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POINTWISE ERGODICITY OF THE BASIC CONTACT PROCESS

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We prove a complete pointwise ergodic theorem for Harris' basic one-dimensional two-sided nearest neighbor contact process, provided that the corresponding one-sided contact process is nonergodic.

In this note we prove a complete pointwise ergodic theorem for Harris' one-dimensional two-sided nearest neighbor contact processes [3] with sufficiently large parameter value. Let $\mathbb{Z}$ = the integers, $\Xi = \{\text{all subsets of } \mathbb{Z}\}$, $\Xi_0 = \{\text{finite subsets of } \mathbb{Z}\}$, $\Xi_\infty = \Xi - \Xi_0$. Attention will focus on two Ξ-valued Markov processes:

(i) the *basic contact process* (ξ_t) with "flip rates" at $x \in \mathbb{Z}$ when the state is ξ given by

at x	*with rate*
$1 \to 0$	1
$0 \to 1$	$\lambda\lvert\{x-1, x+1\} \cap \xi\rvert,$

($|\xi|$ is the cardinality of ξ) and

(ii) the *one-sided contact process* (ζ_t) by

at x	*with rate*
$1 \to 0$	1
$0 \to 1$	$\lambda\lvert\{x-1\} \cap \zeta\rvert.$

Here λ is a positive parameter.

Letting (ξ_t^η) denote the basic contact process started at $\eta \in \Xi$, Harris [5] has shown that all of these systems can be constructed on a joint probability space so that

$$\xi_t^{\eta_1 \cup \eta_2} = \xi_t^{\eta_1} \cup \xi_t^{\eta_2} \qquad \eta_1, \eta_2 \in \Xi,\ t \geq 0. \tag{1}$$

Note that $\varnothing$ is a trap for (ξ_t^η), and define $\tau^\eta = \min\{t: \xi_t^\eta = \varnothing\}$ ($= \infty$ if no such t exists). It is easy to see that

$$\tau^\eta = \infty \quad \text{a.s.} \qquad \forall \eta \in \Xi_\infty. \tag{2}$$

The Markov family $\{(\xi_t^\eta);\ \eta \in \Xi\}$ has a critical value λ^* such that: for $\lambda < \lambda^*$,

$$P(\xi_t^\eta \in \cdot) \Rightarrow \delta_\varnothing \quad \text{as} \quad t \to \infty \qquad \forall \eta \in \Xi, \tag{3a}$$

($\Rightarrow$ denotes weak convergence; $\delta_\varnothing$ is the delta measure at $\varnothing$)

$$P(\tau^\eta < \infty) = 1 \qquad \forall \eta \in \Xi_0; \tag{3b}$$

while, for $\lambda > \lambda^*$,

$$P(\xi_t^{\mathbb{Z}} \in \cdot) \Rightarrow \nu \quad \text{as} \quad t \to \infty, \quad \text{with} \quad \nu(\Xi_0) = 0, \tag{4a}$$

$$P(\tau^\eta = \infty) > 0 \qquad \forall \eta \in \Xi_0 - \{\varnothing\}. \tag{4b}$$

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At $\lambda = \lambda^*$ either properties (3a, b) or (4a, b) hold, but it is not known which. Situation (3a, b) is called the *ergodic* case, (4a, b) the *nonergodic* case.

The remarks of the last paragraph apply equally well to $\{(\zeta_t^\eta);\ \eta \in \Xi\}$. In particular, there is a critical value for the one-sided process, call it λ_0^*. It is known that $1 < \lambda^* \leqslant 2 < \lambda_0^* \leqslant 4$. For details, see [3], [4], [5] and [7].

In [1] it was proved that

$$(5) \qquad P(\xi_t^\eta \in \cdot) \Rightarrow P(\tau^\eta < \infty)\delta_\varnothing + P(\tau^\eta = \infty)\nu \qquad \eta \in \Xi,$$

provided that the one-sided process (ζ_t) with the same parameter λ is nonergodic. A key step in the proof was to consider

$$L_t^x = \min\{y : y \in \xi_t^x\}$$
$$R_t^x = \max\{y : y \in \xi_t^x\},$$

(we write x for $\{x\}$ when convenient) and show that

$$(6) \qquad P(L_t^x \to -\infty \quad \text{and} \quad R_t^x \to +\infty \quad \text{as} \quad t \to \infty \mid \tau^x = \infty) = 1.$$

Liggett [8] then made the useful observation that, because of the nearest neighbor dependence, whenever $x \in \eta \in \Xi$,

$$(7) \qquad \xi_t^x = \{y \in \xi_t^\eta : L_t^x \leqslant y \leqslant R_t^x\} \qquad \text{a.s} \quad \text{on} \quad \{\tau^x > t\}.$$

Equation (7) provides a coupling proof of (5). More importantly, it may be used to show that in the nonergodic case, for *all* parameter values, $\delta_\varnothing$ and ν are the only extreme invariant measures for (ξ_t). An analogous result holds for (ζ_t). See [8].

One of the main results in [5] is a pointwise ergodic theorem for nonergodic contact processes. Namely, Harris shows that if λ is *very* large, then for any $f \in \mathcal{C}$ = continuous functions on Ξ,

$$(8) \qquad \frac{1}{t}\int_0^t f(\xi_s^\eta)\, ds \to \int f d\nu \qquad \text{a.s.}$$

provided that η is "dense." This means there is an $R > 0$ such that η intersects every $\mathbb{Z}$-interval of length R. Actually, Harris' method applies to more general finite range contact processes (including one-sided ones) on any integer lattice $\mathbb{Z}^d$, $d \geqslant 1$. It also applies to the "extralineal" processes, satisfying generalizations of (1), which are discussed in [2].

For the basic contact process a stronger result holds. In [5], Harris mentions an elegant way of proving complete pointwise ergodicity for certain spin systems which admit a "successful coupling." The same strategy, aided by (6) and (7), leads to the following

THEOREM. *Let (ξ_t) be a basic contact process with λ large enough that the corresponding one-sided process (ζ_t) is nonergodic. Then for any $f \in \mathcal{C}$, $\eta \in \Xi$,*

$$(9) \qquad \frac{1}{t}\int_0^t f(\xi_s^\eta)\, ds \to f(\varnothing) \quad \textit{as} \quad t \to \infty \qquad \text{a.s.} \quad \textit{on} \quad \{\tau^\eta < \infty\}$$
$$\to \int f d\nu \quad \textit{as} \quad t \to \infty \qquad \text{a.s.} \quad \textit{on} \quad \{\tau^\eta = \infty\}.$$

The first assertion is trivial. The second extends (8) to arbitrary $\eta \in \Xi_\infty$ (by (2)), and also applies to $\eta \in \Xi_0$. In addition, with the aid of [7], the parameter range is improved considerably.

PROOF. Enlarge the probability space supporting $\{(\xi_t^\eta)\}$ to support an independent ν-distributed random subset γ of $\mathbb{Z}$, and define (ξ_t^ν) by

$$\xi_t^\nu = \xi_t^\eta \quad \text{on} \quad \{\gamma = \eta\} \qquad \eta \in \Xi.$$

By (2) and (5), $P(\xi_t^\eta \in \cdot) \Rightarrow \nu$ as $t \to \infty$ $\forall \eta \in \Xi_\infty$. Since $\nu(\Xi_\infty) = 1$, it follows that the stationary process (ξ_t^ν) is Birkhoff ergodic, and so

$$\frac{1}{t}\int_0^t f(\xi_t^\nu) \to \int f d\nu \qquad \text{as} \quad t \to \infty \qquad \forall f \in L^1(\nu). \tag{10}$$

Fix $\eta \in \Xi$, $\eta_0 \in \Xi_0 - \{\varnothing\}$, and $f \in \mathfrak{F}_{\eta_0} = \{$functions $f(\xi) \in \mathcal{C}$ which depend only on $\xi \cap \eta_0\}$. From the easily proven fact that $\nu(\{\xi : \eta \cap \xi = \varnothing\}) = 0$, we see that $P(\tau^x = \infty$ for some $x \in \eta) = 1$. Using (6) and (7), we get

$$\lim_{t\to\infty} P\big(\xi_s^\eta \cap \eta_0 = \xi_s^{\mathbb{Z}} \cap \eta_0\ \forall s \geqslant t | \tau^\eta = \infty\big) = 1.$$

Hence, for any $\eta_1 \in \Xi_\infty$,

$$\lim_{t\to\infty} P(\xi_s^\eta \cap \eta_0 = \xi_s^{\eta_1} \cap \eta_0\ \forall s \geqslant t | \tau^\eta = \infty) = 1.$$

Therefore,

$$\lim_{t\to\infty} P(\xi_s^\eta \cap \eta_0 = \xi_s^\nu \cap \eta_0\ \forall s \geqslant t | \tau^\eta = \infty) = 1.$$

Thus, writing $\sigma = \min\{t : \xi_s^\eta \cap \eta_0 = \xi_s^\nu \cap \eta_0 \quad \forall s \geqslant t\}$, $\sigma < \infty$ a.s. on $\{\tau^\eta = \infty\}$. We conclude that on $\{\tau^\eta = \infty\}$,

$$\begin{aligned}\frac{1}{t}\int_0^t f(\xi_s^\eta)\, ds &= \frac{1}{t}\int_0^{\sigma\wedge t} f(\xi_s^\eta)\, ds + \frac{1}{t}\int_{\sigma\wedge t}^t f(\xi_s^\nu)\, ds \\ &\to \lim_{t\to\infty} \frac{1}{t}\int_0^t f(\xi_s^\nu)\, ds = \int f d\nu \qquad \text{as} \quad t \to \infty,\end{aligned}$$

the last equality by (10). Since η_0 is arbitrary, (9) is proved for $f \in \mathfrak{F} = \cup_{\eta_0} \mathfrak{F}_{\eta_0}$. The extension from $\mathfrak{F}$ to $\mathcal{C}$ is routine. This completes the proof.

It seems most plausible that (8) holds whenever $\{(\xi_t^\eta)\}$ is nonergodic, but our proof breaks down in the same way that the proof of (5) does if $\lambda < \lambda_0^*$. That (8) will not hold for (ζ_t) is clear from remarks in [1]. Whether or not complete pointwise ergodicity should hold for symmetric nearest neighbor contact processes in several dimensions is not at all clear.

Finally, we remark that the same method applies to various other "additive" nearest neighbor processes on $\mathbb{Z}$. For example, if (β_t) is the *biased voter model* of Schwartz [9], with flip rates

$$\begin{array}{lll} & \textit{at } x & \textit{with rate} \\ (11) & 1 \to 0 & |\{x-1, x+1\} \cap \beta^c| \\ & 0 \to 1 & \lambda|\{x-1, x+1\} \cap \beta| \qquad (\lambda > 1), \end{array}$$

then for any $f \in \mathcal{C}$, $\eta \in \Xi$,

$$\frac{1}{t}\int_0^t f(\beta_s^\eta)\,ds \to f(\varnothing) \quad \text{as} \quad t \to \infty \text{ a.s.} \quad \text{on} \quad \{\beta_t^\eta = \varnothing \text{ eventually}\}$$

$$\to f(\mathbb{Z}) \quad \text{as} \quad t \to \infty \text{ a.s.} \quad \text{on} \quad \{\beta_t^\eta \text{ never } \varnothing\}.$$

The case of (11) where $\lambda = 1$ is the simple one-dimensional Holley-Liggett voter model [6]. Starting from $\eta \in \Xi_\infty$, the pointwise ergodic behavior of this process is unstable. From some irregular initial η the Cesaro averages converge to $f(\varnothing)$ a.s., from others to $f(\mathbb{Z})$ a.s., and from still others a.s. fail to converge. These assertions can be proved by combining (i) the observation of [10] that the "borders" of (β_t) are annihilating random walks, and (ii) the example at the end of Section 1 in [0]. The behavior starting from dense initial states remains open.

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