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CONTROLLED SPIN-FLIP SYSTEMS

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We introduce a new tool into the study of spin-flip processes, which we call *controlled spin-flip systems*. For each spin-flip process we define a related class of controlled spin-flip systems. Our main theorem states that bounds on the behavior of a spin-flip process can be obtained by studying the behavior of the related controlled spin-flip system. Since controlled spin-flip systems are in general easier to work with than regular spin-flip processes (they correspond to finite state space Markov processes), our main theorem has applications to some of the important problems concerning spin-flip processes. In particular, we discuss several applications to the uniqueness problem. These include proofs of some new results, as well as new proofs of earlier results.

1. Introduction and preliminaries. We will start with a brief introduction to the theory of spin-flip process. Let V be a countable set, and let $\Xi = \{-1, 1\}^V$. Each element $\xi = (\xi(x))_{x\in V}$ of Ξ, or *configuration*, represents an assignment of $+$ or $-$ *spins* to each of the points or *sites* in V. A *spin-flip process* (defined rigorously below) is a type of stochastic process $(\xi_t)_{t\in\mathbb{R}+}$, with state space Ξ, whose dynamics are prescribed by certain *flip rates* $c = (c_x(\xi))_{x\in V,\,\xi\in\Xi}$. The flip rate $c_x(\xi)$ describes intuitively the rate at which the spin at a site x changes or *flips* from $\xi(x)$ to $-\xi(x)$ when the process is in state ξ.

To give a precise definition of a spin-flip process we need some notation. Assume that V and $\{-1, 1\}$ are given the discrete topology, and Ξ the product topology. Let $\Omega = \mathbb{D}([0, \infty), \Xi)$ be the path space of right continuous functions with left limits from $[0, \infty)$ to Ξ. For $t \in [0, \infty)$, let $\xi_t : \Omega \to \Xi$ be the evaluation map $\omega \mapsto \omega(t) = (\omega_t(x))_{x\in V}$. Let $\mathcal{B} = \sigma((\xi_t)_{t\in[0,\infty)})$ be the σ-algebra generated by the ξ_t. Also put $\mathcal{B}_0^t = \sigma((\xi_s)_{0\leqslant s\leqslant t})$ for $t \in [0, \infty)$. For any $\mathcal{B}_0^t$-stopping time τ, let $\mathcal{B}_0^\tau$ be the σ-algebra consisting of sets $\mathcal{Q} \in \mathcal{B}$ such that $\mathcal{Q} \cap \{\tau \leqslant t\} \in \mathcal{B}_0^t$ for all $t \in [0, \infty)$. For any bounded function $f : X \to \mathbb{R}$, where X is some arbitrary space, let $\|f\| = \sup_{x\in X}|f(x)|$. Let $\mathcal{C} = \mathcal{C}(\Xi)$ be the Banach space of continuous, real-valued functions with domain Ξ, with the norm $\|\cdot\|$. Let $\mathcal{V}_0$ be the collection of finite subsets of V. For $A \in \mathcal{V}_0$, let $\mathcal{F}^A = \{f \in \mathcal{C} : f(\xi) = f(\xi') \text{ whenever } \xi|_A = \xi'|_A\}$, and let $\mathcal{F} = \cup_{A\in\mathcal{V}_0}\mathcal{F}^A$. For $\xi \in \Xi$, $x \in V$, define ${}_x\xi \in \Xi$ by

$$(1.1)\qquad {}_x\xi(y) = \xi(y) \quad y \neq x$$
$$= -\xi(y) \quad y = x.$$

We are now prepared to give a definition of a spin-flip process. The most useful definition given so far is based on the work of Holley and Stroock in [7]:

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DEFINITION 1.1. Let $c = (c_x)_{x \in V}$ be a collection of bounded, Borel measurable functions from Ξ into $[0, \infty)$, called *flip rates*. Define an operator $G : \mathcal{F} \to \mathcal{C}$, called the *pregenerator with rates c*, by $(Gf)(\xi) = \sum_{x \in V} c_x(\xi)(f({}_x\xi) - f(\xi))$. For $\xi \in \Xi$, a probability P on $(\Omega, \mathcal{B})$ is called a *solution to the martingale problem for G starting from* ξ, or SMP(G, ξ), or a *spin-flip process with rates c, starting at* ξ, if

$$P(\xi_0 = \xi) = 1, \tag{1.2}$$

and

$$\text{for all} \quad f \in \mathcal{F}, \langle f(\xi_t) - \textstyle\int_0^t Gf(\xi_s)ds, P, \mathcal{B}_0^t \rangle \quad \text{is a martingale.} \tag{1.3}$$

For a detailed treatment of some of the properties of SMP's, we refer to [7], particularly Section 1 there. Proposition 2.4 and Lemma 2.9 below also help to give some intuitive feeling for the meaning of the flip rates.

The main purpose of this paper is to introduce a new tool, Theorem 2.14, which allows one to study the behavior of general spin-flip processes in terms of the behavior of certain related systems (*controlled spin-flip systems*) that have finite state spaces. The proof of Theorem 2.14 relies on an estimate (Lemma 2.9) which allows us to use the idea behind Bellman's principle of dynamic programming. This principle has already been applied in a different way by Krylov [11] to get existence theorems for Markovian SMP's in a more general setting.

In Section 2, we prove the main result, Theorem 2.14. Sections 3 and 4 are concerned with applications of Theorem 2.14 to the uniqueness problem for SMP's, i.e., the problem of determining for which G and ξ there is a unique SMP (G, ξ). Section 3 contains a set of sufficient conditions for uniqueness (Theorem 3.1), which turn out to be necessary and sufficient if the flip rates are continuous (Theorem 3.4). Theorems 3.1 and 3.4 both follow easily from Theorem 2.14. Section 3 also contains a discussion of the useful properties possessed by processes which satisfy our uniqueness condition. Section 4 contains some applications of Theorem 3.1 to the problem of showing uniqueness for specific classes of SMP's. In the first application, we prove a new uniqueness result, while in the second, we give a new proof of an earlier result of Liggett. This new proof is more probabilistic and intuitive than earlier proofs.

For some of the basics concerning the martingale approach to spin-flip processes, see [4], [5], and especially [7]. General references on infinite interacting systems are [3], [13], and [17].

2. Controlled spin-flip systems. In this section, certain non-time-homogeneous Markov processes with finite state spaces are described. We call these *controlled spin-flip systems*. We also prove the main theorem, which shows that the behavior of a spin-flip process can be bounded, in a certain sense, by the behavior of a related controlled spin-flip system. First, we define a control:

DEFINITION 2.1. Choose $A \in \mathcal{V}_0$. We call a function $\varphi : [0, \infty) \times \{-1, 1\}^A \to \{-1, 1\}^{V \setminus A}$ a *control for A* (or simply a *control*) if there exists a finite sequence of times $0 = t_0 < t_1 < \dots < t_n$ such that for fixed $\eta \in \{-1, 1\}^A$, $\varphi(t, \eta)$ is constant over each of the intervals $[t_0, t_1), [t_1, t_2), \cdots, [t_{n-1}, t_n), [t_n, \infty)$. Denote the collection of all controls for A by $\mathcal{K}_A$, and let $\mathcal{K} = \cup_{A \in \mathcal{V}_0} \mathcal{K}_A$.

Before we define controlled spin-flip systems, we need four more bits of notation (borrowed from various authors) which will be useful throughout this section. The first is for pasting together two partial configurations: for $A \in \mathcal{V}_0$, $\eta \in \{-1, 1\}^A$, and $\zeta \in \{-1, 1\}^{V \setminus A}$, define $\eta \times \zeta \in \Xi$ by

$$\begin{aligned}(\eta \times \zeta)(x) &= \eta(x) \qquad x \in A\\ &= \zeta(x) \qquad x \in V \setminus A.\end{aligned}$$

The second piece of notation will be used to denote the probability measure on Ω that is obtained by using one probability measure up to a fixed time $t \geqslant 0$, and then continuing after time t according to a family of probabilities. More precisely, let $P^{(1)}$ be a probability measure on $(\Omega, \mathfrak{B})$ and $P^{(2)} = (P_\xi^{(2)})_{\xi \in \Xi}$ a measurable family of probabilities on $(\Omega, \mathfrak{B})$, with $P_\xi^{(2)}(\xi_0 = \xi) = 1$ for each $\xi \in \Xi$. For $t \in [0, \infty)$, denote by $P^{(1)} \otimes_t P^{(2)}$ the measure on $(\Omega, \mathfrak{B})$ determined by the following: for finite sequences $0 \leqslant s_1 < s_2 < \ldots < s_m < t \leqslant s_{m+1} < s_{m+2} < \ldots < s_n < \infty$ and Borel sets $\Lambda_1, \Lambda_2, \cdots, \Lambda_n \subset \Xi$, set

$$\begin{aligned}P^{(1)} \otimes_t P^{(2)}(\xi_{s_1} \in \Lambda_1, \xi_{s_2} \in \Lambda_2, \cdots, \xi_{s_n} \in \Lambda_n)\\ = \int_\Xi P^{(1)}(\xi_{s_1} \in \Lambda_1, \cdots, \xi_{s_m} \in \Lambda_m, \xi_t \in d\xi)\\ \times P_\xi^{(2)}(\xi_{s_{m+1}-t} \in \Lambda_{m+1}, \xi_{s_{m+2}-t} \in \Lambda_{m+2}, \cdots, \xi_{s_n - t} \in \Lambda_n).\end{aligned}$$

Next, we will use the following notational convention concerning expectation operators: whenever we denote a probability measure by a letter (such as "P" or "Q") together with possible subscripts, superscripts, etc., we denote the corresponding expectation operator by replacing the letter by an "E." For example, the expectation operators for $P_\xi^{(1)}$ and $P_\xi^{(1)} \otimes_t P^{(2)}$ would be $E_\xi^{(1)}$ and $E_\xi^{(1)} \otimes_t E^{(2)}$ respectively.

Finally, if P is a SMP (G, ξ), and if τ is a $\mathfrak{B}_0^t$-stopping time which is P a.s. finite, we let $(P^{\omega, \tau})_{\omega \in \Omega}$ be a collection of measures on $(\Omega, \mathfrak{B})$ such that for P- almost all $\omega \in \Omega$, $P^{\omega, \tau}$ is a SMP $(G, \xi_{\tau(\omega)})$, and such that for all $\Lambda \in \mathfrak{B}$, $P^{\omega,\tau}(\Lambda) = P(\Lambda^\tau | \mathfrak{B}_0^\tau)(\omega)$ P a.s., where $\Lambda^\tau = \{\omega \in \Omega : \theta_\tau \omega \in \Lambda\}$. Here, $\theta_\tau : \Omega \to \Omega$ is the shift operator defined by $(\theta_\tau \omega)_t = \omega_{\tau(\omega)+t}$. Such a collection exists by Theorem 1.2 of [7], although our $P^{\omega,\tau}$ is *not* the same as the $P^{\tau,\omega}$ of [7]. Instead, $P^{\omega,\tau}$ is in some sense a "shifted" version of $P^{\tau,\omega}$. Obviously, $(P^{\omega,\tau})_{\omega \in \Omega}$ is not uniquely determined by P and τ, but we will assume that one such collection has been chosen for each P and τ. We are now ready to introduce controlled spin-flip systems:

DEFINITION 2.2. Let G be a pregenerator with rates c. Choose $\varphi \in \mathcal{K}_A$ for some $A \in \mathcal{V}_0$, and let $t_0, t_1, \cdots, t_n$ be as in Definition 2.1. For $k = 0, 1, \cdots, n$, let $P^{(k)} = (P_\xi^{(k)})_{\xi \in \Xi}$ be the (unique) spin-flip system with rates $c^{(k)}$, where

$$\begin{aligned}c_x^{(k)}(\xi) &= c_x(\xi|_A \times \varphi(t_k, \xi|_A)) \qquad x \in A\\ &= 0 \qquad x \in V \setminus A.\end{aligned}$$

Set $P^{\varphi, G} = (P_\xi^{\varphi, G})_{\xi \in \Xi}$, where

$$P_\xi^{\varphi, G} = \left(\left(\left(P_\xi^{(0)} \otimes_{t_1} P^{(1)}\right) \otimes_{t_2} P^{(2)}\right) \otimes_{t_3} \cdots\right) \otimes_{t_n} P^{(n)}.$$

We call $P^{\varphi, G}$ the *spin-flip system with rates c controlled by* φ, or simply a *controlled spin-flip system.*

REMARK 2.3. For each k, the system $P^{(k)}$ corresponds to the Markovian semigroup $e^{t\bar{G}_k}$, where $\bar{G}_k$ is the closure in $\mathcal{C}$ of the pregenerator G_k with rates $c^{(k)}$. Thus, $P_\xi^{\varphi, G}$ is a Markov process on $(\Omega, \mathcal{B}, \mathcal{B}_0^t)$ which is in general not time homogeneous. $P_\xi^{\varphi, G}$ may be thought of as a spin-flip process on the state space $\{-1, 1\}^A$ with a pregenerator that changes at the times t_k according to the control φ; since φ only depends on time and the configuration on the set A, the controlled system also corresponds to a Markov process with finite state space $\{-1, 1\}^A$. On the other hand, $P_\xi^{\varphi, G}$ may be viewed as a spin-flip process with rates c in which the configuration outside of A is controlled by φ. The controlled configuration outside A influences the flip rates for sites in A, affecting the behavior of the system on A. For technical reasons, however, we leave the configuration outside of A fixed in our definition of $P_\xi^{\varphi, G}$. Finally, we note that for $f \in \mathcal{F}$,

$$f(\xi_t) - \int_0^t \Sigma_{x \in A} c_x\left(\xi_{s|A} \times \varphi\left(s, \xi_{s|_A}\right)\right)\left(f\left({}_x\xi_s\right) - f(\xi_s)\right) ds$$

is a $P_\xi^{\varphi, G}$-martingale for each $\xi \in \Xi$, so that $P_\xi^{\varphi, G}$ falls under the general definition of SMP given in [7].

To prove the main result concerning controlled spin-flip systems, Theorem 2.14, we first need a proposition and a lemma. The proposition is a corollary to the results found in Section 1 of [7], and is not new.

PROPOSITION 2.4. *Let P be a* SMP (G, ξ). *The following hold for all $A \in \mathcal{V}_0$, $h > 0$, and $\mathcal{B}_0^t$-stopping times τ such that $P(\tau < \infty) = 1$:*

(2.5) $$E\left(\#\left\{s \in [\tau, \tau + h] : \xi_{s|_A} \neq \xi_{s^-|_A}\right\} | \mathcal{B}_0^\tau\right) \leqslant h \Sigma_{x \in A} \|c_x\| \qquad P \text{ a.s.}$$

(2.6) $$\textit{For all } \; s \in [0, \infty), \quad P(\xi_s \neq \xi_{s^-}) = 0.$$

(2.7) $$P\left(\exists r, s \in [\tau, \tau + h], r \neq s : \xi_{r|_A} \neq \xi_{r^-|_A} \quad \textit{and} \quad \xi_{s|_A} \neq \xi_{s^-|_A} | \mathcal{B}_0^\tau\right) \leqslant \left(h \Sigma_{x \in A} \|c_x\|\right)^2 \qquad P \text{ a.s.}$$

(2.8) $$P\left(\exists s \in [0, \infty) \quad \textit{and} \quad x, y \in V, x \neq y : \xi_s(x) \neq \xi_{s^-}(x) \quad \textit{and} \quad \xi_s(y) \neq \xi_{s^-}(y)\right) = 0.$$

(We have used ξ_{s^-} here to denote $\lim_{r \nearrow s} \xi_r$ and $\#$ to denote cardinality.)

PROOF. The left side of (2.5) is equal to $E^{\omega, \tau}(\#\{s \in [0, h] : \xi_{s|_A} \neq \xi_{s^-|_A}\})P$ a.s., so we only need to prove (2.5) for $\tau \equiv 0$, since $P^{\omega, \tau}$ is a SMP $(G, \xi_{\tau(\omega)})$ for P almost all ω. Similarly, we need only prove (2.7) for $\tau \equiv 0$.

For $A \in \mathcal{V}_0$, define $\theta^A : [0, \infty) \times \Omega \to \mathbb{R}^V$ by

$$\begin{aligned}\left(\theta^A(s, \omega)\right)_x &= 1 \qquad x \in A\\ &= 0 \qquad x \in V \setminus A.\end{aligned}$$

Then (1.11) and (i) of Theorem 1.10 of [7], with $\theta = \theta^A$, imply that

$$\int_0^t \Sigma_{x \in V}\big[(\theta^A(s, \omega))_x\big] d\gamma_x(s, \omega) - \int_0^t \Sigma_{x \in V} c_x(\xi_s(\omega))\big[(\theta^A(s, \omega))_x\big] ds$$

is a P-martingale, where $\gamma_x(t, \omega)$ is the number of sign changes made by $\xi_s(x, \omega)$ in $[0, t]$. Taking the expected value of this martingale at $t = h$ easily implies (2.5), which in turn implies (2.6).

Now define

$$\sigma_A(\omega) = h \wedge (\inf\{s > 0 : \xi_s(\omega)_{|A} \neq \xi_{s^-}(\omega)_{|A}\}).$$

Then the left side of (2.7) is bounded above by

$$E\big(P\big(\exists s \in (\sigma_A, h] : \xi_{s|_A} \neq \xi_{s^-|_A} | \mathcal{B}_0^{\sigma_A}\big)\big) P(\sigma_A < h),$$

since $P(\xi_{h|_A} \neq \xi_{h^-|_A}) = 0$ by (2.6). But by (2.5) both

$$P\big(\exists s \in (\sigma_A, h] : \xi_{s|_A} \neq \xi_{s^-|_A} | \mathcal{B}_0^{\sigma_A}\big) \qquad \textit{and} \qquad P(\sigma_A < h)$$

are P a.s. bounded above by $h\Sigma_{x \in A}\|c_x\|$. This implies (2.7).

We now verify (2.8). Choose $x, y \in V$, $x \neq y$. For $n = 0, 1, 2, \cdots$, define stopping times σ_n inductively as follows: $\sigma_0 \equiv 0$, and for $n > 0$,

$$\sigma_n(\omega) = n \wedge (\inf\{s > \sigma_{n-1}(\omega) : \xi_s(x) \neq \xi_{s^-}(x) \quad or \quad \xi_s(y) \neq \xi_{s^-}(y)\}).$$

It is easy to see that (2.8) follows if for all $n \geq 0$ and for P-almost all ω,

$$P^{\omega, \sigma_n}\big(\xi_{\sigma_1}(x) \neq \xi_0(x) \quad \text{and} \quad \xi_{\sigma_1}(y) \neq \xi_0(y)\big) = 0.$$

Fix $n \geq 0$. Then for P-almost all ω, P^{ω, σ_n} is a SMP $(G, \xi_{\sigma_n(\omega)}(\omega))$. Fix one such ω. Define $f \in \mathcal{F}$ by

$$\begin{aligned} f(\xi') &= 1 \qquad \text{if } \xi'(x) \neq \xi_{\sigma_n(\omega)}(\omega)(x) \text{ and } \xi'(y) \neq \xi_{\sigma_n(\omega)}(\omega)(y) \\ &= 0 \qquad \text{otherwise.} \end{aligned}$$

Then

$$\begin{aligned} P^{\omega, \sigma_n}\big(\xi_{\sigma_1}(x) \neq \xi_0(x) \quad \text{and} \quad \xi_{\sigma_1}(y) \neq \xi_0(y)\big) &= E^{\omega, \sigma_n}\big(f(\xi_{\sigma_1})\big) \\ &= E^{\omega, \sigma_n}\big(\textstyle\int_0^{\sigma_1} Gf(\xi_s) ds\big) = 0. \qquad \square \end{aligned}$$

Before continuing, we set up some notation for certain finite processes which will be useful as approximations for infinite spin-flip processes. Let G be a pregenerator with rates c. Then for $A \in \mathcal{V}_0$, $\eta \in \{-1, 1\}^A$, and $\zeta \in \{-1, 1\}^{V \setminus A}$, let $P_{(G, \eta, \zeta)}$ denote the (unique) SMP $(G^{(\eta, \zeta)}, \eta \times \zeta)$, where $G^{(\eta, \zeta)}$ has rates $c^{(\eta, \zeta)}$ defined by

$$\begin{aligned} c_x^{(\eta, \zeta)}(\xi) &= c_x(\eta \times \zeta) \qquad \text{if } \xi|_A = \eta \text{ and } x \in A \\ &= 0 \qquad\qquad \text{otherwise.} \end{aligned}$$

We are now ready for a lemma which is of key importance to the proof of our main theorem:

LEMMA 2.9. *Let P be a* SMP (G, ξ), *and choose $h > 0$ and $A \in \mathcal{V}_0$. Define a Borel measure μ on $\{-1, 1\}^{V \setminus A}$ by*

$$\mu(\cdot) = (1/h)\int_0^h P(\xi_{s|_{V \setminus A}} \in \cdot) ds.$$

Then

(2.10)
$$\Sigma_{\eta\in\{-1,1\}^A}\left|P(\xi_h|_A=\eta)-\int_{\{-1,1\}^{V\setminus A}}P_{(G,\xi|_A,\zeta)}(\xi_h|_A=\eta)\mu(d\zeta)\right| \leq 6\left(h\Sigma_{x\in A}\|c_x\|\right)^2.$$

PROOF. Let σ_A be as in the proof of Proposition 2.4. Since $c_x^{(\xi_{|A}\times\zeta)}(\xi')=0$ for all $x\notin A$ and all ξ' such that $\xi'_{|A}\neq\xi_{|A}$,

(2.11)
$$P_{(G,\xi_{|A},\zeta)}(\xi_h\in\cdot)=P_{(G,\xi_{|A},\zeta)}(\xi_{\sigma_A}\in\cdot)$$

for all $\zeta\in\{-1,1\}^{V\setminus A}$. Also, by (2.8),

$$\Sigma_{x\in A}P\left(\xi_{\sigma_A}|_A={}_x\xi_{|A}\right)+P\left(\xi_{\sigma_A}|_A=\xi_{|A}\right)=1,$$

so by (2.7),

(2.12)
$$\begin{aligned}
&\Sigma_{\eta\in\{-1,1\}^A}\left|P(\xi_{h|A}=\eta)-\int_{\{-1,1\}^{V\setminus A}}P_{(G,\xi|_A,\zeta)}(\xi_h|_A=\eta)\mu(d\zeta)\right|\\
&\leq 2\Sigma_{\eta\in\{-1,1\}^A;\,\eta\neq\xi_{|A}}\left|P(\xi_h|_A=\eta)-\int_{\{-1,1\}^{V\setminus A}}P_{(G,\xi|_A,\zeta)}(\xi_h|_A=\eta)\mu(d\zeta)\right|\\
&\leq 2\Big[\Sigma_{x\in A}\left|P\left(\xi_{\sigma_A}|_A={}_x\xi_{|A}\right)-\int_{\{-1,1\}^{V\setminus A}}P_{(G,\xi_{|A},\zeta)}\left(\xi_{\sigma_A}|_A={}_x\xi_{|A}\right)\mu(d\zeta)\right|\\
&\qquad+\Sigma_{\eta\in\{-1,1\}^A;\,\eta\neq\xi_{|A}}\left|P(\xi_h|_A=\eta)-P\left(\xi_{\sigma_A}|_A=\eta\right)\right|\Big]\\
&\leq 2\Big[\Sigma_{x\in A}\left|P\left(\xi_{\sigma_A}|_A={}_x\xi_{|A}\right)-\int_{\{-1,1\}^{V\setminus A}}P_{(G,\xi_{|A},\zeta)}\left(\xi_{\sigma_A}|_A={}_x\xi_{|A}\right)\mu(d\zeta)\right|\\
&\qquad+2\left(h\Sigma_{x\in A}\|c_x\|\right)^2\Big].
\end{aligned}$$

For each $x\in V$, define $\theta^x:[0,\infty)\times\Omega\to\mathbb{R}^V$ by

$$\begin{aligned}(\theta^x(s,\omega))_y&=1 &&x=y\\ &=0 &&x\neq y.\end{aligned}$$

Then as in the proof of (2.5),

$$\int_0^t\Sigma_{x\in V}\left[(\theta^x(s,\omega))_x\right]d\gamma_x(s,\omega)-\int_0^t\Sigma_{x\in V}c_x(\xi_s(\omega))\left[(\theta^x(s,\omega))_x\right]ds$$

and

$$\int_0^t\Sigma_{x\in V}\left[(\theta^x(s,\omega))_x\right]d\gamma_x(s,\omega)-\int_0^t\Sigma_{x\in V}c_x^{(\xi_{|A},\zeta)}(\xi_s(\omega))\left[(\theta^x(s,\omega))_x\right]ds$$

are respectively P- and $P_{(G,\xi_{|A},\zeta)}$-martingales for all $x\in A$ and $\zeta\in\{-1,1\}^{V\setminus A}$. By the optional sampling theorem,

$$\begin{aligned}
P\left(\xi_{\sigma_A}|_A={}_x\xi_{|A}\right)&=E\int_0^{\sigma_A}c_x(\xi_s)ds\\
&=E\int_0^{\sigma_A}c_x(\xi_{|A}\times\xi_{s|V\setminus A})ds\quad\text{and}\\
P_{(G,\xi_{|A},\zeta)}\left(\xi_{\sigma_A}|_A={}_x\xi_{|A}\right)&=E(G,\xi_{|A},\zeta)\int_0^{\sigma_A}c_x(\xi_{|A},\zeta)(\xi_s)ds\\
&=E_{(G,\xi_{|A},\zeta)}\int_0^{\sigma_A}c_x(\xi_{|A}\times\zeta)ds
\end{aligned}$$

for all $x\in A$ and $\zeta\in\{-1,1\}^{V\setminus A}$. Substituting these expressions into (2.12) shows that the right side of (2.12) is bounded above by

$$2\Big[\Sigma_{x\in A}\big|\big(E\textstyle\int_0^{\sigma_A}c_x(\xi_{|A}\times\xi_{s|V\setminus A})ds\big)$$
$$-\textstyle\int_{\{-1,1\}^{V\setminus A}}\big(E_{(G,\xi_{|A},\zeta)}\int_0^{\sigma_A}c_x(\xi_{|A}\times\zeta)ds\big)\mu(d\zeta)\big|$$
$$+2(h\Sigma_{x\in A}\|c_x\|)^2\Big]$$

which is in turn bounded above by

$$2\Big[\Sigma_{x\in A}\big|\big(E\textstyle\int_0^{h}c_x(\xi_{|A}\times\xi_{s|V\setminus A})ds\big)-h\int_{\{-1,1\}^{V\setminus A}}c_x(\xi_{|A}\times\zeta)\mu(d\zeta)\big|+3(h\Sigma_{x\in A}\|c_x\|)^2\Big], \tag{2.13}$$

since by (2.5), both

$$P(\sigma_A<h)\quad\text{and}\quad\sup_{\zeta\in\{-1,1\}^{V\setminus A}}P_{(G,\xi_{|A},\zeta)}(\sigma_A<h)$$

are less than or equal to $h\Sigma_{x\in A}\|c_x\|$. By Fubini's theorem and the definition of μ,

$$E\textstyle\int_0^h c_x(\xi_{|A}\times\xi_{s|V\setminus A})ds=h\int_{\{-1,1\}^{V\setminus A}}c_x(\xi_{|A}\times\zeta)\mu(d\zeta).$$

Thus, (2.13) is bounded above by $6(h\Sigma_{x\in A}\|c_x\|)^2$. This implies (2.10). □

Lemma 2.9 says that over short time spans, the distribution of the configuration of an infinite spin-flip process on a finite set A may be approximated by a convex combination of distributions of very simple finite spin-flip processes with state space $\{-1,1\}^A$. This is the key observation necessary to prove the main theorem:

THEOREM 2.14. *Let P be a* SMP (G,ξ). *Choose $A\in\mathcal{V}_0$ and a finite sequence $0\leqslant s_1<s_2<\dots<s_m=T$. Also choose $\{f_i\}_{i=1}^m\subset\mathcal{F}^A$, with $f_i\geqslant 0$ for each $i=1,2,3,\cdots,m$. Then with $\Phi:\Omega\to[0,\infty)$ defined by*

$$\Phi(\omega)=\Pi_{i=1}^m f_i(\xi_{s_i}(\omega)),$$

we have

$$\sup_{\varphi\in\mathcal{K}_A}E_\xi^{\varphi,G}(\Phi)\geqslant E(\Phi)\geqslant\inf_{\varphi\in\mathcal{K}_A}E_\xi^{\varphi,G}(\Phi). \tag{2.15}$$

PROOF. We will first construct a nearly optimal control φ, using the idea behind Bellman's principle of dynamic programming, and then we will use Lemma 2.9 to compare $P_\xi^{\varphi,G}$ with P.

We will prove only the first inequality in (2.15), since the proof of the second is analogous. We also assume without loss of generality that $\|f_i\|\leqslant 1$ for $i=1,2,\cdots,m$, and that $T>0$. Finally, we will assume that s_i is rational for $i=1,2,\cdots,m$. The general case follows easily by approximating general sequences $\{s_i\}_{i=1}^m$ by rational ones and then using (2.5).

Choose $\varepsilon>0$. Pick $N\in\mathbb{Z}^+$ such that

$$12(T\Sigma_{x\in A}\|c_x\|)^2/N<\varepsilon\text{ and }\{s_i\}_{i=1}^m\subset\left\{\frac{kT}{N}\right\}_{k=0}^N.$$

This is possible since the s_i are assumed to be rational. For $k=0,1,\cdots,N$, let

$t_k = kT/N$. Define $\varphi \in \mathcal{K}_A$ as follows: for $t \in [T, \infty)$ and $\eta \in \{-1, 1\}^A$, let $\varphi(t, \eta) = \zeta(N, \eta)$, where $\zeta(N, \eta)$ is some arbitrary element of $\{-1, 1\}^{V \setminus A}$. Now assume that for some $k \in \{1, 2, \cdots, N\}$, $\varphi(t, \eta)$ has been defined for all $t \in [t_k, \infty)$ and $\eta \in \{-1, 1\}^A$. Define $\varphi_k \in \mathcal{K}_A$ by $\varphi_k(t, \eta) = \varphi(t + t_k, \eta)$, and define $\Phi_{k-1} : \Omega \to [0, \infty)$ by

$$\Phi_{k-1}(\omega) = \Pi_{i : s_i > t_{k-1}} f_i(\xi_{s_i - t_{k-1}}(\omega)).$$

For $\eta \in \{-1, 1\}^A$, choose $\zeta(k-1, \eta) \in \{-1, 1\}^{V \setminus A}$ so that

$$E_{(G, \eta, \zeta(k-1, \eta))} \otimes_{T/N} E^{\varphi_k, G}(\Phi_{k-1}) + \varepsilon/N \geqslant \sup_{\zeta \in \{-1, 1\}^{V \setminus A}} E_{(G, \eta, \zeta)} \otimes_{T/N} E^{\varphi_k, G}(\Phi_{k-1})$$

and let $\varphi(t, \eta) = \zeta(k-1, \eta)$ for $t \in [t_{k-1}, t_k)$. This completes (inductively) the definition of φ.

We now show that for $k = 1, 2, \cdots, N$,

$$E \otimes_{t_{k-1}} E^{\varphi_{k-1}, G}(\Phi) + 2\varepsilon/N \geqslant E \otimes_{t_k} E^{\varphi_k, G}(\Phi). \tag{2.16}$$

A simple inductive proof shows that the first inequality in (2.15) follows from (2.16), since $\varepsilon > 0$ is arbitrary. Fix $k \in \{1, 2, \cdots, N\}$. Now write $P^\omega = P^{\omega, t_{k-1}}$ for $\omega \in \Omega$, and let $\mathfrak{N} \in \mathfrak{B}$ be a P-null set such that for $\omega \notin \mathfrak{N}$, P^ω is a SMP $(G, \xi_{t_{k-1}}(\omega))$. We will apply Lemma 2.9 to P^ω for $\omega \notin \mathfrak{N}$ to get (2.16). Fix $\omega \notin \mathfrak{N}$, and to simplify notation, let $\bar{\xi} = \xi_{t_{k-1}}(\omega)$. As in Lemma 2.9, let

$$\mu(\cdot) = \int_0^{T/N} P^\omega(\xi_{s|_{V \setminus A}} \in \cdot) ds / (T/N).$$

For $i = 1, 2, \cdots, m$, we have, by assumption, both $\|f_i\| \leqslant 1$ and $s_i - t_{k-1} \notin (0, T/N)$, so that Lemma 2.9 implies that

$$\begin{aligned}
&E_{\bar{\xi}}^{\varphi_{k-1}, G}(\Phi_{k-1}) - E^\omega \otimes_{T/N} E^{\varphi_k, G}(\Phi_{k-1}) \\
&\quad = \left[E_{\bar{\xi}}^{\varphi_{k-1}, G}(\Phi_{k-1}) - E_{(G, \bar{\xi}_{|A}, \zeta(k-1, \bar{\xi}_{|A}))} \otimes_{T/N} E^{\varphi_k, G}(\Phi_{k-1}) \right] \\
&\qquad + \left[E_{(G, \bar{\xi}_{|A}, \zeta(k-1, \bar{\xi}_{|A}))} \otimes_{T/N} E^{\varphi_k, G}(\Phi_{k-1}) \right. \\
&\qquad \left. - \int_{\{-1, 1\}^{V \setminus A}} E_{(G, \bar{\xi}_{|A}, \zeta)} \otimes_{T/N} E^{\varphi_k, G}(\Phi_{k-1}) \mu(d\zeta) \right] \\
&\qquad + \left[\int_{\{-1, 1\}^{V \setminus A}} E_{(G, \bar{\xi}_{|A}, \zeta)} \otimes_{T/N} E^{\varphi_k, G}(\Phi_{k-1}) \mu(d\zeta) - E^\omega \otimes_{T/N} E^{\varphi_k, G}(\Phi_{k-1}) \right] \\
&\quad \geqslant E_1 + E_2 - 6((T/N) \Sigma_{x \in A} \|c_x\|)^2,
\end{aligned}$$

where E_1 and E_2 are the first two expressions in square brackets. By the definition of $\zeta(k-1, \bar{\xi}_{|A})$, $E_2 \geqslant -\varepsilon/N$. Also

$$\begin{aligned}
E_1 \geqslant -\Sigma_{\eta \in \{-1, 1\}^A} | &P_{\bar{\xi}}^{\varphi_{k-1}, G}(\xi_{T/N|_A} = \eta) \\
&- P_{(G, \bar{\xi}_{|A}, \zeta(k-1, \bar{\xi}_{|A}))}(\xi_{T/N|_A} = \eta)|,
\end{aligned}$$

since for $i = 1, 2, \cdots, m$, $s_i - t_{k-1} \notin (0, T/N)$ and $\|f_i\| \leqslant 1$. Let G' be the pregenerator with rates c' defined by

$$\begin{aligned}
c'_x(\xi') &= c_x(\xi'_{|A} \times \varphi(t_{k-1}, \xi'_{|A})) \qquad && x \in A \\
&= 0 && x \in V \setminus A.
\end{aligned}$$

Then by definition

$$P_{(G', \xi_{|A}, \zeta(k-1, \xi_{|A}))}(\xi_{T/N|_A} = \eta) = P_{(G, \xi_{|A}, \zeta(k-1, \xi_{|A}))}(\xi_{T/N|_A} = \eta) \quad \text{and}$$

$$P'(\xi_{T/N|_A} = \eta) = P_{\xi}^{\varphi_{k-1}, G}(\bar{\xi}_{T/N|_A} = \eta)$$

for all $\eta \in \{-1, 1\}^A$, where P' is the unique SMP $(G', \bar{\xi}_{|A} \times \zeta(k-1, \bar{\xi}_{|A}))$. Substituting these into the above expression for a lower bound for E_1 and applying Lemma 2.9 to P' yields

$$E_1 \geqslant -6((T/N)\Sigma_{x \in A} \|c_x\|)^2.$$

This implies that

$$E_{\xi_{t_{k-1}}(\omega)}^{\varphi_{k-1}, G}(\Phi_{k-1}) + 2\varepsilon/N \geqslant E^{\omega, t_{k-1}} \otimes_{T/N} E^{\varphi_k, G}(\Phi_{k-1}).$$

Thus,

$$\begin{aligned} E_{t_{k-1}} &\otimes E^{\varphi_{k-1}, G}(\Phi) + 2\varepsilon/N \\ &= E_{t_{k-1}} \otimes E^{\varphi_{k-1}, G}\Big[\big(\Pi_{i : s_i \leqslant t_{k-1}} f_i(\xi_{s_i})\big) \\ &\qquad \times E_{t_{k-1}} \otimes E^{\varphi_{k-1}, G}\big(\Pi_{i : s_i > t_{k-1}} f_i(\xi_{s_i}) | \mathfrak{B}_0^{t_{k-1}}\big)\Big] + 2\varepsilon/N \\ &= E\Big[\big(\Pi_{i : s_i \leqslant t_{k-1}} f_i(\xi_{s_i})\big) E_{\xi_{t_{k-1}}}^{\varphi_{k-1}, G}(\Phi_{k-1})\Big] + 2\varepsilon/N \\ &\geqslant E\Big[\big(\Pi_{i : s_i \leqslant t_{k-1}} f_i(\xi_{s_i})\big) E^{\cdot, t_{k-1}} \otimes_{T/N} E^{\varphi_k, G}(\Phi_{k-1})\Big] \\ &= E \otimes_{t_k} E^{\varphi_k, G}(\Phi). \end{aligned}$$

This completes the proof of (2.16). ☐

By being less restrictive in the definition of controls (e.g., allow φ to depend on the past), one can obtain (2.15) for more general Φ. The advantage to defining controls as in Definition 2.1 is that any controlled spin-flip system corresponds to a Markov process on $\{-1, 1\}^A$ for some $A \in \mathcal{V}_0$. Such finite Markov processes can be analyzed using various techniques, such as "coupling," which cannot be used on general spin-flip processes. This fact will be exploited in the next section.

3. Uniqueness. We first derive some general criteria for the uniqueness of SMP's in terms of the behavior of the corresponding controlled spin-flip systems. The first of these, Theorem 3.1, is an immediate corollary to Theorem 2.14. Theorem 3.1 is similar in spirit to Propositions 5.5, 5.6, and 5.7 of [14].

THEOREM 3.1. *Suppose $\exists P$ such that P is a* SMP (G, ξ). *Then P is the unique SMP (G, ξ) if*

(3.2) *for all* Φ *as in Theorem* 2.14, *we have* $\lim_{A \nearrow V;\, A \in \mathcal{V}_0} \sup_{\varphi \in \mathcal{K}_A} E_{\xi}^{\varphi, G}(\Phi) = \lim_{A \nearrow V;\, A \in \mathcal{V}_0} \inf_{\varphi \in \mathcal{K}_A} E_{\xi}^{\varphi, G}(\Phi)$.

REMARK 3.3. A simple modification of the proof of Theorem 2.14 shows that for Φ and A as in Theorem 2.14,

$$\sup_{\varphi \in \mathcal{K}_{A \cup B}} E_\xi^{\varphi, G}(\Phi) \qquad \left(\inf_{\varphi \in \mathcal{K}_{A \cup B}} E_\xi^{\varphi, G}(\Phi)\right)$$

decreases (increases) as B increases, so the limits in (3.2) exist, and are Borel measurable functions of the configuration ξ.

If we have continuous flip rates, we can do even better:

THEOREM 3.4. *Let G be a pregenerator with continuous rates and choose $\xi \in \Xi$. Then there exists a unique* SMP(G, ξ) iff (3.2) *holds for G and ξ.*

PROOF. Let $\{A_k\}_{k=1}^\infty \subset \mathcal{V}_0$ be a sequence such that $A_k \nearrow V$ as $k \to \infty$, and choose $\varphi_k \in \mathcal{K}_{A_k}$ for $k = 1, 2, 3, \cdots$. A standard argument shows that the sequence $\{P_\xi^{\varphi_k, G}\}_{k=1}^\infty$ is relatively weakly compact. The argument is based on the following two facts: Ξ is compact; and for each $A \in \mathcal{V}_0$, $T \in [0, \infty)$, and $\varepsilon > 0$, there exists $N \in \mathbb{Z}^+$ such that $\sup_k P_\xi^{\varphi_k, G}(\#\{s : \xi_{s|A} \neq \xi_{s-}|_A\} > N) < \varepsilon$. The second fact follows from (2.7). This same type of argument has been referred to in Theorem 2.2 of [7] and Theorem 1.15 of [10]. Now, since for each $k = 1, 2, 3, \cdots$, and $f \in \mathcal{F}$,

$$f(\xi_t) - \int_0^t \Sigma_{x \in A_k} c_x\left(\xi_s|_{A_k} \times \varphi_k(s, \xi_s|_{A_k})\right)(f({}_x\xi_s) - f(\xi_s))ds$$

is a $P_\xi^{\varphi_k, G}$-martingale, we can use the proof of the first assertion of Theorem 2.3 in [13] to show that every weak limit point of the sequence $\{P_\xi^{\varphi_k, G}\}_{k=1}^\infty$ is a SMP (G, ξ). These facts together with Theorem 3.1 now imply Theorem 3.4. $\square$

REMARK 3.5. When the flip rates are not continuous, G and ξ can be such that no SMP (G, ξ) exists even though (3.2) holds, or a unique SMP (G, ξ) exists even though (3.2) fails. An example of the first case is (G_1, ξ^-), where $\xi^-(x) = -1$ for all $x \in V$ and G_1 has rates

$$c_x^{(1)}(\xi) = 1 \quad \text{if} \quad \xi = \xi^-$$
$$= 0 \quad \text{otherwise.}$$

For the second case, take (G_2, ξ^-), where G_2 has rates

$$c_x^{(2)}(\xi) = 0 \quad \text{if} \quad \xi(y) = -1 \quad \text{for infinitely many } y$$
$$= 1 \quad \text{otherwise.}$$

In computing the bounds in (2.15), we will rely often on a technique known as coupling, which we describe here. Since this technique has been used often before, we will try to be brief. For some added details, see [13].

We will only use the "basic coupling" of [13], modified slightly to fit controlled spin-flip systems. Suppose we have two controlled spin-flip systems $P_\xi^{\varphi, G}$ and $P_\xi^{\varphi', G'}$, where G and G' have rates c and c' respectively, and where both φ and φ' are in $\mathcal{K}_B$ for some $B \in \mathcal{V}_0$. Assume without loss of generality that the sequence $t_0, t_1, \cdots, t_n$ of Definition 2.1 is the same for φ and φ'. For $k = 0, 1, 2, \cdots, n$, let $P^{(k)}$ and $P^{(k)'}$ be the systems of measures used in Definition 2.2 to define $P_\xi^{\varphi, G}$

and $P_{\xi'}^{\varphi', G'}$ respectively. For each $\xi^{(1)}, \xi^{(2)} \in \Xi$, and for each k, let $\tilde{P}^{(k)}_{(\xi^{(1)}, \xi^{(2)})}$ be the basic coupling of $P^{(k)}_{\xi^{(1)}}$ and $P^{(k)'}_{\xi^{(2)}}$ and let

$$\tilde{P}^{(k)} = \left[\tilde{P}^{(k)}{}_{(\xi^{(1)}, \xi^{(2)})}\right]_{(\xi^{(1)}, \xi^{(2)}) \in \Xi \times \Xi}.$$

We will string the $\tilde{P}^{(k)}$'s together at the times t_k as in Definition 2.2 to get the basic coupling of $P_\xi^{\varphi, G}$ and $P_{\xi'}^{\varphi', G'}$. For $i = 1, 2$, and $t \geq 0$, let $\xi_t^{(i)} : \Omega \times \Omega \to \Xi$ be the coordinate map $(\omega^{(1)}, \omega^{(2)}) \to \omega_t^{(i)}$.

DEFINITION 3.6. Let $P_\xi^{\varphi, G}$, $P_{\xi'}^{\varphi', G'}$, and $(\tilde{P}^{(k)})_{k=0}^n$ be as above. The *basic coupling of* $P_\xi^{\varphi, G}$ *and* $P_{\xi'}^{\varphi', G'}$ is the probability measure $\tilde{P}_{(\xi, \xi')}$ on $\Omega \times \Omega$ given by

$$\tilde{P}_{(\xi, \xi')} = \left(\left(\left(\tilde{P}^{(0)}_{(\xi, \xi')} \otimes_{t_1} \tilde{P}^{(1)}\right) \otimes_{t_2} \tilde{P}^{(2)}\right) \otimes_{t_3} \cdots\right) \otimes_{t_n} \tilde{P}^{(n)},$$

where the symbol "$\otimes_t$" is used in the way that is analogous to the way it was used in Definition 2.2.

Note that the first and second marginals of $\tilde{P}_{(\xi, \xi')}$ are $P_\xi^{\varphi, G}$ and $P_{\xi'}^{\varphi', G'}$ respectively, and that $\tilde{P}_{(\xi, \xi')}$ is Markovian. An intuitive description of the basic coupling is as follows: if the spins for the two coordinate processes in the coupling differ at site $x \in B$, then those spins flip independently, thus bringing them into agreement as fast as possible; if they agree at x, they flip simultaneously at a rate equal to the minimum of the two individual flip rates, so that they stay in agreement for as long as possible; in addition, in this second case, the spin in the coordinate with the higher flip rate must flip alone at a rate equal to the difference of the two flip rates.

We now list two properties of the basic coupling in the form of propositions. These are already known, so we omit the proof of the second and only sketch the proof of the first. Both proofs are based on the following fact: let $\tilde{\mathfrak{F}}$ be the analogue of $\mathfrak{F}$, i.e., the set of all functions $f : \Xi \times \Xi \to \mathbb{R}$ such that there exists $A \in \mathcal{V}_0$ satisfying $f(\xi, \cdot) \in \mathfrak{F}^A$ and $f(\cdot, \xi) \in \mathfrak{F}^A$ for all $\xi \in \Xi$. Let $P_\xi^{\varphi, G}$, $P_{\xi'}^{\varphi', G'}$, and $\tilde{P}_{(\xi, \xi')}$ be as above. For each $k = 0, 1, 2, \cdots, n$, let $\tilde{G}_k$ be the restriction to $\tilde{\mathfrak{F}}$ of the infinitesimal generator of the process associated with $\tilde{P}^{(k)}$ (see [13] for the form of $\tilde{G}_k$). Then

(3.7) for all $f \in \tilde{\mathfrak{F}}$, $\quad f(\xi_t^{(1)}, \xi_t^{(2)}) - \int_0^t \tilde{G}_{k(s)} f(\xi_s^{(1)}, \xi_s^{(2)})\, ds$

is a $\tilde{P}_{(\xi, \xi')}$-martingale, where $k(s) = \max\{k = 0, 1, 2, \cdots, n : t_k \leq s\}$.

PROPOSITION 3.8. *Let* $\tilde{P}$ *be the basic coupling of* $P_\xi^{\varphi, G}$ *and* $P_{\xi'}^{\varphi', G'}$, *where* φ, φ', B, G *and* G' *are as in Definition* 3.6. *Assume that* $\xi|_{B'} = \xi'|_{B'}$ *for some* $B' \subset B$. *Let* $\tau : \Omega \times \Omega \to [0, \infty]$ *be a* $\mathfrak{B}_0^t \times \mathfrak{B}_0^t$*-stopping time which is* $\tilde{P}$*- a.s. finite. Then*

$$(3.9) \quad \tilde{P}\left(\exists t \in [0, \tau] : \xi_t^{(1)}|_{B'} \neq \xi_t^{(2)}|_{B'}\right) \leq \tilde{E}\Big(\int_0^\tau \Sigma_{x \in B'} \big|c_x\big(\xi_t^{(1)}|_B \times \varphi(t, \xi_t^{(1)}|_B)\big) - c_x'\big(\xi_t^{(1)}|_{B'} \times \xi_t^{(2)}|_{B \setminus B'} \times \varphi'(t, \xi_t^{(1)}|_{B'} \times \xi_t^{(2)}|_{B \setminus B'})\big)\big| dt\Big).$$

SKETCH OF PROOF. Define $g \in \tilde{\mathfrak{F}}$ by

$$g(\xi^{(1)}, \xi^{(2)}) = 1 \qquad \text{if } \xi^{(1)}|_{B'} \neq \xi^{(2)}|_{B'},$$
$$= 0 \qquad \text{otherwise,}$$

and let $\tau_{B'}$ be the $\mathfrak{B}_0^t \times \mathfrak{B}_0^t$-stopping time defined by

$$\tau_{B'} = \tau \wedge \left(\inf\left\{t \geq 0 : \xi_t^{(1)}|_{B'} \neq \xi_t^{(2)}|_{B'}\right\}\right).$$

Now evaluate the martingale in (3.7) at time $\tau_{B'}$ with $f = g$, take expected values, and use the optional sampling theorem to get (3.9).

Proposition 3.8 will be useful in verifying (3.2), since with Φ, A, and T as in Theorem 2.14, we have

$$(3.10) \qquad |E_{\xi}^{\varphi, G}(\Phi) - E_{\xi'}^{\varphi', G'}(\Phi)| \leq \|\Phi\| \tilde{P}\left(\exists t \in [0, T] : \xi_t^{(1)}|_A \neq \xi_t^{(2)}|_A\right).$$

PROPOSITION 3.11 (see [13]). *Let everything be as in Proposition* 3.8, *with* $B' = \varnothing$. *Assume further that* $\xi \geq \xi'$ (i.e., $\xi(x) \geq \xi'(x)$ *for all* $x \in V$), *and that for all* $t \geq 0$ *and* $\eta^{(1)}, \eta^{(2)} \in \{-1, 1\}^B$ *such that* $\eta^{(1)} \geq \eta^{(2)}$ *we have*

$$\left[c_x(\eta^{(1)} \times \varphi(t, \eta^{(1)})) - c'_x(\eta^{(2)} \times \varphi'(t, \eta^{(2)}))\right]\left[\eta^{(1)}(x) + \eta^{(2)}(x)\right] \leq 0.$$

Then $\tilde{P}(\xi_t^{(1)} \geq \xi_t^{(2)}$ *for all* $t \in [0, \infty)) = 1$.

We are now ready for some of the consequences of Theorems 2.14 and 3.1. As remarked in [7], uniqueness is important in the case of continuous flip rates because it implies some very useful properties. However, in general, the stronger condition (3.2) is needed to get all the same properties.

We will use Theorem 2.14 to prove the two theorems below, which show that (3.2) implies (a) stability under various approximation procedures, which includes a type of Feller property; and (b) that the stationarity of a measure can be checked using only the pregenerator.

THEOREM 3.12. *Let G be a pregenerator with rates c, choose $\xi \in \Xi$, and suppose that* (3.2) *holds for G and ξ. Assume $\exists P_\xi$ such that P_ξ is a* SMP (G, ξ). *For $k = 1, 2, 3, \cdots$, let G_k be a pregenerator with rates $c^{(k)}$, such that at least one of the following two conditions holds*:

(3.13) *for each* $x \in V$, $\lim_{k\to\infty}\|c_x^{(k)} - c_x\| = 0$;

(3.14) *there exists a sequence* $\{A_k\}_{k=1}^{\infty} \subset \mathcal{V}_0$, $A_k \nearrow V$, *such that for each* $k = 1, 2, 3, \cdots$,

$$c_x^{(k)}(\xi') = \int_{\{-1,1\}^{V\setminus A_k}} c_x(\xi'|_{A_k} \times \zeta)\mu_{\xi'|_{A_k}}^{(k)}(d\zeta) \qquad x \in A_k,$$
$$= 0 \qquad x \in V \setminus A_k,$$

where for each $k = 1, 2, 3, \cdots$, *and* $\eta \in \{-1, 1\}^{A_k}$, $\mu_\eta^{(k)}$ *is a Borel probability on* $\{-1, 1\}^{V\setminus A_k}$. (*One may think of the measures* $\mu_\eta^{(k)}$ *as* boundary conditions.)

Finally let $\{P^{(k)}_{\xi_k}\}_{k=1}^{\infty}$ *be a sequence such that for each* k, $P^{(k)}_{\xi_k}$ *is a* SMP (G_k, ξ_k), *and such that* $\xi_k \to \xi$. *Then* $P_\xi = \text{weak-lim}_k P^{(k)}_{\xi_k}$.

PROOF. By Theorem 2.1 of [7], the sequence $\{P^{(k)}_{\xi_k}\}_{k=1}^{\infty}$ is relatively weakly compact under either condition. Thus it is enough to show in each case that if P' is a weak limit point of $\{P^{(k)}_{\xi_k}\}_{k=1}^{\infty}$ and if Φ is as in Theorem 2.14, then $E_\xi(\Phi) = E'(\Phi)$. Fix such P' and Φ. By Proposition 2.4, Φ is P'-a.s. continuous on Ω, so by Theorem 5.2 of [1], $E_\xi(\Phi) = E'(\Phi)$ if $\lim_k E^{(k)}_{\xi_k}(\Phi) = E_\xi(\Phi)$. To show this last equality, let A and T be as in Theorem 2.14. Choose $\delta > 0$. By (3.2), choose $B \in \mathcal{V}_0$ such that $B \supset A$ and

$$|\sup_{\varphi\in\mathcal{K}_B} E^{\varphi, G}_\xi(\Phi) - \inf_{\varphi\in\mathcal{K}_B} E^{\varphi, G}_\xi(\Phi)| < \delta/6.$$

Now assume (3.13). Choose $K \in \mathbb{Z}$ so that for $k > K$, $\Sigma_{x\in B}\|c^{(k)}_x - c_x\| < \delta/(6T\|\Phi\|)$ and $\xi|_B = \xi_k|_B$. We will show that for $k > K$, $|E^{(k)}_{\xi_k}(\Phi) - E_\xi(\Phi)| < \delta$. Fix $k > K$. Then

$$\begin{aligned}|E^{(k)}_{\xi_k}(\Phi) - E_\xi(\Phi)| &\leqslant \sup_{\varphi\in\mathcal{K}_B}|E^{(k)}_{\xi_k}(\Phi) - E^{\varphi, G_k}_{\xi_k}(\Phi)| \\ &\quad + \sup_{\varphi\in\mathcal{K}_B}|E^{\varphi, G_k}_\xi(\Phi) - E^{\varphi, G}_\xi(\Phi)| + \sup_{\varphi\in\mathcal{K}_B}|E^{\varphi, G}_\xi(\Phi) - E_\xi(\Phi)|,\end{aligned}$$

since $\xi_k|_B = \xi|_B$ for such k. Now apply Theorem 2.14 to the first and third terms on the right side of this last expression, and apply Proposition 3.8 and (3.10) to the second term to get:

$$\begin{aligned}(3.15) \qquad |E^{(k)}_{\xi_k}(\Phi) - E_\xi(\Phi)| &\leqslant |\sup_{\varphi\in\mathcal{K}_B} E^{\varphi, G_k}_{\xi_k}(\Phi) - \inf_{\varphi\in\mathcal{K}_B} E^{\varphi, G_k}_{\xi_k}(\Phi)| \\ &\quad + T\big(\Sigma_{x\in B}\|c^{(k)}_x - c_x\|\big)\|\Phi\| \\ &\quad + |\sup_{\varphi\in\mathcal{K}_B} E^{\varphi, G}_\xi(\Phi) - \inf_{\varphi\in\mathcal{K}_B} E^{\varphi, G}_\xi(\Phi)|.\end{aligned}$$

(In this application of Proposition 3.8, let both the sets B and B' of Proposition 3.8 be the set B of this proof.) By the way in which B and K were chosen, the right side of (3.15) is less than

$$\begin{aligned}&|\sup_{\varphi\in\mathcal{K}_B} E^{\varphi, G_k}_\xi(\Phi) - \inf_{\varphi\in\mathcal{K}_B} E^{\varphi, G_k}_\xi(\Phi)| + \delta/6 + \delta/6 \\ &\quad \leqslant 2\big(\sup_{\varphi\in\mathcal{K}_B}|E^{\varphi, G}_\xi(\Phi) - E^{\varphi, G_k}_\xi(\Phi)|\big) + |\sup_{\varphi\in\mathcal{K}_B} E^{\varphi, G}_\xi(\Phi) - \inf_{\varphi\in\mathcal{K}_B} E^{\varphi, G}_\xi(\Phi)| \\ &\qquad + \delta/3.\end{aligned}$$

Now if we replace the quantities in this last expression by the upper bounds used already above, we see that the whole expression is less than δ, completing this part of the proof.

Now assume (3.14) instead of (3.13). Choose $M \in \mathbb{Z}$ so that for all $m > M$, $\xi|_B = \xi_m|_B$ and $A_m \supset B$, where $B \in \mathcal{V}_0$ is as above. We will show that for $m > M$, $|E^{(m)}_{\xi_m}(\Phi) - E_\xi(\Phi)| < \delta$. By Theorem 2.14 and the choice of B, we will be done if we can show that for all $m > M$,

$$(3.16) \qquad \sup_{\varphi\in\mathcal{K}_B} E^{\varphi, G}_{\xi_m}(\Phi) \geqslant E^{(m)}_{\xi_m}(\Phi) \geqslant \inf_{\varphi\in\mathcal{K}_B} E^{\varphi, G}_{\xi_m}(\Phi),$$

since for $\varphi \in \mathcal{K}_B$, $E^{\varphi, G}_\xi = E^{\varphi, G}_{\xi_m}$. The proof of (3.16) is essentially the same as the proof of Theorem 2.14, and we will therefore be brief. We prove the first inequality in (3.16) only. We will assume that Φ satisfies the additional restrictions imposed at the beginning of the proof of Theorem 2.14. Fix $m > M$ and choose $\varepsilon > 0$. Now define $\varphi \in \mathcal{K}_B$ by following word for word the definition of φ in the proof of Theorem 2.14, using the set B of this proof in place of the set A of the proof of Theorem 2.14. We will prove an analogue to (2.16), namely, for $k = 1, 2, \cdots, N$,

$$E_{\xi_m}^{(m)} \otimes_{t_{k-1}} E^{\varphi_{k-1}, G}(\Phi) + 3\varepsilon/N \geqslant E_{\xi_m}^{(m)} \otimes_{t_k} E^{\varphi_k, G}(\Phi), \tag{3.17}$$

where φ_k, φ_{k-1}, and N are as in the proof of Theorem 2.14. The proof of (3.17) follows closely the proof of (2.16). Note that the first inequality in (3.16) follows from (3.17) just as the first inequality in (2.15) follows from (2.16). Fix $k \in \{0, 1, \cdots, N\}$. Now write $P^\omega = (P_{\xi_m}^{(m)})^{\omega, t_{k-1}}$ for $\omega \in \Omega$, and fix ω such that P^ω is a SMP $(\xi_{t_{k-1}}(\omega), G_m)$. Let $\bar{\xi} = \xi_{t_{k-1}}(\omega)$. Define μ' to be the Borel measure on $\{-1, 1\}^{V\setminus B}$ determined by

$$\mu'(\zeta : \zeta|_{A_m\setminus B} = \eta \text{ and } \zeta|_{V\setminus A_m} \in Z) = \left[\textstyle\int_0^{T/N} P^\omega(\xi_s|_{A_m\setminus B} = \eta)ds/(T/N)\right]\mu_\eta^{(m)}(Z)$$

for all $\eta \in \{-1, 1\}^{A_m\setminus B}$ and Borel sets Z in $\{-1, 1\}^{V\setminus A_m}$. Notice that μ' and the μ of the proof of Theorem 2.14 are different. Let Φ_{k-1} be as in the proof of Theorem 2.14. Then

$$\begin{aligned} E_{\bar{\xi}}^{\varphi_{k-1}, G}(\Phi_{k-1}) - E^\omega \otimes_{T/N} E^{\varphi_k, G}(\Phi_{k-1}) &\geqslant -6((T/N)\Sigma_{x\in B}\|c_x\|)^2 - \varepsilon/N \\ &\quad -\left|\textstyle\int_{\{-1, 1\}^{V\setminus B}} E_{(G, \bar{\xi}|_B, \zeta)} \otimes_{T/N} E^{\varphi_k, G}(\Phi_{k-1})\mu'(d\zeta) \right.\\ &\quad \left. - E^\omega \otimes_{T/N} E^{\varphi_k, G}(\Phi_{k-1})\right|. \end{aligned} \tag{3.18}$$

(See the proof of (2.16), and, in particular, the derivation of lower bounds for E_1 and E_2, which works here even though μ' is different than μ.) Now define μ (as in the proof of Theorem 2.14) to be the Borel measure on $\{-1, 1\}^{V\setminus B}$ determined by

$$\mu(\cdot) = \textstyle\int_0^{T/N} P^\omega(\xi_s|_{V\setminus B} \in \cdot)ds/(T/N).$$

Then the right side of (3.18) is bounded below by

$$\begin{aligned} &-12((T/N)\Sigma_{x\in B}\|c_x\|)^2 - \varepsilon/N \\ &-\left|\textstyle\int_{\{-1, 1\}^{V\setminus B}} E_{(G, \bar{\xi}|_B, \zeta)} \otimes_{T/N} E^{\varphi_k, G}(\Phi_{k-1})\mu'(d\zeta)\right. \\ &\left.-\textstyle\int_{\{-1, 1\}^{V\setminus B}} E_{(G_m, \bar{\xi}|_B, \zeta)} \otimes_{T/N} E^{\varphi_k, G}(\Phi_{k-1})\mu(d\zeta)\right| \end{aligned}$$

by Lemma 2.9 applied to P^ω. By the same sort of reasoning used in getting the first inequality in (2.12) in the proof of Lemma 2.9, and by the way in which N is chosen, the quantity above is bounded below by

$$\begin{aligned} -2\varepsilon/N - 2\Sigma_{x\in B}&\left|\textstyle\int_{\{-1, 1\}^{V\setminus B}} P_{(G, \bar{\xi}|_B, \zeta)}\left(\xi_{\sigma_B} = {}_x\bar{\xi}\right)\mu'(d\zeta)\right. \\ &\left.-\textstyle\int_{\{-1, 1\}^{V\setminus B}} P_{(G_m, \bar{\xi}|_B, \zeta)}\left(\xi_{\sigma_B} = {}_x\bar{\xi}\right)\mu(d\zeta)\right|, \end{aligned}$$

which is bounded below by

$$(3.19) \qquad -2\varepsilon/N - 2\Sigma_{x\in B}\left|\int_{\{-1,1\}^{V\setminus B}}\left[P_{(G,\bar{\xi}|_B,\zeta)}\left(\xi_{\sigma_B} = {}_x\bar{\xi}\right) - (T/N)c_x\left(\bar{\xi}|_B \times \zeta\right)\right]\mu'(d\zeta)\right| - 2\Sigma_{x\in B}\left|\int_{\{-1,1\}^{V\setminus B}}\left[P_{(G_m,\bar{\xi}|_B,\zeta)}\left(\xi_{\sigma_B} = {}_x\bar{\xi}\right) - (T/N)c_x^{(m)}\left(\bar{\xi}|_B \times \zeta\right)\right]\mu(d\zeta)\right|,$$

since by Fubini's theorem,

$$\int_{\{-1,1\}^{V\setminus B}} c_x^{(m)}\left(\bar{\xi}|_B \times \zeta\right)\mu(d\zeta) = \int_{\{-1,1\}^{V\setminus B}} c_x\left(\bar{\xi}|_B \times \zeta\right)\mu'(d\zeta)$$

for all $x \in B$. Now by the same application of (2.5) and the optional sampling theorem which was used to get (2.13), both of the sums in (3.19) are bounded above by $((T/N)\Sigma_{x\in B}\|c_x\|)^2$, so that (3.19) is bounded below by $-3\varepsilon/N$. This yields (3.17). □

REMARK 3.20. By taking $c^{(k)} = c$ for all k, we get a type of Feller property from Theorem 3.12, namely, if Φ is as in Theorem 2.14, if $(P_\xi)_{\xi\in\Xi'}$ is a collection such that for each $\xi \in \Xi'$, P_ξ is a SMP (G, ξ), and if $h : \Xi \to \mathbb{R}$ is defined by $h(\xi) = E_\xi(\Phi)$, then h is continuous at all points ξ such that (3.2) holds for G and ξ.

THEOREM 3.21. *Let Ξ', G, and $(P_\xi)_{\xi\in\Xi'}$ be as in the previous remark. Assume in addition that Ξ' is Borel, and that if $\xi \in \Xi'$ then* (3.2) *holds for G and ξ. Let μ be a Borel measure on Ξ such that $\mu(\Xi') = 1$. Consider the following four conditions*:

$$(3.22) \qquad \int_{\Xi'} E_\xi(f(\xi_t))\mu(d\xi) = \int_{\Xi'} f(\xi)\mu(d\xi) \quad \text{for all } f \in \mathcal{C} \text{ and all } t \geqslant 0;$$

$$(3.23) \qquad \int_{\Xi'} Gf(\xi)\mu(d\xi) = 0 \quad \text{for all } f \in \mathcal{F};$$

$$(3.24) \qquad \int_{\Xi'} f(\xi)(E_\xi(g(\xi_t)))\mu(d\xi) = \int_{\Xi'} E_\xi(f(\xi_t))g(\xi)\mu(d\xi) \quad \text{for all } f, g \in \mathcal{C} \text{ and all } t \geqslant 0;$$

$$(3.25) \qquad \int_{\Xi'} f(\xi)(Gg(\xi))\mu(d\xi) = \int_{\Xi'} Gf(\xi)g(\xi)\mu(d\xi) \quad \text{for all } f, g \in \mathcal{F}.$$

Then (3.22) *is equivalent to* (3.23) *and* (3.24) *is equivalent to* (3.25). (*The measure μ is called an* equilibrium state *for $(P_\xi)_{\xi\in\Xi'}$ if* (3.22) *holds, and a* time-reversible equilibrium state *for $(P_\xi)_{\xi\in\Xi'}$ if* (3.24) *holds.*)

PROOF. The proof that (3.22) ⇒ (3.23) and (3.24) ⇒ (3.25) is standard. We will show that (3.23) ⇒ (3.22). The argument that (3.25) ⇒ (3.24) is virtually the same.

Assume that (3.23) holds. We use a modification of a technique found in [6]. It is enough to show (3.22) for all $f \in \mathcal{F}$. Fix $f \in \mathcal{F}$, and choose $\{A_k\}_{k=1}^\infty \subset \mathcal{V}_0$ such that $A_k \nearrow V$ and such that $f \in \mathcal{F}^{A_1}$. For $\eta \in \{-1, 1\}^{A_k}$, let $\mu_\eta^{(k)}$ be the Borel probability on $\{-1, 1\}^{V\setminus A_k}$ defined for $Z \subset \{-1, 1\}^{V\setminus A_k}$ by

$$\mu_\eta^{(k)}(Z) = \mu(\xi \in \Xi : \xi|_{A_k} = \eta \text{ and } \xi|_{V\setminus A_k} \in Z)/\mu(\xi \in \Xi : \xi|_{A_k} = \eta)$$

when $\mu(\xi \in \Xi : \xi|_{A_k} = \eta) > 0$; otherwise, let $\mu_\eta^{(k)}$ be arbitrary. For $k =$

1, 2, 3, $\cdots$, define pregenerators G_k with rates $c^{(k)}$, where

$$c_x^{(k)}(\xi) = \int_{\{-1,1\}^{V\setminus A_k}} c_x(\xi|_{A_k} \times \zeta)\mu^{(k)}_{\xi|Ak}(d\zeta) \qquad x \in A_k$$
$$= 0 \qquad\qquad x \in V \setminus A_k.$$

Then by Fubini's theorem, we have for $k \geq 1$ and $g \in \mathcal{F}^{A_k}$, $\int G_k g(\xi)\mu(d\xi) = 0$ so that for all $t \geq 0$,

$$\int E_\xi^{(k)} f(\xi_t)\mu(d\xi) = \int f(\xi)\mu(d\xi),$$

where for each $\xi \in \Xi$, $P_\xi^{(k)}$ is the unique SMP (G_k, ξ). By Theorem 3.12, $E_\xi f(\xi_t) = \lim_k E_\xi^{(k)} f(\xi_t)$ for $t \geq 0$, $f \in \mathcal{F}$, and $\xi \in \Xi'$, and so (3.22) follows by the bounded convergence theorem. $\square$

REMARK *(added in revision).* Let $\Xi_G = \{\xi \in \Xi : (3.2)$ holds for ξ and $G\}$. Then if it is known that there exists a SMP(G, ξ) for each $\xi \in \Xi_G$, it would be desirable to take $\Xi' = \Xi_G$ in applications of Theorem 3.21. We therefore remark that Ξ_G is a Borel set for any G (it is in fact a G_δ-set). The reason for this fact is as follows: Fix G. For each Φ as in Theorem 2.14, let

$$\Xi(\Phi) = \left\{\xi \in \Xi : \lim_{A\uparrow V;\, A\in\mathcal{V}_0} \sup_{\varphi\in\mathcal{K}_A} E_\xi^{\varphi, G}(\Phi) = \lim_{A\uparrow V;\, A\in\mathcal{V}_0} \inf_{\varphi\in\mathcal{K}_A} E_\xi^{\varphi, G}(\Phi)\right\}.$$

It follows from Remark 3.3 that $\Xi(\Phi)$ is an open set for each such Φ. Furthermore, it can be easily shown that there is a countable set $\Phi_1, \Phi_2, \Phi_3, \cdots$, such that $\Xi_G = \cap\, \Xi(\Phi_n)$.

4. Applications to specific classes of pregenerators. In [15], Spitzer introduced an interesting class of flip rates on $\Xi = \{-1, 1\}^{\mathbf{Z}}$ called *nearest particle rates*. Under the assumption that a result such as Theorem 3.21 applies, he was able to completely characterize the time-reversible equilibrium states of processes with nearest particle rates. We show that this assumption was valid by proving here that the hypotheses of Theorem 3.21 are satisfied by a class of pregenerators which includes those studied in [15]. See also [2] and [10] for a discussion of nearest particle processes in the more general setting in which $V = \mathbb{R}$. Uniqueness results slightly less general than the ones given here were proved independently in [2] and [10].

Let $V = \mathbb{Z}$, and let $\Xi_1 = \{\xi \in \Xi : \xi(x) = 1$ for infinitely many positive and infinitely many negative $x \in V\}$.

DEFINITION 4.1. Let k be a positive integer. We call rates c *nearest k particle flip rates* if for all $\xi \in \Xi_1$,

$$c_x(\xi) = \beta_x[l_1(x,\xi), l_2(x,\xi), \cdots, l_k(x,\xi), r_1(x,\xi), r_2(x,\xi), \cdots, r_k(x,\xi)] \text{ if } \xi(x) = -1$$
$$= \delta_x[l_1(x,\xi), l_2(x,\xi), \cdots, l_k(x,\xi), r_1(x,\xi), r_2(x,\xi), \cdots, r_k(x,\xi)] \text{ if } \xi(x) = 1$$

where β_x and δ_x are bounded functions from $(\mathbb{Z}^+)^{2k}$ into $[0, \infty)$, and $l_i(x, \xi)$ $(r_i(x, \xi))$ is defined to be the distance to the ith nearest site $y \in V$ strictly to the left (right) of x such that $\xi(y) = 1$. When $k = 1$, we also call c *nearest particle rates.* Here, we are thinking of a 1 as a particle and a -1 as a vacancy. The functions β_x and δ_x are called *birth rates* and *death rates* respectively.

Before continuing with nearest k particle rates, we need a lemma concerning general spin-flip processes.

LEMMA 4.2. *Fix $\xi \in \Xi$ and $M \in \mathbb{Z}^+$. Let $\{A_n\}_{n=1}^N$ be disjoint subsets of V of cardinality less than or equal to M. Suppose that G is a pregenerator with rates c such that for some $\gamma \in [0, \infty)$,*

$$\sup_{\xi' : \xi'(x)=\xi(x)} c_x(\xi') < \gamma, \qquad \textit{for all} \quad x \in \cup_{n=1}^N A_n.$$

Then for any $\varphi \in \mathcal{K}$ and $T \in [0, \infty)$, we have

$$P_\xi^{\varphi, G}\big(\exists t_1, t_2, \cdots, t_N \in [0, T] : \xi_{t_n|A_n} \neq \xi_{|A_n} \quad \textit{for each} \quad n = 1, 2, \cdots, N\big) \leqslant (1 - e^{-\gamma TM})^N. \tag{4.3}$$

PROOF. By relabeling if necessary, we may assume that $\xi(x) = 1$ for all $x \in \cup_{n=1}^N A_n$. Fix $T \in [0, \infty)$ and $\varphi \in \mathcal{K}$. Let G' be the pregenerator with rates c' defined by

$$c_x'(\xi') = \gamma \qquad \text{if } \xi'(x) = 1$$
$$= 0 \qquad \text{otherwise.}$$

Let $\tilde{P}$ be the basic coupling of $P_\xi^{\varphi, G}$ and $P_\xi^{\varphi, G'}$. By Proposition 3.11, $\tilde{P}(\xi_t^{(1)} \leqslant \xi_t^{(2)}$ for all $t \in [0, \infty)) = 1$, so that the left side of (4.3) is bounded above by

$$P_\xi^{\varphi, G'}\big(\exists t_1, t_2, \cdots, t_N \in [0, T] : \xi_{t_n|A_n} \neq \xi_{|A_n} \text{ for } n = 1, 2, \cdots, N\big).$$

But this last quantity is less than

$$P_\xi^{\varphi, G'}\big(\xi_{T|A_n} \neq \xi_{|A_n} \text{ for all } n = 1, 2, \cdots, N\big)$$

since $c_x'(\xi') = 0$ if $\xi'(x) = -1$. Finally, since flips at different sites in A occur independently in the process corresponding to $P_\xi^{\varphi, G'}$,

$$P_\xi^{\varphi, G'}\big(\xi_{T|A_n} \neq \xi_{|A_n} \text{ for all } n = 1, 2, \cdots, N\big) = \Pi_{n=1}^N P_\xi^{\varphi, G'}\big(\xi_{T|A_n} \neq \xi_{|A_n}\big) = (1 - e^{-c\gamma TM})^N. \qquad \square$$

To state and prove the main result concerning processes with nearest k particle flip rates, we need the following notation: for $k = 1, 2, 3, \cdots,$ let

$$\Xi_k = \{\xi \in \Xi : \exists n \in \mathbb{Z}^+ \text{ such that } \#[\{x \in V : \xi(x) = 1\} \cap \{y + 1, y + 2, \cdots, y + n\}] \geqslant k \text{ for infinitely many positive and infinitely many negative } y \in V\}.$$

THEOREM 4.4. *Let G be a pregenerator with nearest k particle rates c such that* $\sup_{x\in V}\|\delta_x\| = \delta < \infty$. *Then for each* $\xi \in \Xi_1$, *there exists a measure* P_ξ *which is a* SMP (G, ξ). *Furthermore, for* $l = 1, 2, 3, \cdots$, *and* $\xi \in \Xi_l$, Ξ_l *is* P_ξ*-stochastically closed. Finally, if* $k = 1$, (3.2) *holds for G and all* $\xi \in \Xi_1$, *while if* $k > 1$, *and if in addition* $\sup_x\|c_x\| = \gamma < \infty$, *then* (3.2) *holds for G and all* $\xi \in \Xi_k$.

REMARK *(added in revision).* Examples can be constructed to show that the condition $\gamma < \infty$ cannot be dropped completely when $k > 1$.

PROOF OF THEOREM 4.4. Fix $\xi \in \Xi_1$. For $n = 1, 2, \cdots$, let $\varphi_n \in \mathcal{K}_{\{x : |x| \leqslant n\}}$ be defined by $\varphi_n(t, \eta) = \xi_{|V\setminus\{x : |x|\leqslant n\}}$. By Theorem 2.1 of [7], there is a measure P_ξ which is the weak limit of some subsequence of $\{P_\xi^{\varphi_n, G}\}_{n=1}^{\infty}$. We will show first that for each l such that $\xi \in \Xi_l$, Ξ_l is P_ξ-stochastically closed. Fix l such that $\xi \in \Xi_l$. By the definition of Ξ_l, choose $M \in \mathbb{Z}^+$, and $y_i \in V$ for $i = 1, 2, 3, \cdots$, such that the sets $A_i = \{y_i + 1, y_i + 2, \cdots, y_i + M\}$ are disjoint, such that $\#(A_i \cap \{x \in V : \xi(x) = 1\}) \geqslant l$, and such that the set $\cup_{i=1}^{\infty} A_i$ contains infinitely many positive and infinitely many negative elements of $\mathbb{Z}$. For each $A \in \mathcal{V}_0$, let $I(A) = \{i : A_i \subset A\}$ and let $N(A) = \#I(A)$. Also, let $A_i^+ = A_i \cap \{x : \xi(x) = 1\}$, for $i = 1, 2, 3, \ldots$. Then by Lemma 4.2, if $A \in \mathcal{V}_0$ and $T \in (0, \infty)$, we get for all $n = 1, 2, 3, \cdots$,

$$(4.5)\qquad P_\xi^{\varphi_n, G}(\forall i \in I(A) \text{ there exist } t_i \in [0, T) \text{ and } x_i \in A_i^+ \text{such that } \xi_{t_i}(x_i) = -1) \leqslant (1 - e^{-\delta TM})^{N(A)}.$$

Since P_ξ is the weak limit of some subsequence of $\{P_\xi^{\varphi_n, G}\}_{n=1}^{\infty}$, (4.5) implies

$$(4.6)\qquad P_\xi(\forall i \in I(A) \text{ there exist } t_i \in [0, T) \text{ and } x_i \in A_i^+ \text{such that } \xi_{t_i}(x_i) = -1) \leqslant (1 - e^{-\delta TM})^{N(A)}.$$

This implies that Ξ_l is P_ξ-stochastically closed.

Now we show that P_ξ is a SMP (G, ξ). Choose $0 \leqslant s < t$, and let ψ be a continuous functional on Ω which is $\mathfrak{B}_0^s$-measurable, and choose $f \in \mathfrak{F}$. We will show

$$(4.7)\qquad E_\xi\big(\psi\big(f(\xi_t) - \textstyle\int_0^t Gf(\xi_u)du\big)\big) = E_\xi\big(\psi\big(f(\xi_s) - \textstyle\int_0^s Gf(\xi_u)du\big)\big).$$

Define G_n to be the pregenerator with rates $c^{(n)}$, where

$$c_x^{(n)}(\xi') = c_x(\xi') \qquad |x| \leqslant n$$
$$= 0 \qquad \text{otherwise}, \qquad n = 1, 2, 3, \ldots .$$

Then $P_\xi^{\varphi_n, G}$ is the unique SMP (G_n, ξ) for $n = 1, 2, 3, \cdots$, so (4.7) follows if we can show

$$(4.8)\quad \lim_n E_\xi^{\varphi_n, G}\big(\psi\big(f(\xi_r) - \textstyle\int_0^r G_n f(\xi_u)du\big)\big) = E_\xi\big(\psi\big(f(\xi_r) - \textstyle\int_0^r Gf(\xi_u)du\big)\big) \text{ for all } r \geqslant 0.$$

Fix $r \geqslant 0$. It follows from (2.5) that $\psi f(\xi_r)$ is a P_ξ-a.s. continuous functional on Ω, since P_ξ is the weak limit of a sequence of SMP's, so that

$$\lim_n E_\xi^{\varphi_n, G}(\psi f(\xi_r)) = E_\xi(\psi f(\xi_r))$$

by Theorem 5.2 of [1]. By the definition of G_n,

$$\lim_n E_\xi^{\varphi_n, G}(\psi(\int_0^r (G_n f(\xi_u) - Gf(\xi_u))du)) = 0,$$

so we have (4.8) if we show that

$$\lim_n E_\xi^{\varphi_n, G}(\psi(\int_0^r Gf(\xi_u)du)) = E_\xi(\psi(\int_0^r Gf(\xi_u)du)).$$

This last identity follows from Theorem 5.2 of [1] provided $\int_0^r Gf(\xi_u)du$ is a P_ξ − a.s. continuous functional on Ω. But this follows from the fact that (4.6) implies that

$$\lim_{A \nearrow V} P_\xi\big(Gf(\xi_u) = Gf(\xi_{u|A} \times \zeta) \quad \text{for all}$$
$$\zeta \in \{-1, u\}^{V \setminus A} \quad \text{and all} \quad u \in [0, r]\big) = 1.$$

The proofs of the last part of the theorem for the cases $k = 1$ and $k > 1$ are so similar that we combine them into one. To do this, we will use the letter ρ for δ if $k = 1$ and for γ if $k > 1$. We will show that (3.2) holds for G and ξ under the assumption that $\rho < \infty$. Let $\{y_i\}_{i=1}^\infty$, $\{A_i\}_{i=1}^\infty$, M, $I(\cdot)$, and $N(\cdot)$ be as above, with $l = k$. If $k = 1$, we may assume that $A_i = A_i^+$ for all $i = 1, 2, 3, \dots$. Let Φ, A, and T be as in Theorem 2.14. Without loss of generality, we can assume that $A = \{-n, -n+1, \cdots, n-1, n\}$ for some $n \in \mathbb{Z}^+$. Choose $\varepsilon > 0$, and pick $B \in \mathcal{V}_0$ such that $B \supset A$ and such that $(1 - e^{-\rho TM})^N < \varepsilon$, where $N = N((B \setminus A) \cap \mathbb{Z}^-) \cap N((B \setminus A) \cap \mathbb{Z}^+)$. We will show that for all $\varphi, \varphi' \in \mathcal{K}_B$,

$$|E_\xi^{\varphi, G}(\Phi) - E_\xi^{\varphi', G}(\Phi)| < 4\varepsilon \|\Phi\|.$$

The idea of the proof is to use those sets A_i which are in $B \setminus A$ as "buffer zones": as long as no flips occur on one such set to the right of A and one such set to the left of A, the influence of the controls φ and φ' cannot get in to affect the behavior of the system on A. With this idea in mind, choose $\varphi, \varphi' \in \mathcal{K}_B$, and let $\tilde{P}$ be the basic coupling of $P_\xi^{\varphi, G}$ and $P_\xi^{\varphi', G}$. Proposition 3.8 implies that if $y_i < y_j$, then

$$\tilde{P}\big(\exists t \in [0, \tau) : \xi_t^{(1)}|_{\{x : y_i + M < x \le y_j\}} \ne \xi_t^{(2)}|_{\{x : y_i + M < x < y_j\}}\big) = 0,$$

where τ is defined on $\Omega \times \Omega$ by

$$\tau = T \wedge \big(\inf\{t > 0 : \xi_{t|A_i}^{(1)} \ne \xi_{|A_i} \text{ or } \xi_{t|A_j}^{(1)} \ne \xi_{|A_j}$$
$$\text{or } \xi_{t|A_i}^{(2)} \ne \xi_{|A_i} \text{ or } \xi_{t|A_j}^{(2)} \ne \xi_{|A_j}\}\big).$$

But this implies that $\tilde{P}(\exists t \in [0, \tau') : \xi_{t|A}^{(1)} \ne \xi_{t|A}^{(2)}) = 0$, where τ' is defined on $\Omega \times \Omega$ by

$$\tau' = T \wedge \big(\inf\{t > 0 : \quad \text{for all} \quad i \in I((B \setminus A) \cap \mathbb{Z}^-) \exists t_i \in [0, t]$$
$$\text{such that} \quad \xi_{t_i|A_i}^{(1)} \ne \xi_{|A_i} \text{ or } \xi_{t_i|A_i}^{(2)} \ne \xi_{|A_i}\}\big)$$
$$\wedge \big(\inf\{t > 0: \quad \text{for all} \quad i \in I((B \setminus A) \cap \mathbb{Z}^+) \exists t_i \in [0, t]$$
$$\text{such that } \xi_{t_i|A_i}^{(1)} \ne \xi_{|A_i} \text{ or } \xi_{t_i|A_i}^{(2)} \ne \xi_{|A_i}\}\big).$$

By Lemma 4.2, $\tilde{P}(\tau' < T) \leqslant 4(1 - e^{-\rho TM})^N$, since the marginals of $\tilde{P}$ are $P_\xi^{\varphi, G}$ and $P_\xi^{\varphi', G}$. Hence, by (3.10),

$$|E_\xi^{\varphi, G}(\Phi) - E_\xi^{\varphi', G}(\Phi)| \leqslant 4\|\Phi\|(1 - e^{-\rho TM})^N < 4\|\Phi\|\varepsilon. \qquad \square$$

One reason that the class of nearest particle processes does not satisfy earlier uniqueness conditions is that in such processes, the flip rate at a site x may depend strongly on the spins at sites which are very far from x. In cases where the long-range dependence is weak enough, the most useful uniqueness conditions are those of Liggett [12]. Liggett's conditions also have the advantage of being applicable when V is not assumed to have any particular structure. We will give a new proof here of a generalization of Liggett's uniqueness result, a version of Corollary 1 of [4]. There are other generalizations of Liggett's theorem in [4] and [16], and we could extend our proof to handle them, but we do not wish to obscure the main idea with technicalities.

THEOREM 4.9. *Let G be a pregenerator with continuous rates c. Suppose there exists* $(\lambda_x)_{x \in v} \subset \mathbb{R}^+$ *such that* $\inf_x \lambda_x = \lambda > 0$ *and such that*

$$\sup_{x \in V} \Sigma_{y \in V;\, y \neq x} \sup_{\xi \in \Xi} \frac{\lambda_y}{\lambda_x} |c_x({}_y\xi) - c_x(\xi)| = K < \infty. \tag{4.10}$$

Then (3.2) *holds for G and all* $\xi \in \Xi$.

REMARK 4.11. Condition (4.10) does not imply continuity.

PROOF OF THEOREM 4.9. Let Φ, A, and T be as in Theorem 2.14. It is enough to show that (3.2) holds for G and all $\xi \in \Xi$ whenever $T \leqslant 1/(2K)$, since once we had that we could argue as follows: for each $\xi \in \Xi$, the measure $P_\xi^{\omega, 1/(2K)}$ is a SMP $(G, \xi_{1/(2K)}(\omega))$ for P_ξ-almost all ω, and as such would be uniquely determined up to time $1/(2K)$. This would uniquely determine P_ξ up to time $1/K$. Then we could continue inductively to show that P_ξ is uniquely determined up to time $n/(2K)$ for all $n \geqslant 0$. Thus, assume $T < 1/(2K)$. Fix $\xi \in \Xi$. For each $B \in \mathcal{V}_0$ and for $\varphi_1, \varphi_2 \in \mathcal{K}_B$, let $\tilde{P}_{\varphi_1, \varphi_2}$ be the basic coupling of $P_\xi^{\varphi_1, G}$ and $P_\xi^{\varphi_2, G}$. We first show by induction that for each $n = 0, 1, 2, 3, \cdots$,

$$\lim_{B \nearrow V;\, B \in \mathcal{V}_0} \sup_{\varphi_1, \varphi_2 \in \mathcal{K}_B} \tilde{P}_{\varphi_1, \varphi_2}\big(\exists t \in [0, T] : \xi_t^{(1)}(x) \neq \xi_t^{(2)}(x)\big) \leqslant \frac{\lambda_x}{\lambda \cdot 2^n} \quad \text{for all } x \in V. \tag{4.12}$$

The case $n = 0$ is obvious. Now suppose (4.12) holds for some $n \geqslant 0$. Choose $\varepsilon > 0$ and $x \in V$, and pick $B \in \mathcal{V}_0$ such that $x \in B$ and $\Sigma_{y \in V \setminus B} \sup_{\xi' \in \Xi} |c_x({}_y\xi') - c_x(\xi')| < \lambda_x \varepsilon$. This last is possible by (4.10). By the inductive hypothesis, pick $C \in \mathcal{V}_0$ such that $C \supset B$ and such that for all $y \in B \setminus \{x\}$

$$\sup_{\varphi_1, \varphi_2 \in \mathcal{K}_C} \tilde{P}_{\varphi_1, \varphi_2}\big(\exists t \in [0, T] : \xi_t^{(1)}(y) \neq \xi_t^{(2)}(y)\big) \leqslant \frac{\lambda_y(1 + \varepsilon)}{\lambda 2^n}.$$

Then by Proposition 3.8, if $\varphi_1, \varphi_2 \in \mathcal{K}_C$, we have

$$\tilde{P}_{\varphi_1, \varphi_2}(\exists t \in [0, T] : \xi_t^{(1)}(x) \neq \xi_t^{(2)}(x))$$
$$\leqslant \tilde{E}_{\varphi_1, \varphi_2}(\textstyle\int_0^T |c_x(\xi_t^{(1)}|_C \times \varphi_1(t, \xi_t^{(1)}|_C))$$
$$- c_x(\xi_t^{(1)}|_x \times \xi_t^{(2)}|_{C\setminus x} \times \varphi_2(t, \xi_t^{(1)}|_x \times \xi_t^{(2)}|_{C\setminus x}))|dt)$$
$$\leqslant T\Sigma_{y \in B; y \neq x}\big[(\sup_{\xi' \in \Xi}|c_x({}_y\xi') - c_x(\xi')|)$$
$$\times \tilde{P}_{\varphi_1, \varphi_2}(\exists t \in [0, T] : \xi_t^{(1)}(y) \neq \xi_t^{(2)}(y))\big]$$
$$+ \Sigma_{y \in V \setminus B} \sup_{\xi' \in \Xi}|c_x({}_y\xi') - c_x(\xi')|$$
$$\leqslant 1/(2K)\Sigma_{y \in B; y \neq x} \sup_{\xi' \in \Xi}\left(\frac{\lambda_y(1+\varepsilon)}{\lambda 2^n}|c_x({}_y\xi') - c_x(\xi')|\right) + \lambda_x \varepsilon.$$

(Here, we write x for $\{x\}$.) But by (4.10), this last expression is bounded by

$$\frac{\lambda_x}{\lambda}\left(\frac{1+\varepsilon}{2^{n+1}} + \varepsilon\right),$$

completing the inductive proof since $\varepsilon > 0$ was chosen arbitrarily. Now when $n \to \infty$ in (4.12), we get (3.2) for Φ, since by (3.10),

$$\lim_{B \nearrow V;\, B \in \mathcal{V}_0}|\sup_{\varphi \in \mathcal{K}_B} E_\xi^{\varphi, G}(\Phi) - \inf_{\varphi \in \mathcal{K}_B} E_\xi^{\varphi, G}(\Phi)|$$
$$\leqslant \Sigma_{x \in A}\|\Phi\| \lim_{B \nearrow V;\, B \in \mathcal{V}_0} \sup_{\varphi_1, \varphi_2 \in \mathcal{K}_B} \tilde{P}_{\varphi_1, \varphi_2}(\exists t \in [0, T] : \xi_t^{(1)}(x) \neq \xi_t^{(2)}(x)).$$

□

The author intends to show in a future paper that the method used in this last proof can be extended to get useful improvements in uniqueness results that have been obtained for a class of spin-flip processes known as stochastic Ising models (see [7], [8], and [9]).

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