=1}^{\infty} D_n$ is not properly essential (cf. Cogburn (1975)).)

PROPOSITION 2.2. *Let (Y_n) be a ϕ-recurrent Markov chain with continuous state space $(\mathcal{Y}, \mathcal{Q})$. Then property* (2.1) *holds for the subsequence (Y_n^*, Z_n^*) of (Y_n, Z_n) where (Y_n^*, Z_n^*) is the process on $D \times \mathbb{Z}_0$.*

PROOF. Let $(y_0, z_0) \in D \times \mathbb{Z}_0$, $z \in \mathbb{Z}^+$. The fact that D is a uniform set for $\mathcal{Y}$ says that given $\varepsilon' > 0$, there exists $\nu \in \mathbb{N}$ such that

$$\sup_{y \in D} P_y[\tau_D > \nu] < \varepsilon'. \quad (2.17)$$

We assume wlog that $\phi(D) < +\infty$. From Lemma 2.1 (since (Y_n^*) is a Markov chain with state space $(D, \mathcal{Q}_D)$), there exists $A \in \mathcal{Q}_D$ with $\phi(A) > 0$, there exists $m' \in \mathbb{N}$ and $\varepsilon > 0$ so that

$$P_{(y', z')}\Big(\bigcup_{j=1}^{m'}[(Y_j^*, Z_j^*) \in D \times \{0\}]\Big) \geq \varepsilon,$$

for all $(y', z') \in A \times \{1, \cdots, n_0\}$ with $n_0 = z + mv$ where $m \in \mathbb{N}$ satisfies

$$(2.18) \qquad P_{(y_0, z_0)}\Big(\bigcup_{i=1}^{m}[Y_{n+j}^* \in A] \big| (Y_n^*, Z_n^*) = (y, z)\Big) \geq \tfrac{1}{2}.$$

(Of course, $(Y_n^*, Z_n^*) = (Y_{\tau_D^{(n)}}, Z_{\tau_D^{(n)}})$.) Such an m can be found because the assumption that D is a uniform set in $\mathcal{Y}$ with positive ϕ-measure implies that the process (Y_n^*) on D is uniformly ϕ-recurrent (cf. Cogburn (1975) and Orey (1971)). Thus Lemma 2.3 can be applied to (Y_n^*) since $\phi(A) > 0$. Furthermore (as in the proof of Theorem 2.2),

$$(2.19) \qquad \Sigma_{n=1}^{\infty} P_{(y_0, z_0)}[(Y_n^*, Z_n^*) \in A \times \{1, \cdots, n_0\}] < +\infty.$$

Now define sets A_j, B_n, C_n as follows:

$$A_j = \{\omega : \tau_D^{(j)} - \tau_D^{(j-1)} > v\}, \qquad j \geq 1,$$

where ω is a sample sequence of (Y_n, Z_n), and

$$(2.20) \qquad B_n = \bigcap_{j=1}^{m} A_{n+j}^c, \qquad C_n = \bigcup_{j=1}^{m}[Y_{n+j}^* \in A].$$

Observe that $B_n, C_n \in \mathcal{F}_{\tau_D^{(n)}}$, the σ-algebra generated by $\{Y_k^*\}$ for $k > \tau_D^{(n)}$.

Note that for $y \in \mathcal{Y}$,

(i) $P_{(y_0, z_0)}[B_n^c | (Y_n^*, Z_n^*) = (y, z)] \leq \Sigma_{j=1}^m P_\phi(A_{n+j})$ by the strong Markov property which applies from observing (2.20). But (2.17) says $P_\phi(A_1) < \varepsilon'$ since ϕ carries D and another application of the strong Markov property gives $P(A_{n+j}) < \varepsilon'$. By choosing $\varepsilon' < 1/4m$ and using (i), we obtain

$$P_{(y_0, z_0)}[B_n^c | (Y_n^*, Z_n^*) = (y, z)] \leq \tfrac{1}{4}.$$

(ii) From (2.18), we have

$$P_{(y_0, z_0)}[C_n | (Y_n^*, Z_n^*) = (y, z)] \geq \tfrac{1}{2}.$$

It follows from (i) and (ii) that

$$P_{(y_0, z_0)}[B_n \cap C_n | (Y_n^*, Z_n^*) = (y, z)] \geq P_{(y_0, z_0)}[C_n | (Y_n^*, Z_n^*) = (y, z)] - P_{(y_0, z_0)}[B_n^c | (Y_n^*, Z_n^*) = (y, z)] \geq \tfrac{1}{4},$$

hence

$$P_{(y_0, z_0)}[B_n \cap C_n | Z_n^* = z] \geq \tfrac{1}{4}.$$

Thus

$$P_{(y_0, z_0)}.([Z_n^* = z] \cap B_n \cap C_n) = P_{(y_0, z_0)}[Z_n^* = z] P_{(y_0, z_0)}[B_n \cap C_n | Z_n^* = z] \geq 4^{-1} P_{(y_0, z_0)}[Z_n^* = z].$$

One clearly sees that $|Z^*_{n+j} - z| \leqslant m\nu$ on $B_n, j \leqslant m$ where we may think of j as the number of steps between returns to A. Thus

$$\begin{aligned} P_{(y_0, z_0)}([Z^*_n = z] \cap B_n \cap C_n) &\leqslant \Sigma_{j=1}^m P_{(y_0, z_0)}([(Y^*_{n+j}, Z^*_n) \in (A, z)] \cap B_n) \\ &\leqslant \Sigma_{j=1}^m \{P_{(y_0, z_0)}[(Y^*_{n+j}, Z^*_{n+j}) \in (A, B)] \\ &\quad + P_{(y_0, z_0)}[Z^*_n = z, Z^*_{n+j} = 0]\} \end{aligned}$$

where

$$B = \{z \pm i : 0 \leqslant i \leqslant m\nu\} \cap \mathbb{Z}^+.$$

Using the bound $P_{(y_0, z_0)}[Z^*_n = z] \leqslant 4P_{(y_0, z_0)}([Z^*_n = z] \cap B_n \cap C_n)$ derived in notes (i) and (ii), and using (2.19), the assertion $\Sigma_{n=1}^\infty P_{(y_0, z_0)}[Z^*_n = z] < +\infty$ now follows from the above inequalities as in Theorem 2.2 (see (2.5)–(2.9)) to obtain property (2.1). □

The following example shows that positive recurrence of the environmental process (Y_n) is not enough for (Z_n) to satisfy the instability condition (2.1). Indeed the example shows that the (Y_n) process may be taken so that all moments of the return of (Y_n) to a distinguished state of $\mathcal{Y}$ exist, yet (2.1) fails to hold.

EXAMPLE 2.1. The state space for (Y_n) is the lattice set $\mathcal{Y} = \{(i, j) \in \mathbb{N} \times \mathbb{N} : 1 \leqslant j \leqslant i, i \geqslant 0\}$, with transition probabilities defined as follows:

$$\begin{aligned} P[Y_{n+1} = (i, i) | Y_n = (0, 0)] &= f_i > 0 \quad \text{for} \quad i = 2, 4, 6, \cdots \\ &= 0 \qquad\quad \text{for} \quad i = 1, 3, 5, \cdots \end{aligned}$$

$$P[Y_{n+1} = (i, j-1) | Y_n = (i, j)] = 1, \qquad 1 < j \leqslant i$$

$$P[Y_{n+1} = (0, 0) | Y_n = (i, 1)] = 1.$$

The successive returns to (0, 0) represent a periodic recurrent event with the distribution $P[\tau = 2k + 1] = f_{2k}, k = 1, 2, \cdots$ for the recurrence time τ of (0, 0). By choosing the f_i's so that the mean recurrence time of (0, 0) is finite, i.e., so that $\Sigma_{k=1}^\infty (2k + 1) f_{2k} < +\infty$, the (Y_n) chain will be made positive recurrent (e.g., let $f_{2k} = 1/(2k)^\alpha, \alpha > 2$).

The birth and death probabilities, i.e., the transition probabilities of (Z_n), are defined as follows:

$$\begin{aligned} q_z^{(i,j)} &= 1 \qquad\quad \text{for} \quad i/2 < j \leqslant i \quad \text{and} \quad z \geqslant 2, \\ p_z^{(i,j)} &= 1 \qquad\quad \text{for} \quad 1 \leqslant j \leqslant i/2 \quad \text{and} \quad z \geqslant 1, \\ q_1^{(i,j)} &= 1/2^i \qquad \text{for} \quad j = i/2 + 1 \\ &= 0 \qquad\quad \text{for} \quad i/2 + 1 < j \leqslant i, \\ r_1^{(i,j)} &= 1 - 1/2^i \quad \text{for} \quad j = i/2 + 1 \\ &= 1 \qquad\quad \text{for} \quad i/2 + 1 < j \leqslant i, \end{aligned}$$

$r_0^{(i,j)} = r_z^{(0,0)} = 1, z \geqslant 1, (i, j) \in \mathcal{Y}$.

Note that these probabilities allow all positive states of (Z_n) to lead to 0. Since the environmental process (Y_n) can be taken to be an irreducible, recurrent Markov chain by a suitable definition of the f_i's as noted above, the example fits the specifications of our model.

Let $\tau(n) = n$th return time to state $y = (0, 0)$ for (Y_n) and fix $(y_0, z_0) \in \mathcal{Y} \times \mathbb{Z}^+$.

CLAIM 1. $Z = \lim_{n\to\infty} Z_{\tau(n)}$ exists a.s. and $P_{(y_0, z_0)}[Z_{\tau(n)} \neq 0$ for all n and $Z < +\infty] = 0$.

PROOF. (For convenience, we will suppress the subscript (y_0, z_0).) Suppose that $Z_{\tau(n)} \neq 0$ for all n. We will show that $Z_{\tau(n)}$ is an increasing sequence. Suppose at time $\tau(n)$, $Z_{\tau(n)} = z_0$ where $z_0 \geqslant 2$ and $Y_{\tau(n)+1} = (i, i)$ where $i/2 < z_0$. (If $z_0 = 1$, i.e., if $Z_{\tau(n)} = 1$, then $Z_{\tau(n+1)} \geqslant 2$ (assuming $Z_n \neq 0$ for all n) since (Z_n) remains at 1 for some period of time, and then moves to the right when $j < i/2$.) During the time span $\tau(n) + 1$ to $\tau(n) + 1 + i/2$, when the control chain (Y_n) has moved deterministically from (i, i) to $(i, i/2)$, the (Z_n) process has moved $i/2$ steps left since $q_z^{(i,j)} = 1$ for $i/2 + 1 \leqslant j \leqslant i$ and $2 \leqslant z \leqslant z_0$. During the time span $\tau(n) + i/2 + 1$ to $\tau(n + 1)$ when the control has moved deterministically from $(i, i/2)$ to $(0, 0)$, (Z_n) has moved right $i/2$ steps since $p_z^{(i,j)} = 1$ for $1 \leqslant j \leqslant i/2$ and $1 \leqslant z \leqslant z_0$. Thus $Z_{\tau(n+1)} = z_0$.

Consider now the case that $i/2 \geqslant z_0 \geqslant 2$. Then because $q_z^{(i,j)} = 1$ for $i/2 < j \leqslant i$, $1 < z \leqslant z_0$, (Z_n) will hit state 1 for some j, $i/2 < j \leqslant i$. It will remain at 1 until $j = i/2 + 1$. But since $Z_{\tau(n)} \neq 0$ for all n, the process will remain at 1 when $j = i/2 + 1$ as well since $r_1^{(i, i/2+1)} = 1 - 1/2^i$. It is at state 1 $(i/2 - z_0 + 1)$ times and will be at state $i/2 + 1 > z_0$ at time $\tau(n + 1)$ since $p_z^{(i,j)} = 1$ for $1 \leqslant j \leqslant i/2$ and $z \geqslant 1$. Thus $Z_{\tau(n+1)} = i/2 + 1 > z_0$ and so $Z_{\tau(n)}$ increases, and $Z = \lim_{n\to\infty} Z_{\tau(n)}$ exists a.s. Suppose ζ is such that $1 \leqslant \zeta < +\infty$. Clearly $[1 \leqslant Z \leqslant \zeta] = \bigcap_{n=1}^{\infty}[1 \leqslant Z_{\tau(n)} \leqslant \zeta]$. We calculate $P\left(\bigcap_{n=1}^{\infty}[1 \leqslant Z_{\tau(n)} \leqslant \zeta]\right)$ and show that this probability is 0. By the chain rule for probabilities,

$$(2.21)\quad P\left(\bigcap_{n=1}^{N}[1 \leqslant Z_{\tau(n)} \leqslant \zeta]\right) = P[1 \leqslant Z_{\tau(1)} \leqslant \zeta] \cdot P[1 \leqslant Z_{\tau(2)} \leqslant \zeta \mid 1 \leqslant Z_{\tau(1)} \leqslant \zeta] \cdots \times P[1 \leqslant Z_{\tau(N)} \leqslant \zeta \mid 1 \leqslant Z_{\tau(j)} \leqslant \zeta,\ 1 \leqslant j \leqslant N - 1]$$

and by the strong Markov property,

$$P[1 \leqslant Z_{\tau(n+1)} \leqslant \zeta \mid 1 \leqslant Z_{\tau(j)} \leqslant \zeta,\ 1 \leqslant j \leqslant n] = P[1 \leqslant Z_{\tau(n+1)} \leqslant \zeta \mid 1 \leqslant Z_{\tau(n)} \leqslant \zeta].$$

From the above discussion, it is clear that $Z_{\tau(n)} > \zeta$ if and only if $i/2 \geqslant \zeta$ so that

$$P[Z_{\tau(n+1)} > \zeta \mid 1 \leqslant Z_{\tau(n)} \leqslant \zeta] = P[Y_{\tau(n)+1} \in \{(i, i) : i \geqslant 2\zeta\}] = \textstyle\sum_{j=\zeta}^{\infty} f_{2j}.$$

Denote the right-hand side of this expression by $t_{2\zeta}$. Then

$$(2.22)\qquad P[1 \leqslant Z_{\tau(n+1)} \leqslant \zeta \mid 1 \leqslant Z_{\tau(n)} \leqslant \zeta] \leqslant 1 - t_{2\zeta}.$$

Using (2.21) and (2.22), we have

$$P\left(\bigcap_{n=1}^{\infty}[1 \leqslant Z_{\tau(n)} \leqslant \zeta]\right) = P[1 \leqslant Z_{\tau(1)} \leqslant \zeta] \cdot \Pi_{n=1}^{\infty} P[1 \leqslant Z_{\tau(n+1)} \leqslant \zeta \mid 1 \leqslant Z_{\tau(n)} \leqslant \zeta]$$
$$\leqslant P[1 \leqslant Z_{\tau(1)} \leqslant \zeta] \cdot \Pi_{n=1}^{\infty}(1 - t_{2\zeta}),$$

where the product $\Pi_{n=1}^{\infty}(1 - t_{2\zeta})$ is understood to be the factor $(1 - t_{2\zeta})$ multiplied by itself infinitely often. Since $1 - t_{2\zeta} < 1$ this product is 0. Hence $P[1 \leqslant Z \leqslant \zeta] = 0$, which proves Claim 1.

CLAIM 2. $[Z_n = 1 \text{ infinitely often}] \supset [\lim_{n\to\infty} Z_{\tau(n)} = +\infty]$.

PROOF. If $\lim_{n\to\infty} Z_{\tau(n)} = +\infty$, then $Z_{\tau(n)}$ increases without bound. This is only possible if Z_n visits state 1 infinitely often, for an increase in $Z_{\tau(n)}$ during the time span $\tau(n)$ to $\tau(n+1)$ meant that Z_n hit state 1 within the time block $\tau(n)$ to $\tau(n+1)$.

CLAIM 3. $P_{(y_0, z_0)}[Z_{\tau(n)} = 0 \text{ for some } n] < 1$.

PROOF. Define a process $\tilde{Y}_{\tau(n)+1}$ by $Y_{\tau(n)+1} = (\tilde{Y}_{\tau(n)+1}, \tilde{Y}_{\tau(n)+1})$. Then $(\tilde{Y}_{\tau(n)+1})$ is an independent, identically distributed sequence of random variables with probability distribution $\{f_i\}$, $i = 2, 4, 6, \cdots$. Let $\sigma(1) = \min\{\inf\{\tau(n) : \tilde{Y}_{\tau(n)+1} > 2z_0\}, +\infty\}$ and $\sigma(k) = \min\{\inf\{\tau(n) : \tau(n) > \sigma(k-1) \text{ and } \tilde{Y}_{\tau(n)+1} > 2(z_0 + k - 1)\}, +\infty\}$. Note that since (Y_n) is a recurrent Markov chain, the state $y_0 \in \mathcal{Y}$ is hit infinitely often. Also $P_{y_0}[\tilde{Y}_n > 2(z_0 + k - 1)] > 0$ for all n and k so that $\sigma(k) < +\infty$ for all k with probability 1. Now

$$P_{(y_0, z_0)}[Z_{\sigma(k+1)} = 0, Z_{\sigma(k)} \neq 0] = P_{(y_0, z_0)}[Z_{\sigma(k+1)} = 0, Z_{\sigma(k)} \geqslant z_0 + k - 1]$$
$$\leqslant \Sigma_{i=2(z_0+k-1)}^{\infty} f_i/2^i.$$

The inequality above follows because in order that the Z_n process be absorbed at 0 during the time interval $\sigma(k)$ to $\sigma(k+1)$, it must happen that $Y_{\tau(n)+1} = (i, i)$ with $i > 2(z_0 + k - 1)$, i.e., it must happen that $\tilde{Y}_n > 2(z_0 + k - 1)$ (see the proof of Claim 1). The factor $1/2^i$ follows from the definition of the $q_1^{(i,j)}$'s. Thus

$$(2.23)\quad \Sigma_{k=1}^{\infty} P_{(y_0, z_0)}[Z_{\sigma(k+1)} = 0, Z_{\sigma(k)} \neq 0] \leqslant \Sigma_{k=1}^{\infty} \Sigma_{i=2(z_0+k-1)}^{\infty} f_i/2^i \leqslant \Sigma_{k=1}^{\infty} 4^{-(z_0+k-1)} < 1.$$

But $Z_{\sigma(1)} > z_0 \neq 0$ and $Z_{\sigma(k)} \neq 0$ implies $Z_{\sigma(j)} \neq 0$ for $j \leqslant k$ so that

$$P_{(y_0, z_0)}[Z_n = 0 \quad \text{for some} \quad n] = P_{(y_0, z_0)}[Z_{\tau(n)} = 0 \quad \text{for some} \quad n]$$
$$= \Sigma_{k=1}^{\infty} P_{(y_0, z_0)}[Z_{\sigma(k+1)} = 0, Z_{\sigma(k)} \neq 0] < 1$$

by (2.23).

This proves Claim 3. Now from Claims 1 and 3, it follows that $P_{(y_0, z_0)}[Z_{\tau(n)} \neq 0$ for all n and $Z = +\infty] > 0$, so that by Claim 2, $P_{(y_0, z_0)}[Z_n = 1$ infinitely often$] > 0$ thus contradicting the instability condition (2.1). □

3. Extinction of the birth and death chain in a random environment. In this section, we state and prove conditions for which $Z_n \to 0$, considering both the cases when the "transition probabilities" $\{p_z^{(y)}, q_z^{(y)}, r_z^{(y)}\}$, $(y, z) \in \mathbb{S}$ do or do not depend on the z-state of (Z_n). When no such dependence is assumed, we refer to these probabilities as homogeneous and denote them as $\{p^{(y)}, q^{(y)}, r^{(y)}\}, y \in \mathcal{Y}$; of course, the state set $\mathcal{Y} \times \{0\}$ remains closed in $\mathbb{S}$. For the homogeneous case, we assume that the state space $\mathcal{Y}$ of (Y_n) is discrete and that (Y_n) possesses an invariant probability distribution $\{\pi_y\}$ (which is certainly true when (Y_n) is uniformly ϕ-recurrent) and we give a necessary and sufficient condition for $Z_n \to 0$ with probability 1. The result is stated as follows:

THEOREM 3.1. *Assume that $p^{(y)}$, $q^{(y)}$, $r^{(y)}$ are homogeneous probabilities associated with (Z_n). Then $P_{(y_0, z_0)}[Z_n \to 0] = 1$ where (y_0, z_0) is any initial state for (Y_n, Z_n) if and only if $\Sigma_{y \in \mathcal{Y}}(p^{(y)} - q^{(y)})\pi_y \leq 0$.*

PROOF. Fix $y_0 \in \mathcal{Y}$ and let $\tau^{(n)}$ be the nth return time to y_0. Let $\tau = \tau^{(1)}$. Let

$$D_n = Z_{\tau^{(n+1)}} - Z_{\tau^{(n)}}.$$

Since the probabilities $\{p^{(y)}, q^{(y)}, r^{(y)}\}$ are independent of the z state, the $\{D_n\}$ form a sequence of independent, identically distributed random variables, and the $Z_{\tau^{(n)}}$ form a generalized random walk. Note that $D_1 = \Sigma_{k=0}^{\tau-1}(Z_{k+1} - Z_k)$. By the smoothing property of expectation,

$$\begin{aligned} E_{(y_0, z_0)}D_1 &= E_{(y_0, z_0)}\{\Sigma_{k=0}^{\tau-1}E[(Z_{k+1} - Z_k)|(Y_k, Z_k)]\} \\ &= E_{(y_0, z_0)}\{\Sigma_{k=0}^{\tau-1}(p^{(Y_k)} - q^{(Y_k)})\} \\ &= (1/\pi_{y_0})\Sigma_{y \in \mathcal{Y}}(p^{(y)} - q^{(y)})\pi_y. \end{aligned}$$

(The last equality follows from formula (16), Chung (1960), page 82.)

By the general results in the theory of random walks, $E_{(y_0, z_0)}D_1 = 0$ if and only if $(Z_{\tau^{(n)}})$ forms a recurrent chain and $E_{(y_0, z_0)}D_1 < 0$ if and only if $(Z_{\tau^{(n)}})$ forms a transient chain with negative drift. Now place an absorbing barrier at 0 and consider D_n in this context. If $\Sigma_{y \in \mathcal{Y}}p^{(y)}\pi_y = \Sigma_{y \in \mathcal{Y}}q^{(y)}\pi_y$, then we know from the above that $(Z_{\tau^{(n)}})$ will be a recurrent chain and thus it is sure to absorbed at 0. If $\Sigma_{y \in \mathcal{Y}}p^{(y)}\pi_y < \Sigma_{y \in \mathcal{Y}}q^{(y)}\pi_y$ so that a drift left occurs, then 0 is certain to be entered starting from a positive z-state. Conversely, if extinction is certain, then either the process $(Z_{\tau^{(n)}})$ is transient with a drift to $-\infty$ or recurrent. Thus $E_{(y_0, z_0)}D_1 \leq 0$, and so $\Sigma_{y \in \mathcal{Y}}p^{(y)}\pi_y \leq \Sigma_{y \in \mathcal{Y}}q^{(y)}\pi_y$. □

Let us now consider the extinction problem in the case that the "transition probabilities" $\{p_z^{(y)}, q_z^{(y)}, r_z^{(y)}\}$, $(y, z) \in \mathbb{S}$ of (Z_n) are nonhomogeneous. The following result, which gives a condition for $P_{(y_0, z_0)}[Z_n \to 0] = 1$ for any initial state (y_0, z_0), makes no assumption on the environmental process so that (Y_n) may be taken to be any stochastic process on general state space. Define the following quantities: $\underline{q}_z = \inf_{y \in \mathcal{Y}} q_z^{(y)}, \bar{p}_z = \sup_{y \in \mathcal{Y}} p_z^{(y)}, \bar{r}_z = 1 - (\bar{p}_z + \underline{q}_z)$. Analogously define quantities $\bar{q}_z, \underline{p}_z, \underline{r}_z$.

THEOREM 3.2. *Assume that the quantities $\bar{p}_z$ and $\underline{p}_z$ are positive for all $z \in \mathbb{Z}^+$, and that $q_z^{(y)} + r_z^{(y)} \geqslant q_{z+1}^{(y)}$ for all $(y, z) \in \mathcal{Y} \times \mathbb{Z}^+$.*

(1) *If $\Sigma_{z=1}^{\infty} \Pi_{k=1}^{z}(\underline{q}_k/\bar{p}_k) = +\infty$, then $P_{(y_0, z_0)}[Z_n \to 0] = 1$ for any initial state $(y_0, z_0) \in \mathbb{S}$.*

(2) *If $\Sigma_{z=1}^{\infty} \Pi_{k=1}^{z}(\bar{q}_k/\underline{p}_k) < +\infty$, then $P_{(y_0, z_0)}[Z_n \to 0] < 1$ for any initial state $(y_0, z_0) \in \mathbb{S}$.*

PROOF. (1) Let $(\underline{Z}_n)$ be an ordinary birth and death chain with transition probabilities $\{\underline{q}_z, \bar{p}_z, \bar{r}_z\}$. Denote by $\mathfrak{B}$ the σ-algebra generated by $Y_0, Y_1, Y_2, \cdots$ and define $\underline{F}_n(z) = P_{z_0}[\underline{Z}_n \leqslant z]$ and $F_n^{\mathfrak{B}}(z) = P_{(y_0, z_0)}[Z_n \leqslant z | \mathfrak{B}]$, $n = 0, 1, 2, \cdots$ so that $\underline{F}_n$ and $F_n^{\mathfrak{B}}$ are, respectively, the distribution function and conditional distribution function of the random variables $\underline{Z}_n$ and Z_n. This definition sets $\underline{Z}_0 = z_0 = Z_0$ so the two chains start at the same state z_0. Thus both $\underline{F}_0$ and $F_0^{\mathfrak{B}}$ are distributions degenerate at z_0 so that $\underline{F}_0 = F_0^{\mathfrak{B}}$. We will show by induction on n that $F_n^{\mathfrak{B}}(z) \geqslant \underline{F}_n(z)$ for all z and for all $(y_0, y_1, \cdots)$ realizations of $Y_0, Y_1, \cdots$. Assume $F_n^{\mathfrak{B}} \geqslant \underline{F}_n$. Set $p_{-1}^{(y)} = 0$ for all y by convention. Now

$$(3.1)\quad F_{n+1}^{\mathfrak{B}}(z) = \Sigma_{\zeta=0}^{z} P_{(y_0, z_0)}[Z_{n+1} = \zeta | \mathfrak{B}]$$

$$= \Sigma_{\zeta=0}^{z}\{p_{\zeta-1}^{(y_n)} P_{(y_0, z_0)}[Z_n = \zeta - 1 | \mathfrak{B}] + r_{\zeta}^{(y_n)} P_{(y_0, z_0)}[Z_n = \zeta | \mathfrak{B}]$$
$$+ q_{\zeta+1}^{(y_n)} P_{(y_0, z_0)}[Z_n = \zeta + 1 | \mathfrak{B}]\}.$$

Collecting terms with $Z_n = \zeta$ for $0 \leqslant \zeta \leqslant z - 1$ and noting that $r_0^{(y_n)} = 1$, expression (3.1) becomes

$$(3.2)\quad \Sigma_{\zeta=0}^{z-1} P_{(y_0, z_0)}[Z_n = \zeta | \mathfrak{B}] + (q_z^{(y_n)} + r_z^{(y_n)}) P_{(y_0, z_0)}[Z_n = z | \mathfrak{B}]$$
$$+ q_{z+1}^{(y_n)} P_{(y_0, z_0)}[Z_n = z + 1 | \mathfrak{B}]$$

$$= F_n^{\mathfrak{B}}(z - 1) + (q_z^{(y_n)} + r_z^{(y_n)})(F_n^{\mathfrak{B}}(z)$$
$$- F_n^{\mathfrak{B}}(z - 1)) + q_{z+1}^{(y_n)}(F_n^{\mathfrak{B}}(z + 1) - F_n^{\mathfrak{B}}(z))$$
$$= p_z^{(y_n)} F_n^{\mathfrak{B}}(z - 1) + (q_z^{(y_n)} + r_z^{(y_n)} - q_{z+1}^{(y_n)}) F_n^{\mathfrak{B}}(z) + q_{z+1}^{(y_n)} F_n^{\mathfrak{B}}(z + 1)$$

$$(3.3)\quad \geqslant p_z^{(y_n)} \underline{F}_n(z - 1) + (q_z^{(y_n)} + r_z^{(y_n)} - q_{z+1}^{(y_n)}) \underline{F}_n(z) + q_{z+1}^{(y_n)} \underline{F}_n(z + 1)$$

by the induction hypothesis and the assumption that $q_z^{(y)} + r_z^{(y)} \geqslant q_{z+1}^{(y)}$ for all y. The above expression (3.3) becomes (in similar rearrangements used to derive the expression (3.2))

$$\begin{aligned}
&\underline{F}_n(z-1) + (q_z^{(y_n)} + r_z^{(y_n)})(\underline{F}_n(z) - \underline{F}_n(z-1)) + q_{z+1}^{(y_n)}(\underline{F}_n(z+1) - \underline{F}_n(z)) \\
&= \underline{F}_n(z-1) + (1 - p_z^{(y_n)})(\underline{F}_n(z) - \underline{F}_n(z-1)) + q_{z+1}^{(y_n)}(\underline{F}_n(z+1) - \underline{F}_n(z)) \\
&\geqslant \underline{F}_n(z-1) + (1 - \bar{p}_z)(\underline{F}_n(z) - \underline{F}_n(z-1) + \underline{q}_{z+1}(\underline{F}_n(z+1) - \underline{F}_n(z)) \\
&= \Sigma_{\zeta=0}^{z-1} P_{z_0}[\underline{Z}_n = \zeta] + (\underline{q}_z + \bar{r}_z) P_{z_0}[\underline{Z}_n = z] + \underline{q}_{z+1} P_{z_0}[\underline{Z}_n = z+1] \\
&= \Sigma_{\zeta=0}^{z} \{\bar{p}_{\zeta-1} P_{z_0}[\underline{Z}_n = \zeta - 1] + \bar{r}_\zeta P_{z_0}[\underline{Z}_n = \zeta] + \underline{q}_{\zeta+1} P_{z_0}[\underline{Z}_n = \zeta + 1]\} \\
&= \Sigma_{\zeta=0}^{z} P_{z_0}[\underline{Z}_{n+1} = \zeta] = \underline{F}_{n+1}(z).
\end{aligned}$$

Thus $F_{n+1}^{\mathfrak{B}} \geqslant \underline{F}_{n+1}$ for $n = 0, 1, \cdots$. Now by the classical theory of birth and death chains (see Karlin and Taylor (1975)), we know if $\Sigma_{z=1}^{\infty} \Pi_{k=1}^{z} [\underline{q}_k / \bar{p}_k] = +\infty$, then $\underline{F}_n(0) \uparrow 1$ as $n \to \infty$ so that $F_n^{\mathfrak{B}}(0) \uparrow 1$ and thus $P_{(y_0, z_0)}[Z_n = 0] = E_{(y_0, z_0)}(F_n^{\mathfrak{B}}(0)) \to 1$ by the monotone convergence theorem. But $P_{(y_0, z_0)}[Z_m = 0] \to P_{(y_0, z_0)}[Z_n = 0$ for some $n]$ as $m \to \infty$ so that $P_{(y_0, z_0)}[Z_n = 0$ for some $n] = 1$ and part (1) of the theorem is proved.

(2) In a similar fashion, we consider an ordinary birth and death chain $(\bar{Z}_n)$ with transition probabilities $\{\bar{q}_z, \underline{p}_z, \underline{r}_z\}$, and using obvious modifications of the argument in part (1), we can show $F_n^{\mathfrak{B}}(z) \leqslant \bar{F}_n(z)$ where $\bar{F}_n$ is the distribution function of the variate $\bar{Z}_n$, $n = 0, 1, 2, \cdots$. Now the classical theory tells us that if $\Sigma_{z=1}^{\infty} \Pi_{i=1}^{z} (\bar{q}_k / \underline{p}_k) < +\infty$, then $\bar{F}_n(0) < 1$ for all n and hence $F_n^{\mathfrak{B}}(0) < 1$ for all n. But then $P_{(y_0, z_0)}[Z_n = 0] = E_{(y_0, z_0)}(F_n^{\mathfrak{B}}(0)) < 1$ for all n so that $P_{(y_0, z_0)}[Z_n = 0$ for some $n] < 1$. □

REMARK. Note that this leads to a much stronger result: conditionally, given any sequence of Y_n's, i.e., any realization of the environmental process,

$$\begin{aligned}
u^{\mathfrak{B}}(y_0, z_0) &= P_{(y_0, z_0)}[Z_n = 0 \quad \text{for some} \quad n | \mathfrak{B}] \\
&\leqslant P_{z_0}[\bar{Z}_n = 0 \quad \text{for some} \quad n] = \bar{u}(z_0).
\end{aligned}$$

Since $\bar{u}(z_0)$ can be computed (indeed,

$$\bar{u}(z_0) = \frac{\Sigma_{z=z_0}^{\infty} \Pi_{k=1}^{z} (\bar{q}_k / \underline{p}_k)}{1 + \Sigma_{z=1}^{\infty} \Pi_{k=1}^{z} (\bar{q}_k / \underline{p}_k)}$$

(cf. Karlin and Taylor (1975))), a nice upper bound for the probability of extinction of the birth and death chain in a random environment initially at (y_0, z_0) is obtained.

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