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GAUSSIAN AND THEIR SUBORDINATED SELF-SIMILAR RANDOM GENERALIZED FIELDS

By R. L. Dobrushin

Institute of Problems of Transmission of Information, Moscow

A large class of generalized random fields is defined, containing random elements F of $\mathscr{S}'$, where $\mathscr{S}'$ is the dual of the Schwartz space $\mathscr{S} = \mathscr{S}(\mathbb{R}^\nu)$. Such a generalized random field is translation-invariant if $F\psi$ is the same as $F\phi$ for any translate ψ of ϕ; it is invariant under the renormalization group with index κ (or self-similar with index κ) if $F\phi_\lambda = \lambda^{-\alpha}F\phi$ for all $\lambda > 0$ and $\phi \in \mathscr{S}$, where ϕ_λ is the rescaled test function $\phi_\lambda(x) = \lambda^{-\nu}\phi(x/\lambda)$. Recent work of several authors has shown that self-similar generalized random fields on $\mathbb{R}^\nu$, and self-similar random fields on $\mathbb{Z}^\nu$ which can be constructed from them, arise naturally in problems of statistical mechanics and limit laws of probability theory. They generalize the theory of stable distributions. Here the class of all translation-invariant self-similar Gaussian generalized random fields on $\mathbb{R}^\nu$ is completely described. By the discretization of such fields the class of self-similar Gaussian fields with discrete arguments (found by Sinai) is extended. Finally, a class of generalized random fields subordinated to the self-similar translation-invariant Gaussian ones is constructed. These non-Gaussian generalized random fields are Wick powers (multiple Itô integrals) of the Gaussian ones.

1. Introduction. In recent years many investigations have attempted to clarify mathematically the applications of renormalization group theory to statistical physics: see [1], [3], [6], [7], [8], [16] and [26]. In particular, a mathematical definition of the class of random fields with discrete argument which are invariant under renormalization transformations was introduced in the papers of Gallavotti and Jona-Lasinio [6] and Sinai [26]. This generalizes the well-known definition of random variables with stable probability distribution. (See for example [13].) We shall call such fields self-similar. (We translate the term *automodel* of [26] by *self-similar*. The term *stable random fields* used in [6] is less convenient because of the polysemantic mathematical use of the word *stable*.) However, there exists also another tradition for the study of what is in essence the same set of ideas. Lamperti [18] has introduced, in connection with limit theorems for "strongly dependent" random processes, the notion of semistable random process, which is a direct analog of the notion of a self-similar random field in its application to usual (i.e., nongeneralized) one-dimensional random fields (i.e., random processes) with continuous argument. Here some interesting results have been obtained (see, for example, a recent paper of Taqqu [27]); however,

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the possibilities of direct application of these ideas to physical problems are limited. This is seen, for example, from the fact that there exist no nontrivial stationary random fields of the usual kind.

In this connection it was proposed in a recent paper [4] of the author to begin the study of generalized self-similar random fields,[1] which seems to us natural and useful for several reasons. In particular we explain below and in more detail in [4] that probably (i.e., if some plausible but unproved hypotheses are true) the notion of self-similar field with discrete argument (in the sense of [6] and [26]) can be embedded in the notion of generalized self-similar random field. Self-similar random fields with discrete argument can be constructed by the discretization of generalized self-similar fields if we consider the values of generalized fields at the indicators of cubes which arise from the partitioning of Euclidean space. The general properties of self-similar fields were studied in [4]. In particular in [4] were introduced the notions of the large-scale limit of the field, which describes its behavior at infinity, and of the short-scale limit of the field, which describes its local behavior near the point zero of the argument space. It was shown in [4] that in the case of any (not necessarily power) normalization, these limits are always self-similar random fields. (The large-scale limits are natural from the point of view of statistical physics, and the short-scale limits can be useful in the Euclidean quantum field theory.)

The fundamental problem of looking for the stationary self-similar fields defined over Euclidean space of arbitrary dimension ν was considered in [4] only for the case of fields with independent values, leading to stable probability distributions. In Section 3 of this paper we describe all generalized stationary Gaussian self-similar fields. By the discretization of such fields we extend the class of self-similar Gaussian fields with discrete arguments found by Sinai in [26]. For $\nu = 1$ this description reduces to the results of Kolmogorov [17] (which were obtained as early as 1940) and Pinsker [22], who investigated the Gaussian random processes with stationary increments invariant with respect to the similarity transformations, provided we translate their results into the language of the theory of generalized random processes. In Sections 5 and 6 we consider generalized random fields subordinated to the Gaussian random fields (i.e., random fields which are functionally dependent on the Gaussian ones and stationarily connected with them). The main results state that Wick powers (or Itô integrals in the language of probability theory) of self-similar stationary Gaussian fields are also self-similar stationary fields (not Gaussian, of course). For $\nu = 1$ the random process corresponding to Wick power 2 is the derivative of the so-called Rosenblatt process. Its one-dimensional distributions were found by Rosenblatt in [24] (see also Section 19.5 of [13]) and multidimensional ones

[1] We follow here the tradition of the Russian mathematical terminology by using the term *generalized random field* instead of *random distribution*, the term more customary in English. The main reason is the wish to avoid the expression "the probability distribution of a random distribution."

by Taqqu in [27] in connection with limit theorems of probability theory.[2] The discretization of the random fields thus constructed leads to random fields with discrete argument which are self-similar in the Gallavotti–Jona-Lasinio–Sinai sense. The possibility of constructing such fields with the help of the Rosenblatt process was noted in the Gallavotti and Jona-Lasinio paper [6].

In another paper we shall prove that, after a corresponding normalization, the constructed self-similar fields define in the space of probability distributions of random fields a family of curves which branch out of the curve of Gaussian self-similar fields at the point corresponding to white noise. Such points of bifurcation were studied on the level of methods of asymptotic expansions by Sinai [26]. The question of the possibility of physical applications of such fields is open. But these fields are adequate for the problem of studying the limit distributions of functionals of Gaussian fields (compare [27]). Results in such directions will be published in another paper of the author. We finally note that we give in Section 5 a general representation of generalized random fields subordinated to a Gaussian one which apparently has some independent interest.

2. Generalized random fields. A *generalized random field* is usually defined (see [10]) as a probability measure on the space of real generalized functions (distributions) $\mathscr{S}'$ which is dual to the Schwartz space $\mathscr{S} = \mathscr{S}(\mathbb{R}^\nu)$ of all infinitely differentiable rapidly decreasing functions on ν-dimensional Euclidean space (see, for example, [9] and [10]). However it is more natural for our aims to consider a wider class of random fields, the study of which corresponds, roughly speaking, to the study of derivatives of different orders of random fields in the usual sense.

Let $\mathscr{J}^\nu$, $\nu = 1, 2, \cdots$ be the set of all ν-dimensional multi-indices $j = (j_1, \cdots, j_\nu)$, $j_1, \cdots, j_\nu = 0, 1, \cdots$; $|j| = j_1 + \cdots + j_\nu$ for $j \in \mathscr{J}^\nu$. Also define the sets $\mathscr{J}_r^\nu = \{j \in \mathscr{J}^\nu : |j| = r\}$, $r = 0, 1, \cdots$ and $\mathscr{J}_{<r}^\nu = \{j \in \mathscr{J}^\nu : |j| < r\}$, $r = 1, 2, \cdots$. We shall write here and in the following

$$(2.1) \qquad x^j = (x_1)^{j_1} \cdots (x_\nu)^{j_\nu}, \qquad x = (x_1, \cdots, x_\nu) \in \mathbb{R}^\nu, \quad j \in \mathscr{J}^\nu .$$

For each $r = 1, 2, \cdots$ let us denote by $\mathscr{S}_r = \mathscr{S}_r(\mathbb{R}^\nu)$ the closed subspace of $\mathscr{S}(\mathbb{R}^\nu)$ consisting of all functions $\phi \in \mathscr{S}(\mathbb{R}^\nu)$ such that

$$(2.2) \qquad \int_{\mathbb{R}^\nu} \phi(x) x^j \, dx = 0, \quad j \in \mathscr{J}_{<r}^\nu .$$

We note that by using the Fourier transform $\tilde{\phi}$ of the function $\phi \in \mathscr{S}(\mathbb{R}^\nu)$ we can write the condition (2.2) in the form

$$(2.3) \qquad D^j \tilde{\phi}(0) = 0, \quad j \in \mathscr{J}_{<r}^\nu ,$$

where

$$(2.4) \qquad D^j = (D_1)^{j_1} \cdots (D_\nu)^{j_\nu}, \quad j = (j_1, \cdots, j_\nu) \in \mathscr{J}_r^\nu ;$$

[2] It follows apparently from a note in [27] that in his dissertation (unpublished and regrettably inaccesible to the author) Taqqu found formally the moments of such processes corresponding to any Wick power but could not prove that these moments define a unique random process.

here D_k is the differentiation operator in the kth variable, and D^0 is the identity operator. We note also that it is possible to derive from one of Vilenkin's results (see the addendum to Section 2.4 in [10]) that for all ν the subspace $\mathscr{S}_r$ can be described as the closure of the set of all linear combinations of functions of the set

$$\hat{\mathscr{S}}_r = \{D^j\phi,\ \phi \in \mathscr{S},\ j \in \mathscr{J}_r^\nu\}, \tag{2.5}$$

and for $\nu = 1$ it coincides with the set (2.2) itself. The space $\mathscr{S}$ will be denoted $\mathscr{S}_0$ for the sake of unified notation.

Probability measures P on the σ-algebra $\mathscr{B}_r$ of Borel subsets (with respect to the weak topology) of the dual space $\mathscr{S}_r'$ we shall call *states of a random field over the space* $\mathscr{S}_r$. The random fields over $\mathscr{S}_1$ were introduced earlier by Jaglom [15]. We shall denote the set of all states of random fields over the space $\mathscr{S}_r$ by $\mathscr{P}_r(\mathbb{R}^\nu) = \mathscr{P}_r$.

As $\mathscr{S}_r \subset \mathscr{S}$, there is defined a natural injection of dual spaces

$$\tau_r\colon \mathscr{S}' \to \mathscr{S}_r'\colon F \to F_r\colon F_r(\phi) = F(\phi)\,, \quad \phi \in \mathscr{S}_r\,. \tag{2.6}$$

The pre-image of the point F_r in $\mathscr{S}_r'$ under the injection τ_r is a class of generalized functions for which all the derivatives of order r or greater coincide. We shall say that the state $P_r \in \mathscr{P}_r$ is a *restriction to* $\mathscr{S}_r$ *of the state* $P \in \mathscr{P}$ if

$$P_r(B) = P(\tau_r^{-1}B)\,, \quad B \in \mathscr{B}_r\,. \tag{2.7}$$

Thus the description of the state $P_r \in \mathscr{P}_r$ which is a restriction of a state $P \in \mathscr{P}$ can be interpreted as the description of the statistical properties of partial derivatives of order r or greater of the random field with state P.

PROPOSITION 2.1. *For each state $P_r \in \mathscr{P}_r$ where $r > 0$ there exists a state $P \in \mathscr{P}$ such that P_r is a restriction of P.*

PROOF. Fix a function $\psi \in \mathscr{S}$ such that $\tilde{\psi}(0) \neq 0$. It is easy to check by Fourier transforms that each function $\phi \in \mathscr{S}$ can be represented in a unique way in the form

$$\phi = \textstyle\sum_{j \in \mathscr{J}_{<r}^\nu} c_j^\phi D^j\psi + \bar{\phi}\,, \tag{2.8}$$

where $\bar{\phi} \in \mathscr{S}_r$ and the coefficients c_j^ϕ, $j \in \mathscr{J}_{<r}^\nu$, can be uniquely determined from the system of linear equations

$$\textstyle\sum_{j \in \mathscr{J}_{<r}^\nu} c_j^\phi \int_{\mathbb{R}^\nu} (x)^k D^j\psi(x)\,dx = \int_{\mathbb{R}^\nu} (x)^k\phi(x)\,dx\,, \quad k \in \mathscr{J}_{<r}^\nu\,. \tag{2.9}$$

(By Fourier transforms it is easy to check that the integrals on the left in this system of equations vanish for $|j| \geqq |k|$, $j \neq k$, but not for $j = k$, which implies that the corresponding determinant does not vanish.) Let us now define the measurable transform

$$U_\psi\colon \mathscr{S}_r' \to \mathscr{S}'\colon F_r \to F\colon F(\phi) = F_r(\bar{\phi})\,, \quad \phi \in \mathscr{S}\,. \tag{2.10}$$

It is clear that the transform $\tau_r U_\phi$ is the identity. So the state P defined from P_r by the relation

$$P(B) = P_r(U_\phi^{-1}B)\,, \quad B \in \mathscr{B}_0 \tag{2.11}$$

has P_r as its restriction.

It is convenient to describe the states of random fields by the *characteristic functionals*

$$L^P(\phi) = \int_{\mathscr{S}_r'} \exp\{iF(\phi)\}P(dF)\,, \quad \phi \in \mathscr{S}_r\,. \tag{2.12}$$

As the $\mathscr{S}_r$ are closed subspaces of the nuclear space $\mathscr{S}$, they are also nuclear (see [21], Sections 6.2.5 and 5.1.1). So the Minlos theorem is true for them (see [2], Section 6.10 and [10], Section 4.4). In particular this theorem implies that the transformation $P \to L^P$ is a one-to-one mapping of the space $\mathscr{P}_r$ onto the set of all continuous nonnegative-definite functionals $\{L^P(\phi), \phi \in \mathscr{S}_r\}$ such that $L^P(0) = 1$.

We introduce the *group of shift transformations* $E = \{E_a, a \in \mathbb{R}^\nu\}$ on the space $\mathscr{S}_r(\mathbb{R}^\nu)$:

$$E_a\phi(x) = \phi(x - a)\,, \qquad \phi \in \mathscr{S}_r\,, \quad x \in \mathbb{R}^\nu\,, \quad a \in \mathbb{R}^\nu\,. \tag{2.13}$$

We introduce the *group of Euclidean rotations* $Q = \{Q_G, G \in \mathscr{G}(\mathbb{R}^\nu)\}$ with respect to the origin:

$$Q_G\phi(x) = \phi(G^{-1}x)\,, \quad \phi \in \mathscr{S}_r\,, \quad x \in \mathbb{R}^\nu\,, \quad G \in \mathscr{G}(\mathbb{R}^\nu)\,; \tag{2.14}$$

here $\mathscr{G}(\mathbb{R}^\nu)$ is the set of Euclidean rotations with respect to the origin in $\mathbb{R}^\nu$. We introduce further the *group of similarity transformations* $U = \{U_\lambda, \lambda \in (0, \infty)\}$:

$$U_\lambda\phi(x) = \lambda^{-\nu}\phi(\lambda^{-1}x)\,, \quad \phi \in \mathscr{S}_r\,, \quad x \in \mathbb{R}^\nu\,, \quad \lambda \in (0, \infty)\,. \tag{2.15}$$

We also introduce similar groups of transformations $\hat{E} = \{\hat{E}_a, a \in \mathbb{R}^\nu\}$, $\hat{Q} = \{\hat{Q}_G, G \in \mathscr{G}(\mathbb{R}^\nu)\}$, $\hat{U} = \{U_\lambda, \lambda \in (0, \infty)\}$ on the space of generalized functions $\mathscr{S}_r'$:

$$\begin{aligned} \hat{E}_a F(\phi) &= F(E_a\phi)\,, && a \in \mathbb{R}^\nu\,, \\ \hat{Q}_G F(\phi) &= F(Q_G\phi)\,, && G \in \mathscr{G}(\mathbb{R}^\nu)\,, \\ \hat{U}_\lambda F(\phi) &= F(U_\lambda\phi)\,, && \lambda \in (0, \infty)\,, \quad F \in \mathscr{S}_r'\,, \quad \phi \in \mathscr{S}_r\,. \end{aligned} \tag{2.16}$$

The sense of these definitions is explained by the fact that for ordinary functions $f(x)$ of $x \in \mathbb{R}^\nu$, treated as generalized functions, the transformations (2.16) reduce to the transformations

$$\begin{aligned} \hat{E}_a f(x) &= f(x + a)\,, && a \in \mathbb{R}^\nu\,, \\ \hat{Q}_G f(x) &= f(Gx)\,, && G \in \mathscr{G}(\mathbb{R}^\nu)\,, \\ \hat{U}_\lambda f(x) &= f(\lambda x)\,, && \lambda \in (0, \infty)\,, \quad x \in \mathbb{R}^\nu\,. \end{aligned} \tag{2.17}$$

A random field over $\mathscr{S}_r$, described by the state $P \in \mathscr{P}_r$, we shall call *stationary* if

$$P(B) = P((\hat{E}_a)^{-1}B)\,, \qquad B \in \mathscr{B}_r\,, \quad a \in \mathbb{R}^\nu\,. \tag{2.18}$$

We shall call such a field *isotropic* if

$$P(B) = P((\hat{Q}_G)^{-1}B), \qquad B \in \mathscr{B}_r, \quad G \in \mathscr{G}(\mathbb{R}^\nu). \tag{2.19}$$

In terms of characteristic functionals stationarity means that

$$L^P(\phi) = L^P(E_a\phi), \qquad \phi \in \mathscr{S}_r, \quad a \in \mathbb{R}^\nu, \tag{2.20}$$

and isotropy means that

$$L^P(\phi) = L^P(Q_G\phi), \qquad \phi \in \mathscr{S}_r, \quad G \in \mathscr{G}(\mathbb{R}^\nu). \tag{2.21}$$

We must remember that the *partial derivative of order* j ($j \in \mathscr{J}^\nu$) of the random field over $\mathscr{S}$ described by the state $P \in \mathscr{P}_0$ is by definition the random field over $\mathscr{S}$ described by the state $\hat{D}^j P \in \mathscr{P}_0$ such that

$$\hat{D}^j P(B) = P((D^j)^{-1}B), \quad B \in \mathscr{B}_0; \tag{2.22}$$

here D^j is the differentiation operator (2.4) applied to a generalized function from $\mathscr{S}'$ (see for example [9]). We shall say that a random field over $\mathscr{S}$ is a *random field with stationary* rth *increments* (where $r > 0$) if all its partial derivatives of order $j \in \mathscr{J}_r^\nu$ are stationary random fields.

PROPOSITION 2.2. *The random field described by the state* $P \in \mathscr{P}_0$ *is a random field with stationary* rth *increments if and only if the restriction* P_r *of the state* P *to* $\mathscr{S}_r$ *describes a stationary random field over* $\mathscr{S}_r$.

PROOF. The definition (2.22) implies that the state P describes a random field with stationary rth increments if and only if

$$L^P(\phi) = L^P(E_a\phi), \qquad \phi \in \hat{\mathscr{S}}_r, \quad a \in \mathbb{R}^\nu, \tag{2.23}$$

where the set of functions $\hat{\mathscr{S}}_r$ was introduced in (2.5). Further, stationarity of the field described by the state P_r means that (2.23) is true for all $\phi \in \mathscr{S}_r$. But the set of linear combinations of functions of $\hat{\mathscr{S}}_r$ is dense in $\mathscr{S}_r$, so that these two conditions are equivalent.

Proposition 2.2 shows that the problem of investigating the random fields with stationary rth increments is in essence equivalent to the problem of investigating the stationary random fields over $\mathscr{S}_r$, and it justifies the study of this class of random fields.

Fixing the number $\kappa \in \mathbb{R}^1$, we shall introduce the group of transformations $\hat{S}_\kappa = \{\hat{S}_{\kappa,\lambda}, \lambda \in (0, \infty)\}$ on the space $\mathscr{S}_r'$ of generalized functions by the relation

$$\hat{S}_{\kappa,\lambda}F = \lambda^\kappa \hat{U}_\lambda F, \quad F \in \mathscr{S}_r'. \tag{2.24}$$

In the case of ordinary functions $f(x)$, treated as generalized functions, these transformations reduce to the transformations

$$\hat{S}_{\kappa,\lambda}f(x) = \lambda^\kappa f(\lambda x), \tag{2.25}$$

and this explains their meaning.

Further, we shall introduce the transformation group $S_\kappa^* = \{S_{\kappa,\lambda}^*, \lambda \in (0, \infty)\}$ in the space of states $\mathscr{P}_r$ by the relations

$$S_{\kappa,\lambda}^* P(B) = P((\hat{S}_{\kappa,\lambda})^{-1}B), \quad B \in \mathscr{B}_r. \tag{2.26}$$

In terms of characteristic functionals this means that

$$L^{S_{\kappa,\lambda}^* P}(\phi) = L^P(\lambda^\kappa U_\lambda \phi), \quad \phi \in \mathscr{S}_r. \tag{2.27}$$

The group S_κ^* will be called (in the tradition of physics) the *renormalization group of order* κ. The random field over $\mathscr{S}_r$ described by the state $P \in \mathscr{P}_r$ we shall call a *self-similar field of order* κ if

$$P = S_{\kappa,\lambda}^* P, \quad \lambda \in (0, \infty). \tag{2.28}$$

In terms of characteristic functionals this means that

$$L^P(\phi) = L^P(\lambda^\kappa U_\lambda \phi), \quad \phi \in \mathscr{S}_r. \tag{2.29}$$

This notion was introduced in a somewhat more general situation in the author's paper [4], which contains a more detailed discussion.

The following sections of this paper are devoted to the construction of examples of self-similar stationary random fields and self-similar random fields with stationary increments. We note for use below that trivial examples of self-similar fields of order $-r$ with stationary rth increments (where $r > 0$) are the homogeneous random polynomials of order r, i.e., random fields for which the state describing them is concentrated on the functions of the type

$$f(x) = \textstyle\sum_{j \in \mathscr{J}_r^\nu} c_j x^j, \quad x \in \mathbb{R}^\nu \tag{2.30}$$

for real numbers c_j, $j \in \mathscr{J}_r^\nu$. If the state P is concentrated on the function $f(x) \equiv 0$, $x \in \mathbb{R}^1$, then this state will be called *the zero state*. It is self-similar for all values of the order $\kappa \in \mathbb{R}^1$.

We shall give below a general construction which was introduced in [4] and which permits us to construct the self-similar fields with discrete argument (in the sense of [26] and [6]) with the help of generalized self-similar fields. For this we shall introduce a linear functional space $M_w(\mathbb{R}^\nu)$ consisting of the finite-range functions $\phi \in L_1(\mathbb{R}^\nu)$ such that for $k = (k_1, \cdots, k_\nu) \in \mathbb{R}^\nu$

$$\|\phi\| = \sup_{k \in \mathbb{R}^\nu} \{|\tilde{\phi}(k)|(\textstyle\prod_{j=1}^\nu (1 + |k_j|)w(|k_j|))\} < \infty \tag{2.31}$$

and the function of $k \in \mathbb{R}^\nu$ under the supremum tends to 0, $\lambda \to \infty$; here $w(k)$, $0 < k < \infty$, is a bounded, positive, monotone nonincreasing function of k tending to 0 as $k \to \infty$ and $\tilde{\phi}$ is the Fourier transform of the function ϕ. We introduce a topology in the subspaces $M_w(S_r)$, where $M_w(S_r)$ is the set of functions $\phi \in M_w(\mathbb{R}^\nu)$ with support in the sphere $S_r \subset \mathbb{R}^\nu$ with center at the origin and radius r in the norm (2.31). The topology in $M_w(\mathbb{R}^\nu)$ is defined as the inductive limit (see, for example, [23]) of topologies in $M_w(S_r)$, $r = 1, 2, \cdots$. It is easy to prove (see [4]) that the Schwarz space $\mathscr{D}(\mathbb{R}^\nu)$ of finite-range functions is a dense subspace of the space $M_w(\mathbb{R}^\nu)$, and that the topology of the space

$\mathscr{D}(\mathbb{R}^\nu)$ is stronger than the topology induced by the topology in $M_w(\mathbb{R}^\nu)$. Let χ_t be the indicator of the unit cube with center at the point $t = (t_1, \cdots, t_\nu) \in \mathbb{Z}^\nu$, where $\mathbb{Z}^\nu$ is the ν-dimensional integer lattice:

$$\begin{aligned} \chi_t(x) &= 1 \qquad \text{if} \quad t_j - \tfrac{1}{2} < x_j \leqq t_j + \tfrac{1}{2}, \quad 1 \leqq j \leqq \nu, \\ &= 0 \qquad \text{otherwise}; \end{aligned} \tag{2.32}$$

these are elements of the spaces $M_w(\mathbb{R}^\nu)$. We shall say that the state $P \in \mathscr{P}$ is *discretizable* if for some choice of the function w its characteristic functional $\{L^P(\phi), \phi \in \mathscr{S}\}$ (see (2.12)) can be extended to a continuous functional $\{\hat{L}^P(\phi), \phi \in M_w(\mathbb{R}^\nu)\}$. To each discretizable state P we make correspond a random field $\{\xi_t, t \in \mathbb{Z}^\nu\}$ with discrete argument, such that its joint probability distributions are described by the system of joint characteristic functions

$$\begin{aligned} L_{t_1,\cdots,t_m}(s_1, \cdots, s_m) &= E \exp\{i \textstyle\sum_{j=1}^m s_j \xi_{t_j}\} \\ &= \hat{L}^P(s_1\chi_{t_1} + \cdots + s_m\chi_{t_m}), \\ &\qquad s_1, \cdots, s_m \in \mathbb{R}^1, \quad t_1, \cdots, t_m \in \mathbb{Z}^\nu, \\ &\qquad m = 1, 2, \cdots. \end{aligned} \tag{2.33}$$

It is easy to show (see [4]) that if the random field described by the state P is self-similar, then the corresponding random field $\{\xi_t, t \in \mathbb{Z}^\nu\}$ will also be self-similar in the sense of Gallavotti–Jona-Lasinio–Sinai. The self-similar parameter α used in the papers [6] and [26] is connected with our self-similar parameter κ by the relation

$$\alpha = -\frac{2\kappa}{\nu} + 2. \tag{2.34}$$

In [4] it is shown that the previous construction relates the different self-similar fields with discrete argument to the different generalized self-similar fields.

3. Gaussian random fields. We recall (compare for example [10]) that the random field described by the state P is called *Gaussian* (with zero means—obviously the inclusion of nonzero means would only lead to trivial complications) if its characteristic functional is

$$L^P(\phi) = \exp\{-\tfrac{1}{2}B^P(\phi, \phi)\}, \quad \phi \in \mathscr{S}_r, \tag{3.1}$$

where $B^P(\phi, \psi)(\phi, \psi \in \mathscr{S}_r)$ is a continuous nonnegative-definite real-valued bilinear functional. The functional B^P is called the *correlation functional of the random field described by the state P*. It follows from the relations (2.29) and (3.1) that the Gaussian random field described by the correlation functional B^P is a self-similar field with parameter κ if and only if

$$B^P(U_\lambda\phi, U_\lambda\psi) = \lambda^{-2\kappa}B^P(\phi, \psi), \qquad \phi, \psi \in \mathscr{S}_r, \quad \lambda \in (0, \infty). \tag{3.2}$$

The explicit description of the stationary Gaussian random fields over the spaces $\mathscr{S}_r$ is given by the following proposition.

PROPOSITION 3.1. *The functional $B^P(\phi, \psi)$ is the correlation functional of a stationary Gaussian random field over $\mathscr{S}_r$ if and only if*

$$B^P(\phi, \psi) = \sum_{j,j' \in \mathscr{J}_r^\nu} a^P_{j,j'} D^j\tilde{\phi}(0)\overline{D^{j'}\tilde{\psi}(0)} + \int_{\mathbf{R}^\nu\setminus\{0\}} \tilde{\phi}(k)\overline{\tilde{\psi}(k)}G^P(dk)\,, \quad \phi, \psi \in \mathscr{S}_r\,, \tag{3.3}$$

(here $\tilde{\phi}$ and $\tilde{\psi}$ are the Fourier transforms of ϕ and ψ), where the matrix $A^P = \{a^P_{j,j'}, j, j' \in \mathscr{J}_r^\nu\}$ is symmetric and nonnegative-definite, and the measure G^P on the σ-algebra $\mathscr{B}_{\mathbf{R}^\nu\setminus\{0\}}$ of Borel subsets of $\mathbb{R}^\nu\setminus\{0\}$ is such that

$$G^P(E) = G^P(-E)\,, \quad E \in \mathscr{B}_{\mathbf{R}^\nu\setminus\{0\}}\,, \tag{3.4}$$

such that for some $q > 0$

$$\int_{|k|>1} |k|^{-q}G^P(dk) < \infty\,, \tag{3.5}$$

and such that

$$\int_{0<|k|\leq 1} |k|^{2r}G^P(dk) < \infty\,. \tag{3.6}$$

The measure G^P and the matrix A^P are determined uniquely by the correlation functional B.

This proposition is a direct consequence of a result of Vilenkin (see [10], Section 3.5.2) and of the Propositions 2.1 and 2.2. For the case $r = 1$ it was proven by Jaglom [15]. For the case $r = 0$ it is a reformulation of the well-known Bochner–Schwartz theorem (see [10], Section 3.1). The measure G^P is called a *spectral measure* of the corresponding Gaussian field and its density is called a *spectral density.*

We shall now describe the self-similar stationary Gaussian random field in its spectral form. For this purpose we write

$$Q^\nu = \{e \in \mathbb{R}^\nu : |e| = 1\} \tag{3.7}$$

for the unit sphere in the space $\mathbb{R}^\nu$, and we introduce the one-to-one correspondence

$$\chi : \mathbb{R}^\nu\setminus\{0\} \to Q^\nu \times (0, \infty) : \qquad k \to (e, \alpha)\,, \quad k = \alpha e\,. \tag{3.8}$$

We shall sometimes interpret the spectral measure G^P as a measure on the direct product of the measurable spaces $(Q^\nu \times (0, \infty), \mathscr{B}_{Q^\nu} \times \mathscr{B}_{(0,\infty)})$, where $\mathscr{B}_{Q^\nu}$ and $\mathscr{B}_{(0,\infty)}$ are σ-algebras of Borel subsets of the corresponding space (i.e., we shall use "spherical coordinates").

THEOREM 3.2. *Let $P \in \mathscr{P}_r$ be a state that describes a Gaussian stationary field over $\mathscr{S}_r$. This field will be self-similar of order $\kappa > -r$ if and only if $A^P = 0$ and there exists a finite measure $\bar{G}^P$ on the σ-algebra $\mathscr{B}_{Q^\nu}$ such that*

$$\bar{G}^P(\bar{E}) = \bar{G}^P(-\bar{E})\,, \quad E \in \mathscr{B}_{Q^\nu} \tag{3.9}$$

and

$$G^P(\bar{E} \times C) = \bar{G}^P(\bar{E}) \int_C \alpha^{2\kappa-1}\, d\alpha\,, \qquad \bar{E} \in \mathscr{B}_{Q^\nu}\,, \quad C \in \mathscr{B}_{(0,\infty)}\,. \tag{3.10}$$

The field will be self-similar of order $\kappa = -r$ if and only if the spectral measure $G^P \equiv 0$. If the field is self-similar of order $\kappa < -r$, then it is the zero field.

PROOF. A comparison of the relations (2.27) and (3.1) shows that the renormalization-transformation $S^*_{\kappa,\lambda}$ transforms the stationary Gaussian field over $\mathscr{S}_r$ described by the state P into a stationary Gaussian field. And it follows easily from the relations (2.27), (3.1), and (3.3), together with the relation

$$(U_\lambda \phi)^{\sim} = \lambda^{-\nu} U_{\lambda^{-1}} \tilde{\phi}\,, \tag{3.11}$$

that

$$\begin{gathered} a^{S^*_{\kappa,\lambda}P}_{j,j'} = \lambda^{2r+2\kappa} a^P_{j,j'}\,, \quad j, j' \in \mathscr{J}_r^{\,\nu}\,, \\ G^{S^*_{\kappa,\lambda}P}(E) = \lambda^{2\kappa} G^P(\lambda^{-1}E)\,, \qquad E \in \mathscr{B}_{\mathbf{R}^\nu\setminus\{0\}}\,, \quad \lambda \in (0, \infty)\,. \end{gathered} \tag{3.12}$$

The relations (3.12) imply the sufficiency statement in the theorem. To prove the necessity we let

$$\hat{G}^P(g, \bar{E}) = \textstyle\int_{\bar{E}\times(0,g)} \alpha^{2r+\nu-1} G^P(de, d\alpha)\,, \qquad \bar{E} \in \mathscr{B}_{Q^\nu}\,, \quad g \in (0, \infty)\,. \tag{3.13}$$

The condition (3.6) implies the convergence of the integral (3.13). Let P be a self-similar state of order κ. To the representation of the integral (3.13) as a limit of its approximating sums apply the relation (3.12) for $\lambda = g^{-1}$; this gives

$$\begin{aligned} \hat{G}^P(g, \bar{E}) &= \lim_{\rho\to1-0} \textstyle\sum_{j=-\infty}^0 (g\rho^j)^{2r+\nu-1} G^P([g\rho^{j-1}, g\rho^j] \times \bar{E}) \\ &= \lim_{\rho\to1-0} \textstyle\sum_{j=-\infty}^0 (g\rho^j)^{2r+\nu-1} g^{2\kappa} G^P([\rho^{j-1}, \rho^j] \times \bar{E}) \\ &= g^{2r+\nu+2\kappa-1} \hat{G}^P(1, \bar{E})\,, \qquad g \in (0, \infty)\,, \quad \bar{E} \in \mathscr{B}_{Q^\nu}\,. \end{aligned} \tag{3.14}$$

By differentiating this identity with respect to g and using (3.13) we now obtain the formula (3.10) for

$$\bar{G}^P(\bar{E}) = (2r + \nu + 2\kappa - 1)\hat{G}^P(1, \bar{E})\,, \quad \bar{E} \in \mathscr{B}_{Q^\nu}\,. \tag{3.15}$$

The relation (3.4) implies (3.9), and so the necessity of the condition (3.10) follows. The condition (3.6) implies that $\bar{G}^P(\bar{E}) = 0$ if $\kappa \leqq -r$, and therefore the corresponding self-similar field is zero if $\kappa < -r$.

The state of the nonzero Gaussian self-similar field described by the spectral measure (3.10) will be denoted by $P^\kappa_{\bar{G}}$.

The self-similar Gaussian field of order $\kappa = -r$ is a homogeneous polynomial of order r (see Section 2) giving a stationary field over $\mathscr{S}_r$. The isotropic self-similar Gaussian fields are especially important. As follows from (3.10), such fields (with $\kappa > -r$) are described by the spectral densities

$$g(k) = c|k|^{2\kappa-\nu}\,, \quad k \in \mathbf{R}^\nu\setminus\{0\}\,, \tag{3.16}$$

where $c \geqq 0$. (For more details about such fields see [15].) In particular, for $\kappa = \nu/2$ and $r = 0$ this field is the white noise field, and for $\kappa = -\frac{1}{2}$ and $r = 1$ it is the ν-parameter Brownian motion introduced by Lévy. For $\kappa = (\nu/2) - 1$, $r = 0$, and $\nu \geqq 3$ it is the massless free field usually used in Euclidean quantum-field theory. (For $\nu = 1, 2$ it can be interpreted as a field over $\mathscr{S}_1$.) For $\nu = 1$

all stationary self-similar Gaussian fields have spectral density of the type (3.16); this fact was in essence proved by Kolmogorov [17] for $r = 1$ and then by Pinsker [22] for general r. The versions of the self-similar process and the process with stationary increments used in these papers are the integrated versions of the processes in this paper and therefore do not require the use of generalized functions. (Kolmogorov and Pinsker used the phrase "process invariant with respect to the similarity transformation.") We note that even in the case $\nu \geqq 2$ the set of self-similar Gaussian states described in the theorem is infinite dimensional. By means of the operations of convolution and closure (that agrees with the general hypothesis formulated in Section 5 of [4]), it can be generated by the finite-dimensional set of those states for which the measure $\bar{G}$ is concentrated in two opposite points.

By using the general construction described in Section 2 it is possible to apply Theorem 3.2 to the construction of the self-similar random fields with discrete argument.

PROPOSITION 3.3. *The self-similar Gaussian state $P_{\bar{G}}^{\kappa}$ $\kappa > 0$ over $\mathcal{S}$ is discretizable if and only if for $k = (k_1, \cdots, k_\nu) \in \mathbb{R}^\nu$*

$$(3.17) \qquad \int_{\mathbb{R}^\nu \setminus \{0\}} \left(\prod_{j=1}^{\nu} (1 + |k_j|)\right)^{-2} G(dk) < \infty ,$$

where the measure $G = G^{P_{\bar{G}}^{\kappa}}$ is given by the relation (3.10).

Suppose that the measure $\bar{G}$ is given by density $\bar{g}(e)$, $e \in Q^\nu$, with respect to the uniform measure on Q^ν and that for some $c > 0$, $C < \infty$

$$(3.18) \qquad 0 < c \leqq \bar{g}(e) \leqq C < \infty , \quad e \in Q^\nu ;$$

then the integral (3.17) *converges if and only if*

$$(3.19) \qquad \kappa < \frac{\nu + 1}{2} .$$

A measure $\bar{G}$ such that the integral (3.17) *converges for some fixed κ exists if and only if*

$$(3.20) \qquad \kappa < \nu .$$

The integral (3.17) *converges for all measures $\bar{G}$ if and only if*

$$(3.21) \qquad \kappa < 1 .$$

PROOF. The convergence of integral (3.17) implies that there exists a positive, bounded, nonincreasing function $w(k)$, going to 0 as $k \to \infty$, such that

$$(3.22) \qquad \int_{\mathbb{R}^\nu} \left(\prod_{j=1}^{\nu} (1 + |k_j|)(w(|k_j|)\right)^{-2} G(dk) < \infty .$$

This relation and the condition (2.31) imply that if

$$(3.23) \qquad B(\phi, \psi) = \int_{\mathbb{R}^\nu \setminus \{0\}} \tilde{\phi}(k)\overline{\tilde{\psi}(k)} G(dk) , \quad \phi, \psi \in M_w(\mathbb{R}^\nu) ,$$

then the integral in this relation converges and defines a continuous bilinear functional on the space $M_w(\mathbb{R}^\nu)$. It is possible to define the continuation of the characteristic functional to the space $M_w(\mathbb{R}^\nu)$ again by the relation (3.1), and this proves the sufficiency of the condition (3.17).

On the other hand it is easy to see that if the condition (3.17) is not true, then for $\phi = \psi = \chi_t$ (see (2.32)) the integral (3.23) is infinite even though $\chi_t \in M_w(\mathbb{R}^\nu)$ for every choice of w. By approximating the function χ_t by functions of the space $\mathscr{S}(\mathbb{R}^\nu)$ it is easy to check that the functional B is unbounded on the intersection of $\mathscr{S}(\mathbb{R}^\nu)$ with any sphere in $M_w(\mathbb{R}^\nu)$, whatever its radius. Since $L^P(0) = 1$, we see from the relation (3.1) that the functional $L^P(\phi)$ cannot be continuously extended to $M_w(\mathbb{R}^\nu)$, and this proves the necessity of the condition (3.17).

If the condition (3.18) holds then the spectral measure G is given by a density $g(k)$ such that for some $c_1 > 0$, $C_1 < \infty$ (compare (3.16))

$$c_1|k|^{2\kappa-\nu} \leqq g(k) \leqq C_1|k|^{2\kappa-\nu}\,, \quad k \in \mathbb{R}^\nu\backslash\{0\}\,. \tag{3.24}$$

So the sufficiency statement of the condition (3.19) can be obtained by the application of a variant of a power counting theorem well known in mathematical physics (see (3.4) in Lowenstein and Zimmerman's paper [19]). The necessity of the condition (3.19) will be evident if we note that the integral of the function $g(k)\prod_{j=1}^{\nu}(1 + |k_j|)^{-2}$ diverges along every line parallel to the coordinate axis if $\kappa \geqq (\nu + 1)/2$.

The divergence of the integral (3.17) for $\kappa \geqq \nu$ and all $\bar{G}$ follows from (3.10) and the fact that the integrated function in (3.17) restricted to any ray αe, $0 < \alpha < \infty$ (here $e \in Q^\nu$), decreases when $\alpha \to \infty$ as $\alpha^{-2\nu}$ or slower. Along the rays which do not lie on any coordinate hyperflats this function decreases exactly as $\alpha^{-2\nu}$. So if $\kappa < \nu$ and the measure $\bar{G}$ is concentrated on the closed set corresponding to such rays, then the integral (3.17) converges. Along any ray the integrated function decreases as α^{-2} or more rapidly, so for $\kappa < 1$ the relation (3.10) implies that the integral converges if restricted to any ray, and this means that it converges. If $\kappa \geqq 1$ and the measure $\bar{G}$ is concentrated on the set corresponding to the set of rays parallel to the coordinate axis, then the integral (3.17) diverges.

The self-similar fields with discrete argument defined by the application of Propposition 3.3 and the relation (2.33) are again stationary and Gaussian. It is easy to compute the spectral measure of such a self-similar field with discrete argument; it is

$$\begin{gathered}\hat{G}(E) = \sum_{t\in Z^\nu}\int_{E'+2\pi t}\left(\prod_{j=1}^{\nu}(k_j + 2\pi t_j)^2|e^{ik_j} - 1|^2\right)G(dk)\,, \\ E \subset \{t = (t_1, \cdots, t_\nu) \in \mathbb{R}^\nu\{0\} : -\pi < t_j \leqq \pi, j = 1, \cdots, \nu\}\,, \\ E' = E\backslash\{0\} \in \mathscr{B}_{\mathbb{R}^\nu\backslash\{0\}}\,,\end{gathered} \tag{3.25}$$

where the measure G satisfies the condition (3.17), which guarantees the finiteness of the measure $\hat{G}$. At the formal level (without investigation of the condition for the finiteness of the measure $\hat{G}$), and in the case where there exists a density, the formula (3.25) was obtained by Sinai [26]. He proved that for $\nu = 1$ it exhausts the entire class of self-similar stationary Gaussian fields.

4. Multiple Itô integrals. Let $L_2(P)$ be the Hilbert space of real functions $\Phi(F)$ of $F \in \mathscr{S}_r'$ that are square-integrable with respect to P. For the construction of non-Gaussian self-similar fields it is necessary to use the description of the space $L_2(P)$ in terms of multiple Itô integrals with respect to the spectral measure of the stationary Gaussian state P (see [14]). (The same ideas are usually described in the mathematical physics literature in the language of the Wick polynomials; see for example [25], and for the connection between the two approaches see [5].)

We shall assume in this and following sections that a Gaussian stationary state $P \in \mathscr{P}_r$ is given for which the matrix A^P in its spectral description (see Proposition 3.1) is zero and the spectral measue is continuous. This last condition is introduced for the simplification of some constructions and is not essential.

Let us denote by $L_2^c(P)$ the complex Hilbert space of functions $\Phi(F)$ square-integrable with respect to P. Let us denote by H_G^c the complex Hilbert space of complex-valued functions of $k \in \mathbb{R}^\nu$ integrable with respect to the measure G and by $H_G^1 \subset H_G^c$ the real Hilbert space of (in general complex-valued) even elements of the space H_G^c. For each ϕ in the set $\mathscr{S}_r^c$ of functions of the type $\phi_1 + i\phi_2$, $\phi_1, \phi_2 \in \mathscr{S}_r$, we define a random variable

$$\Phi_\phi(F) = F(\operatorname{Re}\phi) + iF(\operatorname{Im}\phi)\,, \quad F \in \mathscr{S}_r'\,; \tag{4.1}$$

this is an element of $L_2^c(P)$. Consider the mapping $\tilde{\mathscr{S}}_r^c \to L_2^c(P): \tilde{\phi} \to \Phi_\phi$, where $\tilde{\phi}$ is the Fourier transform of the function ϕ and the set $\tilde{\mathscr{S}}_r^c$ of the Fourier transforms of the functions in $\mathscr{S}_r^c$ is treated as a subspace of the Hilbert space H_G^c. The relation (3.3) implies that this mapping is isometric. The usual considerations of the theory of generalized functions (see [9], Chapter 1, Appendix 1) shows that the set $\tilde{\mathscr{S}}_r^c$ is dense in H_G^c. Therefore the mapping introduced above can be extended uniquely to an isometry $Z: H_G^c \to L_2^c(P): \phi \to Z_\phi$ such that

$$Z_{\tilde{\phi}} = \Phi_\phi\,, \quad \phi \in \mathscr{S}_r^c\,. \tag{4.2}$$

The restriction of such a mapping Z to H_G^1 is an isometry from H_G^1 into $L_2(P)$.

Let $\mathscr{B}_G$ be the family of Borel sets $\Delta \in \mathscr{B}_{\mathbb{R}^\nu \setminus \{0\}}$ such that $G(\Delta) < \infty$. Let

$$Z_G(\Delta) = Z_{\chi_\Delta}\,, \quad \Delta \in \mathscr{B}_G\,, \tag{4.3}$$

where χ_Δ is the indicator of the set Δ. The set of random variables $\{Z_G(\Delta), \Delta \in \mathscr{B}_G\}$ is called in probability theory a *random orthogonal measure of the random field* described by the state P (see [10], Section 3.3.4, and [15]).

We shall point out some properties of the spectral measure Z_G useful in the sequel. All these properties of Z_G can be easily obtained from its definition.

(1) For any set $\Delta \in \mathscr{B}_G$,

$$Z_G(\Delta) = \overline{Z_G(-\Delta)}\,. \tag{4.4}$$

(2) For any nonintersecting $\Delta_1, \Delta_2, \cdots, \Delta_s \in \mathscr{B}_G$, $s = 2, 3, \cdots$

$$Z_G(\Delta_1 \cup \cdots \cup \Delta_s) = Z_G(\Delta_1) + \cdots + Z_G(\Delta_s)\,. \tag{4.5}$$

(3) For any sets $\Delta_1, \Delta_2 \in \mathscr{B}_G$,

$$E(Z_G(\Delta_1)\overline{Z_G(\Delta_2)}) = G(\Delta_1 \cap \Delta_2) . \tag{4.6}$$

(4) For any set $\Delta \in \mathscr{B}_G$,

$$EZ_G(\Delta) = 0 . \tag{4.7}$$

(5) For any sets $\Delta_1, \Delta_2, \cdots, \Delta_s$, $s = 2, 3, \cdots$, such that the sets $\Delta_1 \cup (-\Delta_1), \cdots, \Delta_s \cup (-\Delta_s)$ are nonintersecting, the random variables $Z_G(\Delta_1), \cdots, Z_G(\Delta_s)$ are independent.

(6) For any set $\Delta \in \mathscr{B}_G$ such that $\Delta \cap (-\Delta)$ is empty, the real part $\operatorname{Re} Z_G(\Delta)$ and the imaginary part $\operatorname{Im} Z_G(\Delta)$ of the random variable $Z_G(\Delta)$ are independent Gaussian random variables with mean 0 and variance $G(\Delta)/2$.

It will be necessary for us to modify Itô's construction [14] a little because property (1) above shows that Z_G is not a measure with independent values; besides, the measure $Z_G(\Delta)$ is defined only for $\Delta \in \mathscr{B}_G$. This modification reduces, roughly speaking, to the fact that besides the diagonals $k_i = k_j$, usually discarded, it is also necessary to discard hyperplanes $k_i = -k_j$; in addition it is necessary to include in the integral-approximating sums only the terms corresponding to the sets $\Delta \in \mathscr{B}_G$. A detailed exposition of this modification (for the case $\mathscr{B}_G = \mathscr{B}_0$) can be found in the lectures [28]. But because the lectures are not easily available we shall give a brief exposition of the construction here. Let us denote by H_G^n, $n = 1, 2, \cdots$, the real Hilbert space of complex-valued symmetric functions $h(k_1, \cdots, k_n)$ of $k_1, \cdots, k_n \in \mathbb{R}^\nu$ such that

$$h(k_1, \cdots, k_n) = \overline{h(-k_1, \cdots, -k_n)} \tag{4.8}$$

and such that the norm

$$\|h\|_G^n = \left(\frac{1}{n!} \int_{\mathbb{R}^\nu} \cdots \int_{\mathbb{R}^\nu} |h(k_1, \cdots, k_n)|^2 G(dk_1) \cdots G(dk_n)\right)^{\frac{1}{2}} \tag{4.9}$$

is finite. Let us denote by H_G^0 the one-dimensional space of real constants and by $\operatorname{Exp} H_G$ the orthogonal direct sum of the spaces H_G^n, $n = 0, 1, \cdots$. The space $\operatorname{Exp} H_G$ is usually called the *Fock space* in the mathematical physics literature. The elements of $\operatorname{Exp} H_G$ will be interpreted as sequences

$$h = (h_0, h_1, \cdots) , \qquad h_n \in H_G^n , \quad n = 0, 1, \cdots . \tag{4.10}$$

Let $\hat{H}_G^n \subset H_G^n$ be the subspace of functions $h \in H_G^n$ which can be represented in the following form: Suppose $A_1, \cdots, A_s \in \mathscr{B}_G$, $s = 1, 2, \cdots$ and $A_{-i} = -A_i$, $i = 1, \cdots, s$ are sets such that $A_{-s}, \cdots, A_{-1}, A_1, \cdots, A_s$ are disjoint, and suppose $\hat{h}(i_1, \cdots, i_n)$ are complex numbers. The condition is that

$$h(k_1, \cdots, k_n) = \hat{h}(i_1, \cdots, i_n) \qquad \text{if} \quad k_1 \in A_{i_1}, \cdots, k_n \in A_{i_n} , \tag{4.11}$$

where the indices $i_1, \cdots, i_n$ take values $\pm 1, \cdots, \pm s$ such that $i_j \neq \pm i_{j'}$, $j \neq j'$, $j, j' = 1, \cdots, n$, and that $h(k_1, \cdots, k_n) = 0$ for the other $k_1, \cdots, k_n$. It is easy to check that each $\hat{H}_G^n$ is dense in H_G^n. Let us make the definition

$$(4.12)\qquad I_G(h) = \frac{1}{n!}\int \cdots \int h(k_1, \cdots, k_n) Z_G(dk_1) \cdots Z_G(dk_n)$$

$$= \frac{1}{n!}\sum'_{i_1,\cdots,i_n} \hat{h}(i_1, \cdots, i_n) Z_G(A_{i_1}) \cdots Z_G(A_{i_n}), \quad h \in \hat{H}_G^n,$$

where $\sum'$ indicates that the summation extends over the set of indices $i_1, \cdots, i_n$ used in (4.11). The conditions (4.4) and (4.8) imply that $I_G(h)$ is a real random variable, and so $I_G(h) = \operatorname{Re} I_G(h) \in L_2(P)$. Furthermore, a simple calculation based on the properties of the measure Z given above shows that the mapping $\hat{H}_G^n \to L_2(P): h \to I(h)$ is isometric. Therefore it can be extended uniquely to an isometry $H_G^n \to L_2(P)$. The image of $h \in H_G^n$ under this transformation will be called the *n-tuple Itô integral* and will be denoted $I_G(h)$ (or by the integral notation used in (4.12)). For a vector $h \in \operatorname{Exp} H_G$ of the type (4.10), we set

$$(4.13)\qquad I_G(h) = h_0 + \sum_{n=1}^{\infty} \frac{1}{n!}\int \cdots \int h_n(k_1, \cdots, k_n) Z_G(dk_1) \cdots Z_G(dk_n).$$

The usual constructions (see [5], [14], [25] and [28]) prove that the transformation $\operatorname{Exp} H_G \to L_2(P): h \to I_G(h)$ is a unitary mapping of $\operatorname{Exp} H_G$ onto $L_2(P)$.

We shall describe some further relations for Itô integrals which are important for future use. First we shall give the formula for Itô integrals in terms of Hermite polynomials (see [5], [14], [25] and [28]). The proof of this formula in the case of the usual Itô integrals can be adapted without any change to our case. For any orthonormal functions $h^1, \cdots, h^m \in H_G^1$, $m = 1, 2, \cdots$, and any integers $j_1, \cdots, j_m$, if

$$(4.14)\qquad g^i = h^s \quad \text{for} \quad j_1 + \cdots + j_{s-1} < i \leqq j_1 + \cdots + j_s,$$

where $i = 1, \cdots, j_1 + \cdots + j_m = n$, we have

$$(4.15)\qquad \int \cdots \int \left(\frac{1}{n!}\sum_{(i_1,\cdots,i_n)\in \Pi_n} g^{i_1}(k_1) \cdots g^{i_n}(k_n)\right) Z_G(dk_1) \cdots Z_G(dk_n)$$

$$= H_{j_1}\left(\int h^1(k) Z_G(dk)\right) \cdots H_{j_m}\left(\int h^m(k) Z_G(dk)\right),$$

where Π_n is the set of all $n!$ permutations of the indices $1, 2, \cdots, n$ and H_j, $j = 1, 2, \cdots$, are Hermite polynomials with highest coefficient 1.

Other formulas which are important for our aims give the "diagram" expression for the product of multiple Itô integrals. Suppose we are given integers $n_1, \cdots, n_m$. We shall use the term *diagram*[3] *of order* $(n_1, \cdots, n_m)$ for an undirected graph of $\mathscr{N} = n_1 + \cdots + n_m$ vertices such that its vertices are indexed by the pairs of integers (j, l), $j = 1, \cdots, n_l$, $l = 1, \cdots, m$, such that no more than one

[3] In the description of diagrams traditional in theoretical physics, the set all vertices (j, l) of the graph with fixed l corresponds to one vertex of the diagram together with legs coming out of it, and the vertices connected by the branches correspond to the connected legs. From the unprejudiced point of view such a description seems inconvenient for a mathematical exposition.

branch enters into each vertex, and such that branches can connect only pairs of vertices (j_1, l_1), (j_2, l_2) for which $l_1 \neq l_2$. The set of all diagrams containing t branches will be denoted $\Gamma_t = \Gamma_t(n_1, \cdots, n_m)$, $t = 0, 1, \cdots, [\mathcal{N}/2]$. For even $\mathcal{N}$ the diagrams $\gamma \in \Gamma_{\mathcal{N}/2}(n_1, \cdots, n_m)$ in which one branch enters into each vertex will be called *complete* and the set $\Gamma_{\mathcal{N}/2}(n_1, \cdots, n_m)$ will be denoted also by $\bar{\Gamma} = \bar{\Gamma}(n_1, \cdots, n_m)$. Let there be given a set of functions $h_1 \in H_G^{n_1}, \cdots, h_m \in H_G^{n_m}$. We introduce the function $\hat{h}$ of $\mathcal{N}$ real variables $k_{j,l}$ corresponding to the vertices of the diagram by the formula

$$\text{(4.16)} \qquad \hat{h}(k_{j,l}, j = 1, \cdots, n_l, l = 1, \cdots, m) = \prod_{l=1}^{m} h_l(k_{j,l}, j = 1, \cdots, n_l) \,.$$

Fixing the diagram $\gamma \in \Gamma_t$ we enumerate the variables $k_{j,l}$ in such a way that the variables corresponding to those vertices into which no branches enter will have the numbers $1, 2, \cdots, \mathcal{N} - 2t$, and the variables corresponding to the vertices connected by a branch will have numbers p and $p + t$, where $p = \mathcal{N} - 2t + 1, \cdots, \mathcal{N} - t$. Let

$$\text{(4.17)} \qquad \begin{aligned} h_\gamma'(k_1, \cdots, k_{\mathcal{N}-2t}) &= \int_{R^\nu} \cdots \int_{R^\nu} \hat{h}(k_1, \cdots, k_{\mathcal{N}-t}, -k_{\mathcal{N}-2t+1}, \cdots, -k_{\mathcal{N}-t}) \\ &\quad \times G(dk_{\mathcal{N}-2t+1}) \cdots G(dk_{\mathcal{N}-t}) \,, \end{aligned}$$

and let $h_\gamma \in H_G^{\mathcal{N}-2t}$ be the function obtained from h_γ' by means of symmetrization (i.e., by means of averaging over all $(\mathcal{N} - 2t)!$ permutations of its arguments). It is easy to see, by changing the sign of some of the variables k_s with $s = \mathcal{N} - 2t + 1, \cdots, \mathcal{N} - t$ in the integration and by using the evenness of the measure G, that the function h_γ does not depend on the order of enumeration of the variables of the type considered. Then the following result obtains.

PROPOSITION 4.1. *For any* $h_1 \in H_G^{n_1}, \cdots, h_m \in H_G^{n_m}$, $n_1, \cdots, n_m = 1, 2, \cdots$, *the following formula is true*:

$$\text{(4.18)} \qquad I(h_1) \cdots I(h_m) = \sum_{t=0}^{[\mathcal{N}/2]} \frac{(\mathcal{N} - 2t)!}{n_1! \cdots n_m!} \Big(\sum_{\gamma \in \Gamma_t} I(h_\gamma)\Big) \,.$$

Variants of this formula (all to some degree different in the literature known to the author) are well known in the literature of probability theory (see [14], [28] and [29]) and mathematical physics (the "diagram method"). This formula can be obtained in a purely analytical way from the formula (4.15). We shall indicate shortly another proof of the formula which shows its probability meaning.

PROOF. It is enough to consider the case $m = 2$ because then the relation (4.18) can be proved by induction with respect to m. Let $m = 2$ and let $h_l \in \hat{H}_G^{n_l}$ be the functions given by the relation (4.11) by means of the sets $A_1, \cdots, A_s$ (independent of l) and of the numbers $\hat{h}^{(l)}(i_1, \cdots, i_n)$. We shall enumerate the variables and the sets in the same order as the vertices of the diagrams. From the formula (4.12) we obtain

$$(4.19)\quad \begin{aligned} I(h_1)I(h_2) &= \frac{1}{n_1!\,n_2!}\prod_{l=1}^{2}\left(\sum{}' \hat{h}^l(i_{1,l},\cdots,i_{n_l,l})Z_G(A_{1,l})\cdots Z_G(A_{n_l,l})\right) \\ &= \sum_{t=0}^{[\mathcal{N}/2]}\frac{1}{n_1!\,n_2!}\sum_{\gamma\in\Gamma_t}\Big[\sum{}^{\gamma}\hat{h}(i_{j,l},\, j=1,\cdots,n_l, \\ &\qquad l=1,2)\Big(\prod_{l=1}^{2}\prod_{j=1}^{n_l}Z(A_{j,l})\Big)\Big]\,; \end{aligned}$$

here

$$(4.20)\quad \hat{h}(i_{j,l},\, j=1,\cdots,n_l,\, l=1,2)=\prod_{l=1}^{2}\hat{h}^l(i_{1,l},\cdots,i_{n_l,l})\,,$$

and $\sum^{\gamma}$ indicates that the summation extends over the entire set of indices $i_{j,l}$ such that $i_{j_1,l_1}=\pm i_{j_2,l_2}$ if the vertices (j_1,l_1) and (j_2,l_2) lie on the same branch of the graph γ and such that indices of any other pair of indices i_{j_1,l_1}, i_{j_2,l_2} are different in absolute value. By enumerating the vertices of the diagram in the same way as in the formula (4.17) it is possible to rewrite the term in the square brackets in (4.19) as

$$(4.21)\quad \begin{aligned} &\sum{}'_{i_1,\cdots,i_{\mathcal{N}-2t}}\hat{h}(i_1,\cdots,i_{\mathcal{N}})\prod_{\alpha=1}^{\mathcal{N}-2t}Z_G(A_{i_\alpha}) \\ &\qquad\times\Big[\sum{}'_{i_{\mathcal{N}-2t+1},\cdots,i_{\mathcal{N}-t}}\Big(\sum{}^{*}_{i_{\mathcal{N}-t+1},\cdots,i_{\mathcal{N}}}\prod_{\beta=\mathcal{N}-2t+1}^{\mathcal{N}}Z_G(A_{i_\beta})\Big)\Big]\,, \end{aligned}$$

where $\sum^*$ indicates that the summation extends over all the indices $i_{\mathcal{N}-t+1},\cdots,i_{\mathcal{N}}$ such that $i_{p+t}=\pm i_p$, $p=\mathcal{N}-2t+1,\cdots,\mathcal{N}-t$. By using the properties (1), (3) and (5) of the spectral measure Z_G it is easy to calculate the mathematical expectation

$$(4.22)\quad \begin{aligned} &E\Big(\prod_{\beta=\mathcal{N}-2t+1}^{\mathcal{N}}Z_G(A_{i_\beta})\Big) \\ &\quad = G(A_{i_{\mathcal{N}-2t-1}})\cdots G(A_{i_{\mathcal{N}-t}}) \\ &\qquad\qquad \text{if}\quad i_{p+t}=-i_p\,,\quad p=\mathcal{N}-2t+1,\cdots,\mathcal{N}-t \\ &\quad = 0 \qquad \text{otherwise.} \end{aligned}$$

Elementary calculations show that the random variable in square brackets in (4.21) is close to its mathematical expectation in the $L_2(P)$ metric if $\max_i G(A_i)$ is small enough. So we obtain by a limiting argument the number $(\mathcal{N}-2t)!\,I(h_\gamma)$ and the formula (4.18) from the formula (4.19).

The formula (4.18) can be also used for the evaluation of the moments of the Itô integrals. Because the mean value $EI(g)$ is 0 if $g\in H_G^s$, $s>0$, we find from the relation (4.18) that

$$(4.23)\quad \begin{aligned} &E(I(h_1)\cdots I(h_m)) \\ &\quad = \frac{1}{n_1!\cdots n_m!}\sum_{\gamma\in\bar{\Gamma}(n_1,\cdots,n_m)}h_\gamma \qquad \text{if}\quad \mathcal{N}\ \text{is even}\,, \\ &\quad = 0 \qquad\qquad\qquad\qquad\qquad\qquad \text{if}\quad \mathcal{N}\ \text{is odd}\,; \end{aligned}$$

here the number $\mathcal{N}=n_1+\cdots+n_m$, the set of complete diagrams, and the function $\hat{h}$ are defined the same way as above, and (see (4.17))

$$h_\gamma = \int_{\mathbf{R}^\nu} \cdots \int_{\mathbf{R}^\nu} \hat{h}(k_1, \cdots, k_{\mathscr{N}/2}, -k_1, \cdots, -k_{\mathscr{N}/2}) G(dk_1) \cdots G(dk_{\mathscr{N}/2}) \tag{4.24}$$

if we use the enumeration introduced above for the arguments of the function $\hat{h}$, depending on the choice of γ.

We shall now find the "formula for change of variables" in the Itô integral:

PROPOSITION 4.2. *Let G and G' be continuous spectral measures on $\mathscr{B}_{\mathbf{R}^\nu\setminus\{0\}}$ such that the measure G is absolutely continuous with respect to the measure G', and let $f(k)$ be a complex-valued function such that*

$$f(k) = \overline{f(-k)} \tag{4.25}$$

$$|f(k)|^2 = \frac{dG(k)}{dG'(k)}, \quad k \in \mathbf{R}^\nu\setminus\{0\}.$$

For any $h = (h_0, h_1, \cdots) \in \operatorname{Exp} H_G$ we let

$$h_n'(k_1, \cdots, k_n) = h_n(k_1, \cdots, k_n) f(k_1) \cdots f(k_n), \qquad n = 1, 2, \cdots \tag{4.26}$$

$$h_0' = h_0, \qquad h' = (h_0', h_1', \cdots) \in \operatorname{Exp} H_{G'}.$$

Then the probability distribution of the random variable $I_{G'}(h')$ coincides with the probability distribution of the random variable $I_G(h)$.

PROOF. The formula (4.23) implies the coincidence of all the moments of the variables $I_G(h)$ and $I_{G'}(h')$, but it is regrettably unclear whether the conditions of the uniqueness of the moment problem are fulfilled here. In view of this we use another method of proof. It is clear that the transformation $\operatorname{Exp} H_G \to \operatorname{Exp} H_{G'}$: $h \to h'$ is isometric. From this fact and the fact that the mean-square convergence of random variables implies the weak convergence of their probability distributions, it follows that it is sufficient to prove the proposition only for functions h such that $h_n = 0$ if $n > \mathscr{N}$ and $h_n \in \hat{H}_G^n$, $n = 1, \cdots, \mathscr{N}$. Suppose $h_n \in \hat{H}_G^n$ and $\varepsilon > 0$. Without any loss of generality it is possible to suppose that the sets $A_i = A_i^\varepsilon$, $i = 1, \cdots. s^\varepsilon$ used for the construction of the function h_n are such that for some fixed numbers f_i^ε, $i = 1, \cdots, s^\varepsilon$

$$\operatorname{ess\,sup}_{k \in A_i^\varepsilon} |f(k) - f_i^\varepsilon| \leqq \varepsilon, \qquad i = 1, \cdots, s^\varepsilon \tag{4.27}$$

and $G(A_i^\varepsilon) \neq 0$, $i = 1, \cdots, s^\varepsilon$. The properties (1) through (6) of the measure Z_G listed above imply that two sets of random variables

$$Z_G(A_i^\varepsilon), \qquad i = -s^\varepsilon, \cdots, -1, 1, \cdots, s^\varepsilon \tag{4.28}$$

and (we suppose here that $\frac{0}{0} = 1$).

$$Z_{G'}(A_i^\varepsilon)\left(\frac{G(A_i^\varepsilon)}{G'(A_i^\varepsilon)}\right)^{\frac{1}{2}} \frac{f_i^\varepsilon}{|f_i^\varepsilon|}, \qquad i = -s^\varepsilon, \cdots, -1, 1, \cdots, s^\varepsilon \tag{4.29}$$

have the same joint probability distributions. It is necessary to use here the fact that the property (6) implies that the random variables $Z_G(\Delta)$ and $\zeta Z_G(\Delta)$ are identically distributed if ζ is a complex constant with $|\zeta| = 1$. So the probability distributions of the random variables $I_G(h_n)$ and $I_{G'}(h_n^\varepsilon)$ coincide, where

$$(4.30)\qquad h_n^\varepsilon(k_1, \cdots, k_n) = h_n(k_1, \cdots, k_n) \prod_{l=1}^n \left[\left(\frac{G(A_{i_l}^\varepsilon)}{G'(A_{i_l}^\varepsilon)}\right)^{\frac{1}{2}} \frac{f_{i_l}^\varepsilon}{|f_{i_l}^\varepsilon|}\right], \quad \text{if} \quad k_1 \in A_{i_1}, \cdots, k_n \in A_{i_n}$$

and $h_n^\varepsilon(k_1, \cdots, k_n) = 0$ if $h_n(k_1, \cdots, k_n) = 0$. It is clear that as $\varepsilon \to 0$, $h_n^\varepsilon \to h_n'$ in the Hilbert space $H_{G'}^n$. This implies that the random variables $I_G(h_n)$ and $I_{G'}(h_n')$ are identically distributed. Analogous considerations prove the coincidence of the joint probability distribution of N random variables $(I_G(h_1), \cdots, I_G(h_{\mathscr{N}}))$ and $(I_{G'}(h_1'), \cdots, I_{G'}(h'_{\mathscr{N}}))$, and it implies the coincidence of the probability distributions of the random variables $I_G(h)$ and $I_{G'}(h')$ and therefore proves the proposition.

5. Random fields subordinated to Gaussian fields. Let $P \in \mathscr{P}_r$ be the state of a Gaussian stationary random field given by a spectral measure G, and let $L_2(P)$ be the Hilbert space of the real functions $\Phi(F)$ of $F \in \mathscr{S}_r'$ square-integrable with respect to the measure P. We shall use the term *random functional over the state P* for a system of random variables $\{\Phi_\phi, \phi \in \mathscr{S}_{r'}\}$ such that $\Phi_\phi \in L_2(P)$, $\phi \in \mathscr{S}_{r'}$ and such that the transformation $\mathscr{S}_{r'} \to L_2(P)$ is linear and continuous. The Minlos theorem (see the references in Section 2) implies that for any random functional $\{\Phi_\phi, \phi \in \mathscr{S}_{r'}\}$ there exists a unique state $P' \in \mathscr{P}_{r'}$ such that

$$(5.1)\qquad \begin{aligned} &P(F \in \mathscr{S}_r' : \Phi_\phi(F) \in B) \\ &\quad = P'(F \in \mathscr{S}_{r'}' : F(\phi) \in B), \qquad \phi \in \mathscr{S}_{r'}, \quad B \in \mathscr{B}_1, \end{aligned}$$

where $\mathscr{B}_1$ is the σ-algebra of Borel sets in $\mathbb{R}^1$. The state P' will be called *a state of the random functional* $\{\Phi_\phi, \phi \in \mathscr{S}_{r'}\}$. The random functional can be defined by means of a measurable transformation $H: \mathscr{S}_r' \to \mathscr{S}_{r'}'$ if

$$(5.2)\qquad \Phi_\phi(F) = HF(\phi), \quad F \in \mathscr{S}_r',$$

and in this way we obtain a function $\Phi_\phi \in L_2(P)$, $\phi \in \mathscr{S}_{r'}'$.[4] The shift transformations $\hat{E}_a$ in the space $\mathscr{S}_r'$ of generalized functions (see (2.16)) induce the shift transformation in the space $L_2(P)$ which we shall again denote $\hat{E}_a$:

$$(5.3)\qquad \hat{E}_a \Phi(F) = \Phi(\hat{E}_a F), \qquad \Phi \in L_2(P), \quad F \in \mathscr{S}_r', \quad a \in \mathbb{R}^\nu.$$

We shall say that the random functional describes *a random field subordinate to the field having the state P* if

$$(5.4)\qquad \hat{E}_a \Phi_\phi = \Phi_{E_a\phi}, \qquad \phi \in \mathscr{S}_{r'}, \quad a \in \mathbb{R}^\nu.$$

By using the fact that the space $\mathscr{S}_{r'}$ contains a countable dense subset (see [9], Sections 1.6.5 and 2.2.3) it is easy to show that it is necessary and sufficient for

[4] By applying the Minlos theorem to a "two-dimensional" system of random variables $\{\Phi_{\phi'}, \phi' \in \mathscr{S}_{r'}, F(\phi), \phi \in \mathscr{S}_r\}$ it is possible to show that any random functional can be described in such a way. But it is not essential for what follows.

the subordination of the field of the type (5.2) that

$$H\hat{E}_a = \hat{E}_a H\,, \quad a \in \mathbb{R}^\nu\,. \tag{5.5}$$

An important class of subordinated fields is given as

$$\Phi_\phi = \int_{\mathbb{R}^\nu} \phi(x)\Psi(\hat{E}_x F)\,dx\,, \qquad \phi \in \mathscr{S}_{r'}\,, \quad F \in \mathscr{S}_r' \tag{5.6}$$

where $\Psi \in L_2(P)$ and it is necessary to understand the integral as an integral in the sense of convergence in $L_2(P)$. Such a generalized random field can be interpreted as the ordinary random field given by the system of random variables $\xi(x) = \Psi(\hat{E}_x F)$, $x \in \mathbb{R}^\nu$, on the probability space $(\mathscr{S}_r', \mathscr{B}_r, P)$. We shall give in this section a description of subordinated random fields in terms of Itô's integrals. We begin by the description of the operators $\hat{E}_a$, $a \in \mathbb{R}^\nu$.

PROPOSITION 5.1. *For any vector* $h \in \operatorname{Exp} H_G$ *of the kind* (4.10) *and for any* $a \in \mathbb{R}^\nu$,

$$\hat{E}_a I(h) = h_0 + \sum_{n=1}^\infty \frac{1}{n!} \int \cdots \int h_n(k_1, \cdots, k_n) \times \exp\{ia(k_1 + \cdots + k_n)\} Z_G(dk_1) \cdots Z_G(dk_n)\,. \tag{5.7}$$

PROOF, For the vectors $h \in H_G^1$ the formula (5.7) follows immediately from the definition (5.3) and the fact that the functions $\tilde{\phi}$ with $\phi \in \mathscr{S}_r$ are dense in H_G^1. Then by using the formula (4.15) we can check the relation (5.7) for the vectors $h \in H_G^n$ used in (4.15) under the integral sign. Because it is possible to construct a basis of H_G^n consisting of such vectors the relation (5.7) is true for all $h \in H_G^n$ and hence for all $h \in \operatorname{Exp} H_G$.

We note here the following useful fact (used in Maruyama's report [20]). We shall consider a (usual) Gaussian field of the type (5.6) subordinated to the Gaussian stationary field having the spectral measure G. By choosing $\hat{h} = (\hat{h}_n, n = 0, 1, \cdots)$ such that $I(\hat{h}) = \Psi$ we can describe this field by the formula

$$\Phi_\phi = \hat{h}_0 \tilde{\phi}(0) + \sum_{n=1}^\infty \frac{1}{n!} \int \cdots \int \tilde{\phi}(k_1 + \cdots + k_n) \times \hat{h}_n(k_1, \cdots, k_n) Z_G(dk_1) \cdots Z_G(dk_n)\,. \tag{5.8}$$

To check this formula it is necessary to rewrite the integral-approximating sums as Itô integrals and then to go to the limit. The following theorem shows that if we consider a more general class of functions $\hat{h}_n$ then we can extend this representation to generalized subordinated random fields.

THEOREM 5.2. *A random functional of a stationary Gaussian random field having the spectral measure G is a field subordinated to this Gaussian field if and only if one has the representation*

$$\Phi_\phi = \sum_{j \in \mathscr{J}_{r'}^\nu} i^{r'} \hat{h}_j D^j \tilde{\phi}(0) + \sum_{n=1}^\infty \frac{1}{n!} \int \cdots \int \tilde{\phi}(k_1 + \cdots + k_n) \times \hat{h}_n(k_1, \cdots, k_n) Z_G(dk_1) \cdots Z_G(dk_n)\,, \quad \varphi \in \mathscr{S}_{r'} \tag{5.9}$$

where we use the following notations. Functions $\hat{h}_n$, $n = 1, 2, \cdots$, satisfy the conditions (4.8) *and are functions of the space $L_0(G^n)$, and G^n is the direct product of n copies of the measure G. The measure G is such that the conditions*

$$(5.10)\qquad \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\{(k_1, \cdots, k_n):\, |k_1 + \cdots + k_n| \leqq 1\}} |k_1 + \cdots + k_n|^{2r'} |\hat{h}_n(k_1, \cdots, k_n)|^2 G(dk_1) \cdots G(dk_n) < \infty$$

and

$$(5.11)\qquad \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\{(k_1, \cdots, k_n):\, |k_1 + \cdots + k_n| > 1\}} |k_1 + \cdots + k_n|^{-q} |\hat{h}_n(k_1, \cdots, k_n)|^2 G(dk_1) \cdots G(dk_n) < \infty$$

for some $q < \infty$ are satisfied. Moreover, $\hat{h}_j$, $j \in \mathscr{J}_{r'}^{\nu}$ are real numbers.

PROOF. The conditions (5.10), (5.11) and (2.3) imply that

$$(5.12)\qquad \sum_{n=1}^{\infty} \frac{1}{n!} \int_{\mathbf{R}^\nu \setminus \{0\}} \cdots \int_{\mathbf{R}^\nu \setminus \{0\}} |\tilde{\phi}(k_1 + \cdots + k_n) \hat{h}_n(k_1, \cdots, k_n)|^2 \times G(dk_1) \cdots G(dk_n) < \infty\,, \quad \phi \in \mathscr{S}_{r'}\,,$$

and that the expression (5.9) depends continuously on $\phi \in \mathscr{S}_{r'}$ in $L_2(P)$. Therefore the relation (5.9) defines correctly a random functional over the state P. The relation (5.4) follows from the relation (5.7), and so the sufficiency is proved.

For the proof of necessity, by using the representation of the space $L_2(P)$ through Itô integrals, we consider the random variables

$$(5.13)\qquad \Phi_\phi = h_0^\phi + \sum_{n=1}^{\infty} \frac{1}{n!} \int \cdots \int h_n^\phi(k_1, \cdots, k_n) Z_G(dk_1) \cdots Z_G(dk_n)\,.$$

Let the function $\phi_0 \in \mathscr{S}$ be such that $\tilde{\phi}_0(k) > 0$ for all $k \in \mathbb{R}^\nu$. For any $j \in \mathscr{J}_{r'}^{\nu}$ we denote by $\phi_j \in \mathscr{S}_{r'}$ the function whose Fourier transform is $\tilde{\phi}_j(k) = k^j \tilde{\phi}_0(k)$. Our next aim is to obtain the relation

$$(5.14)\qquad h_n^\phi(k_1, \cdots, k_n) = \frac{h_n^{\phi_j}(k_1, \cdots, k_n)}{\tilde{\phi}_j(k_1 + \cdots + k_n)} \tilde{\phi}(k_1 + \cdots + k_n)\,, \quad \phi \in \mathscr{S}_{r'}\,.$$

(The denominator $\tilde{\phi}_j$ in (5.14) can vanish only on a set of G^n-measure 0 and so the right side of the equality (5.14) is well defined.) Let us denote by $\tilde{\mathscr{D}}^j \subseteq \mathscr{S}_{r'}$ the set of all functions ϕ such that $\phi = D^j \gamma$, where $\gamma \in \mathscr{S}$ and $\tilde{\gamma}$ has finite range. If $\phi \in \tilde{\mathscr{D}}^j$, then the ratio $\tilde{\phi}/\tilde{\phi}_j$ has an inverse Fourier transform $\chi_j \in \mathscr{S}$. Then $\phi = \phi_j * \chi_j$ where $*$ denotes convolution. So it is possible to construct by the usual discretization of the function χ_j a sequence of functions

$$(5.15)\qquad \phi^l(k) = \sum_{\alpha=1}^{s_l} c_\alpha{}^l \phi_j(k - a_\alpha{}^l)\,, \qquad l = 1, 2, \cdots$$

such that $\phi^l \to \phi$ in the sense of convergence in $\mathscr{S}_{r'}$ and the sequence

$$(5.16)\qquad \tilde{\chi}_j^l(k) = \sum_{\alpha=1}^{s_l} c_\alpha{}^l e^{i a_\alpha^l k}\,, \qquad l = 1, 2, \cdots$$

converges at each point to $\tilde{\chi}_j(k)$ and is uniformly bounded for $k \in \mathbb{R}^\nu$ and $l = 1, 2, \cdots$. The relation (5.14) for $\phi = \phi^l$ follows from the formulas (5.13), (5.4) and (5.7). By considering the limit in (5.14) when $l \to \infty$ and ϕ is changed to ϕ^l, we find using the continuity of the transformation $\mathscr{S}_{r'} \to H_G^n : \phi \to h_n^\phi$ that the relation (5.14) is true for all functions $\phi \in \tilde{\mathscr{D}}_j$. We note now that a function $\phi \in \mathscr{S}_{r'}$ such that the support of its Fourier transform has finite range and satisfies

$$\operatorname{supp} \tilde{\phi} \subset \{k \in \mathbb{R}^\nu : k^j \neq 0, j \in \mathscr{J}_{r'}^\nu\} \tag{5.17}$$

is an element of each of the spaces $\tilde{\mathscr{D}}^j$, $j \in \mathscr{J}_{r'}^\nu$. Because the complement of the set on the right in (5.17) has G-measure 0, we can prove by applying (5.14) to such a function ϕ that there exists a function $\hat{h}_n \in L_0(G^n)$ such that

$$\hat{h}_n(k_1, \cdots, k_n) = \frac{h_n^{\phi_j}(k_1, \cdots, k_n)}{\tilde{\phi}_j(k_1 + \cdots + k_n)}, \quad j \in \mathscr{J}_{r'}^\nu \tag{5.18}$$

and that for all functions $\phi \in \bigcup_{j \in \mathscr{J}_{r'}^\nu} \tilde{\mathscr{D}}^j$

$$h_n^\phi(k_1, \cdots, k_n) = \hat{h}_n(k_1, \cdots, k_n)\tilde{\phi}(k_1 + \cdots + k_n) . \tag{5.19}$$

By applying this relation to $\phi = \phi_j$, $j \in \mathscr{J}_{r'}^\nu$, using the fact that

$$|k_1 + \cdots + k_n|^{2r'} = \sum_{j \in \mathscr{J}_{r'}^\nu} c_j[(k_1 + \cdots + k_n)^j]^2 \tag{5.20}$$

for some integers c_j, and finally using the inequalities (see (4.8))

$$\sum_{n=1}^\infty (\|h_n^{\phi_j}\|_G^n)^2 < \infty , \quad j \in \mathscr{J}_{r'}^\nu , \tag{5.21}$$

we find that the condition (5.10) is satisfied. The condition (5.11) can be proved in the same way as a similar condition is checked in the construction of the general expression for conditionally positive functions (see [10], Sections 2.2.2 and 2.3.3). It is known (see [10], the addendum to Section 2.4) that the set of linear combinations of functions of $\bigcup_{j \in \mathscr{J}_{r'}^\nu} \tilde{\mathscr{D}}^j$ is dense in the space $\mathscr{S}_{r'}$. So by using the continuity of the transformation $\phi \to h_n^\phi$ and the conditions (5.10) and (5.11) we can extend the relation (5.19) to all functions $\phi \in \mathscr{S}_{r'}$.

As regards the constant h_0^ϕ, we have first to note that it is a functional on $\phi \in \mathscr{S}_{r'}$ which is invariant under the shifts from $\mathbb{R}^\nu$. Therefore the functionals

$$h_j^\phi = h_0^{D^j\phi} , \qquad \phi \in \mathscr{S}, \quad j \in \mathscr{J}_{r'}^\nu \tag{5.22}$$

are also invariant and a known theorem about general description of the generalized functions which are invariant with respect to shifts (see [9], Section 1.2.6) implies that there exist constants h_j such that

$$h_0^{D^j\phi} = h_j\tilde{\phi}(0) = (j_1! \cdots j_r!)^{-1} i^{r'} R_j D^j (D^j\phi)^\sim(0) . \tag{5.23}$$

By using the fact that $D^j\tilde{\phi}(0) = 0$ if $\phi = D^{j'}\psi$, where $j \neq j'$, $j' \in \mathscr{J}_{r'}^\nu$, and by using (5.23) we find that if $\hat{h}_j = (j_1! \cdots j_r!)^{-1} h_j$

$$h_0^\phi = i^{r'} \sum_{j \in \mathscr{J}_{r'}^\nu} \hat{h}_j D^j \tilde{\phi}(0) \tag{5.24}$$

for all functions $\phi \in \mathscr{S}_{r'}$ which are linear combinations of the functions which are equal to $D^j\psi$, $j \in \mathscr{J}_{r'}^\nu$, $\psi \in \mathscr{S}$. The manifold of such functions is dense in $\mathscr{S}_{r'}$ (see Section 2) and because the transformation $\mathscr{S}_{r'} \to \mathbb{R}^1 : \phi \to h_0^\phi$ is continuous the relation (5.23) is true for all $\phi \in \mathscr{S}_{r'}$, and this completes the proof of the theorem.

The set of numbers and functions $\{\hat{h}_j, j \in \mathscr{J}_{r'}^{\nu}, \hat{h}_n, n = 1, 2, \cdots\}$ will be called *the spectral description of the random functional* $\{\Phi_\phi, \phi \in \mathscr{S}_{r'}\}$ and the expression (5.9) will be called its *spectral representation.* For readers accustomed to them, we note that by using the notations of mathematical physics it is possible in the case $r' = 0$ $\hat{h}_0 = 0$, $\hat{h}_n(k_1, \cdots, k_n) = \hat{h}_n^0$, where $\hat{h}_n^0$ are some constants, to express the random field as

$$\Phi_\phi = \sum_{n=1}^{\infty} \hat{h}_n^0 \int_{\mathbf{R}^\nu} :\Phi_0(x)^n: \phi(x)\, dx\,, \quad \phi \in \mathscr{S}, \tag{5.25}$$

where $\Phi_0(x)$ is the Gaussian field described by the state P.

6. Self-similar random fields subordinated to Gaussian ones. We now describe in terms of the spectral representation the action of the renormalization-group (see (2.26)). We shall drop the inessential constant terms in (5.9).

PROPOSITION 6.1. *Let*

$$\Phi_\phi = \sum_{n=1}^{\infty} \frac{1}{n!} \int \cdots \int \tilde{\phi}(k_1 + \cdots + k_n)\hat{h}_n(k_1, \cdots, k_n) Z_G(dk_1) \cdots Z_G(dk_n) \tag{6.1}$$

be the spectral representation of a random field subordinated to the Gaussian field having the spectral measure G and the state $P' \in P_{r'}$. *Consider the measure*

$$G_\lambda^{\kappa_0}(A) = \lambda^{2\kappa_0} G(\lambda^{-1}A)\,, \qquad A \in \mathscr{B}_{\mathbf{R}^\nu \setminus \{0\}}\,, \quad \lambda \in (0, \infty)\,, \tag{6.2}$$

where $\kappa_0 \in \mathbf{R}^1$ *is some constant. Then the stationary functional* $\{\Phi_\phi^\lambda, \phi \in \mathscr{S}_{r'}\}$ *over the Gaussian stationary field having the spectral measure* $G_\lambda^{\kappa_0}$ *which is described by the spectral representation*

$$\Phi_\phi^\lambda = \sum_{n=1}^{\infty} \frac{1}{n!} \lambda^{\kappa - n\kappa_0} \int \cdots \int \tilde{\phi}(k_1 + \cdots + k_n) \times \hat{h}_n(\lambda^{-1}k_1, \cdots, \lambda^{-1}k_n) Z_{G_\lambda^{\kappa_0}}(dk_1) \cdots Z_{G_\lambda^{\kappa_0}}(dk_n) \tag{6.3}$$

has the state $S_{\kappa,\lambda}^* P'$.

PROOF. The definition (2.26) and the relation (3.11) imply directly that the random field

$$\hat{\Phi}_\phi^\lambda = \sum_{n=1}^{\infty} \frac{1}{n!} \lambda^{\kappa} \int \cdots \int \tilde{\phi}(\lambda(k_1 + \cdots + k_n))\hat{h}_n(k_1, \cdots, k_n) \times Z_G(dk_1) \cdots Z_G(dk_n) \tag{6.4}$$

has the state $S_{\kappa,\lambda}^* P'$. By using the Proposition 4.2 with $G'(A) = \lambda^{2\kappa_0} G(A)$, $A \in \mathscr{B}_{\mathbf{R}^\nu \setminus \{0\}}$ and then using the evident "change of variables of the Itô integral" we can obtain the relation (6.3) from the relation (6.4).

Now it is easy to obtain the following result.

THEOREM 6.2. *Suppose that the spectral measure G defines a self-similar Gaussian random field with parameter* κ_0 *and let* $\{\Phi_\phi, \phi \in \mathscr{S}_{r'}\}$ *be the random field subordinated* *to it and described by formula* (6.1). *If the functions* $\hat{h}_n(k_1, \cdots, k_n)$ *are such that*[5] *for some* $\kappa \in \mathbb{R}^1$

$$(6.5) \qquad \hat{h}_n(\lambda k_1, \cdots, \lambda k_n) = \lambda^{-n\kappa_0+\kappa}\hat{h}_n(k_1, \cdots, k_n),$$

then $\{\Phi_\phi, \phi \in \mathscr{S}_{r'}\}$ *is a self-similar stationary random field with parameter* κ.

PROOF. The relation (3.12) implies that $G_\lambda^{\kappa_0} \equiv G$, $\lambda \in (0, \infty)$, in the conditions of the theorem, and so the theorem follows from Proposition 6.1.

We note that this theorem seems unfinished in several respects. First of all, the following question remains open and seems very difficult: Is it true that the theorem exhausts the entire class of self-similar stationary fields subordinated to stationary Gaussian fields? Secondly, different self-similar stationary functionals can have the same state (see for example the proposition below), and so the question of the description of the set of all self-similar states obtained with the help of the theorem remains open. (A very difficult question, also open, lies on the path to the solution of both problems: How is it possible to describe in spectral terms the classes of subordinated processes having the same state?) Thirdly, the theorem leaves open the following question. To what extent is the homogeneity condition (6.5) consistent with the conditions (5.10) and (5.11), which are necessary for the existence of the subordinated random fields? By restricting somewhat the generality of our considerations we can avoid cumbersome calculations and reduce the question to a variant of the power counting theorem, well known in mathematical physics. It seems reasonable to suspect that the class of self-similar fields explicitly described in the following theorem generates by means of the operations of convolution and closure the entire class of such fields (this is yet another open question). (The question of the possibility of using in such generation only a finite-dimensional set of fields (see [4], Section 5) also remains open.)

THEOREM 6.3. *Consider a spectral measure* G_0 *defined by the relation* (3.10), *with* κ *changed to* κ_0 *and* $\tilde{G}^P$ *to* $\tilde{G}_0$, *which defines a Gaussian stationary self-similar field with parameter* κ_0. *We suppose that the measure* $\tilde{G}_0$ *is described by a bounded density with respect to the uniform measure on* Q^ν. *Let* κ *be a number in* $(0, \nu/2)$. *Then the relation*

$$(6.6) \qquad \Phi_\phi = \sum_{n=1}^{M} \frac{1}{n!} \int \cdots \int \tilde{\phi}(k_1 + \cdots + k_n) \prod_{j=1}^{n} \left(|k_j|^{-\kappa_0+\kappa/n} g_n\left(\frac{k_j}{|k_j|}\right)\right) \times Z_{G_0}(dk_1) \cdots Z_{G_0}(dk_n), \quad \phi \in \mathscr{S},$$

where $g_n(e)$, $e \in Q^\nu$, $n = 1, \cdots, M$, *are real bounded even functions defined on the* ν*-dimensional sphere* Q^ν *and* M *is an integer, defines a self-similar random field with parameter* κ *subordinated to the field having the spectral measure* G_0.

PROOF. The additional statement of this theorem (in comparison with the

[5] The explicit description of the class of functions having the homogeneity property (6.5) can be obtained by using "spherical coordinates" of the type (3.8) (compare Theorem 6.3).

previous theorem) is only the existence of the fields in (6.6). Theorem 5.2 implies that it is necessary and sufficient for the proof of this theorem to prove the finiteness of the integral

$$(6.7)\qquad I_n = \int_{\mathbf{R}^\nu\setminus\{0\}} \cdots \int_{\mathbf{R}^\nu\setminus\{0\}} (1 + |k_1 + \cdots + k_n|)^{-q} \times \prod_{j=1}^n \left(|k_j|^{-2\kappa_0+2\kappa/n} \left|g_n\left(\frac{k_j}{|k_j|}\right)\right|^2\right) G_0(dk_1) \cdots G_0(dk_n)$$

for some $q < \infty$.

By using an evident change of variables and the boundedness of the functions g_n and the density of the measure $\bar{G}_0$, we find that for some $C < \infty$

$$(6.8)\qquad I_n \leqq C \int_{\mathbf{R}^\nu\setminus\{0\}} \cdots \int_{\mathbf{R}^\nu\setminus\{0\}} (1 + |k_1 + \cdots + k_n|)^{-q} \times \left(\prod_{j=1}^n |k_j|^{2\kappa/n-\nu}\right) dk_1 \cdots dk_n .$$

A variant of the power-counting theorem due to Lowensten and Zimmerman (see [19], equation (3.4)) implies that if $0 < \kappa < \nu/2$ and $q > \nu$, then the integral converges.

The state of the self-similar random field described by the formula (6.6) will be denoted $P^\kappa_{G_0,g_1,\cdots,g_M}$. If $\kappa = M\kappa_0$, $g_M \equiv 1$, $g_i \equiv 0$, $i \neq M$, this random field can be written in the notation (5.25) of mathematical physics as

$$(6.9)\qquad \Phi_\phi = \int_{\mathbf{R}^\nu} : \Phi_0(x)^M : \phi(x)\, dx , \quad \phi \in \mathscr{S} .$$

So it is a Wick power of a self-similar Gaussian field Φ_0.

By using the relation (4.23) we can immediately find explicit expressions for the moments of the fields.

PROPOSITION 6.4. *For any* $\phi_1, \cdots, \phi_m \in \mathscr{S}$,

$$(6.10)\qquad \int_{\mathscr{S}'} F(\phi_1) \cdots F(\phi_m) P^\kappa_{G_0,g_1,\cdots,g_M}(dF) = \sum^M_{n_1,\cdots,n_m=1:n_1+\cdots+n_m-\text{even}} (n_1!\, n_2! \cdots n_m!)^{-1} \sum_{\gamma\in\bar{\Gamma}(n_1,\cdots,n_m)} h_\gamma ,$$

where, as in Section 4, $\bar{\Gamma}(n_1, \cdots, n_m)$ *is the set of all complete diagrams of order* $(n_1, \cdots, n_m)$. *Here* (*compare* (4.24), (4.16))

$$(6.11)\qquad h_\gamma = \int_{\mathbf{R}^\nu} \cdots \int_{\mathbf{R}^\nu} \hat{h}(k_1, \cdots, k_{\mathscr{N}/2}, -k_1, \cdots, -k_{\mathscr{N}/2}) \times G_0'(dk_1) \cdots G_0'(dk_{\mathscr{N}/2}) , \quad \gamma \in \bar{\Gamma}(n_1, \cdots, n_m) , \quad \mathscr{N} = n_1 + \cdots + n_m$$

and

$$(6.12)\qquad \hat{h}(k_{j,l}, j = 1, \cdots, n_l, l = 1, \cdots, m) = \prod_{l=1}^m \tilde{\phi}_l(k_{1,l} + \cdots + k_{n_l,l}) g_{n_l}\left(\frac{k_{1,l}}{|k_{1,l}|}\right) \cdots g_{n_l}\left(\frac{k_{n_l,l}}{|k_{n_l,l}|}\right) ,$$

where the arguments $k_{j,l}$ *in* (6.11) *are enumerated in such a way that the vertices of diagrams corresponding to the variables with numbers p and* $p + \mathscr{N}/2$, $p = 1, \cdots, \mathscr{N}/2$, *are connected by branches of the diagram, and finally, the measure* G_0' *is* *described in spherical coordinates* (3.8) *by the relation*

$$G_0'(\bar{E} \times C) = \bar{G}_0(\bar{E}) \int_C \alpha^{2\kappa/n-1}\, d\alpha \,. \tag{6.13}$$

The formula for moments shows that there are many coinciding states among the $P^{\kappa}_{G_0,g_1,\cdots,g_M}$.

PROPOSITION 6.5. *If the states* $P^{\kappa_1}_{G_0^1,g_1^1,\cdots,g_M^1}$ *and* $P^{\kappa_2}_{G_0^2,g_1^2,\cdots,g_M^2}$ *are such that the measures* G_0^i, $i = 1, 2$, *are given by the formula* (3.10) (*where* κ *is changed to* κ_0^i *and* $\bar{G}^P$ *is changed to* $\bar{G}_0^i$) *and are also such that*

$$|g_n^1(e)|^2 = |g_n^2(e)|^2 \frac{d\bar{G}_0^1(e)}{d\bar{G}_0^2(e)}\,, \qquad n = 1, \cdots, M\,, \tag{6.14}$$

almost everywhere with respect to the measure $\bar{G}_0^2$, *then these states coincide.*

This proposition follows immediately from Proposition 4.2. It implies that in the case $\nu = 1$ it is possible without loss of generality to consider only the case where G_0 is Lebesgue measure, i.e., P is the state of white noise. It is natural to conjecture that the states of the random fields of Theorem 6.3 are different if Proposition 6.5 does not imply that they coincide; this is not yet proven. We can note only that the application of the equality (6.10) to positive $\tilde{\phi}_l$ shows that if g_n does not vanish for some even n then the moments of odd order do not vanish and therefore the field is not Gaussian.

With the help of the general construction described in Section 2, the generalized self-similar field with states $P^{\kappa}_{G_0,g_1,\cdots,g_M}$ can be used for the construction of self-similar fields with discrete arguments. It is necessary only to check that the states are discretizable.

PROPOSITION 6.6. *Under the conditions of Theorem* 6.3 *all states* $P^{\kappa}_{G_0,g_1,\cdots,g_M}$ *are discretizable.*

PROOF. We define the random variables Φ_ϕ by the equality (6.6) for any function $\phi \in M_w$, where $w(k) = (|k| + 1)^{-\varepsilon}$, $\varepsilon > 0$ is small enough. The definition (2.31) implies that in order to check the correctness of the definition of the variables Φ_ϕ it is enough to show that for small enough positive ε the integrals (see (6.8))

$$\int_{\mathbb{R}^\nu\setminus\{0\}} \cdots \int_{\mathbb{R}^\nu\setminus\{0\}} \prod_{l=1}^{\nu} [(\prod_{j=1}^{\nu} (1 + |k_j^l|))^{-2+2\varepsilon} \prod_{j=1}^{n} |k_j|^{2\kappa/n-\nu}]\, dk_1 \cdots dk_n\,, \tag{6.15}$$
$$k_j = (k_j^1, \cdots, k_j^\nu) \in \mathbb{R}^\nu\,, \quad j = 1, \cdots, n\,,$$

are finite. This follows again from Lowenstein and Zimmerman's result [19].

Let

$$L^P(\phi) = \int_{\mathscr{S}'} \exp\{i\Phi_\phi(F)\} P(dF)\,, \quad \phi \in M_w\,, \tag{6.16}$$

where P is the state of Gaussian field with the spectral measure G_0. We see that this formula describes the necessary continuation of the characteristic functional of the states $P^{\kappa}_{G_0,g_1,\cdots,g_M}$. The continuity of the functional follows from the isometry of the transformation $\operatorname{Exp} H_G \to L_2(P)\colon h \to I(h)$ (see Section 4), from the fact that the convergence of random variables in the mean-square sense implies the convergence of their characteristic functions, and finally from the finiteness of the integral (6.15).

Let the dimension be $\nu = 1$. Because the indicators γ_t, $t \in \mathbb{R}^1$, of the interval $(0, t)$ for $t \geqq 0$ and of $(t, 0)$ for $t < 0$, are in the space M_w, the construction of Proposition 6.6 makes it possible to define the random process $\xi = \{\xi_t, t \in \mathbb{R}^1\}$ by the relation

$$\xi_t = \Phi_{\gamma_t}, \quad t \in \mathbb{R}^1, \tag{6.17}$$

where Φ_ϕ is defined by the relation (6.6). It is clear that the derivative of the random process ξ is the corresponding self-similar stationary process of Theorem 6.3, and so (see Proposition 2.2) ξ is a process with stationary increments of order 1. (In the terminology of Lamperti [18], this means that it is semi-stable.) A comparison of the expression for the moments given in Proposition 6.4 with that found by Taqqu [27] shows that if $g_n \equiv 0$, $n \neq 2$, it is the Rosenblatt process mentioned in Section 1.

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