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MULTIVARIATE SHOCK MODELS FOR DISTRIBUTIONS WITH INCREASING HAZARD RATE AVERAGE[1]

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Suppose that n devices are subjected to shocks occurring randomly in time as events in a Poisson process. Upon occurrence of the ith shock the devices suffer nonnegative random damages with joint distribution F_i. Damages from successive shocks are independent and accumulate additively. Failure of the jth device occurs at the time T_j when its accumulated damage first exceeds its breaking threshold x_j. If τ is the life function of a coherent system, then the system life length $\tau(T_1, \cdots, T_n)$ has a distribution with increasing hazard rate average providing that $F_1, F_2, \cdots$ satisfy a multivariate stochastic ordering condition that depends upon τ. If $F_1 = F_2 = \cdots$ and $\bar{H}$ is the joint survival function of $T_1, \cdots, T_n$, then $[\bar{H}(\alpha\mathbf{t})]^{1/\alpha}$ is decreasing in α for all $\mathbf{t} > 0$. $\bar{H}$ also satisfies a multivariate "new better than used" property. Moreover $T_1, \cdots, T_n$ are associated when $F_1 = F_2 = \cdots$. Examples of specific distributions are given which arise from the shock model, including a new bivariate gamma distribution which reduces to the bivariate exponential distribution of Marshall and Olkin as a special case.

1. Introduction. Shock models have been used by a number of authors to derive representations for life distributions. This paper is concerned with properties derived from a shock model for multivariate life distributions. Some univariate results of Esary, Marshall and Proschan (1973) are generalized, and some multivariate results without univariate analogs are obtained. The model studied here is the following *cumulative damage shock model*:

> n devices are subjected to shocks occurring randomly in time as events in a Poisson process. Upon occurrence of the ith shock, the devices suffer nonnegative random damages with joint distribution F_i. Damages from successive shocks are independent and accumulate additively. Failure of a device occurs when its accumulated damage first exceeds its breaking threshold.

This model was previously formulated and studied by A-Hameed and Proschan (1973). Most results depend upon the damages from successive shocks being stochastically increasing. The appropriate notion of multivariate stochastic ordering depends upon the result to be proved, as indicated particularly in Section 3.

A related model has been discussed by Esary and Marshall (1974), who assume that the damage distributions F_i are all equal to a distribution F which concentrates

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on the diagonal $z_1 = \cdots = z_n$. This means that upon occurrence of a shock, the same damage is inflicted upon all n devices. This is an interesting multivariate model if the breaking thresholds are random. Further results for random thresholds are obtained by A-Hameed and Proschan (1973), but in this paper only nonrandom thresholds are considered.

It is easy to imagine practical situations in which there are actually several sources of shocks each independently governed by a Poisson process and each with its own sequence of damage distributions F_i. Some sources of damage may affect only a subset of devices. Although such circumstances may be recognized in practice, they do not lead to more general mathematical models: the several Poisson processes can be superimposed to yield a new Poisson process and the several sequences of damage distributions can be replaced by a sequence of appropriate mixtures.

It is possible, in the multivariate shock model, that the n devices have independent life lengths. To see this, suppose that there are n independent Poisson processes and the ith process governs (identically distributed) damages which affect only the ith device. As pointed out above, this is equivalent to the model with only one Poisson process in which F_i is independent of i and F_i has all its mass on the coordinate axes.

Throughout this paper, the joint life distribution derived from the cumulative damage shock model is denoted by H, and the corresponding random variables are denoted by $T_1, \cdots, T_n$. "Increasing" is used to mean "nondecreasing" and "decreasing" is used to mean "nonincreasing." For any vector $\mathbf{z} = (z_1, \cdots, z_n)$, $\mathbf{z} \geqslant 0$ means $z_i \geqslant 0$ for all i and $\mathbf{u} \leqslant \mathbf{v}$ means $\mathbf{v} - \mathbf{u} \geqslant 0$. The terminology of reliability theory, which is sometimes used, is explained in detail by Barlow and Proschan (1975).

In the univariate case the survival function $\bar{H} = 1 - H$, generated by the shock model has the form

$$\bar{H}(t) = \Sigma_{k=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!} \bar{F}^{[k]}(x), \tag{1.1}$$

where the convolution $F^{[k]} = F_1 * \cdots * F_k$ is usually written $F^{(k)}$ when the damage distributions F_i are independent of i, and where x is the breaking threshold. The following theorem gives conditions under which H has an *increasing hazard rate average* (IHRA), i.e., $\bar{H}(z) = 1$ for all $z < 0$ and $-t^{-1} \log \bar{H}(t)$ is increasing in $t > 0$:

THEOREM 1.1. (Esary, Marshall and Proschan (1973)). *If the damages are nonnegative* ($z < 0$ *implies* $F_i(z) = 0$) *and stochastically increasing* ($F_i(z)$ *is decreasing in* i *for all* z), *then* H *is* IHRA.

The proof of Theorem 1.1 has two parts:

LEMMA (1.2). *If* $\bar{H}(t) = \Sigma_{k=0}^{\infty}[e^{-\lambda t}(\lambda t)^k/k!]\bar{P}_k$ *and* $\bar{P}_k^{1/k}$ *is decreasing in* $k = 1, 2, \cdots$, *then* H *is* IHRA.

LEMMA 1.3. *If* $F_i(z) = 0$ *for all* $z < 0$, $i = 1, 2, \cdots$ *and if* $F_i(z)$ *is decreasing in* $i = 1, 2, \cdots$ *for all* z, *then* $[F^{[k]}(x)]^{1/k}$ *is decreasing in* $k = 1, 2, \cdots$.

The multivariate versions of Theorem 1.1 given in Sections 3 and 4 also depend upon Lemma 1.2. However, they require generalizations of Lemma 1.3.

Multivariate analogs of Theorem 1.1 involve multivariate analogs of the IHRA property. Several such analogs have been discussed by Esary and Marshall (1979), which are described in terms of the *hazard function* $R = -\log \bar{H}$ or in terms of the random variables $T_1, \cdots, T_n$ as follows:

CONDITION A. $R(\alpha \mathbf{t})/\alpha$ is increasing in $\alpha > 0$ whenever $\mathbf{t} \geqslant 0$.

CONDITION B. For all coherent life functions τ, $\tau(T_1, \cdots, T_n)$ has an IHRA distribution.

Coherent life functions are discussed by Esary and Marshall (1970).

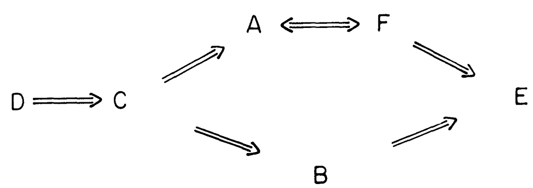
CONDITION C. For some independent IHRA random variables $X_1, \cdots, X_k$, and some coherent life functions $\tau_1, \cdots, \tau_n$ of order k, $T_i = \tau_i(X_1, \cdots, X_k)$, $i = 1, 2, \cdots, n$.

CONDITION D. For some independent IHRA random variables $X_1, \cdots, X_k$ and nonempty subsets B_i of $\{1, 2, \cdots, k\}$, $T_i = \min_{l \in B_i} X_l$, $i = 1, 2, \cdots, n$.

CONDITION E. For all nonempty subsets B of $\{1, 2, \cdots, n\}$, $\min_{i \in B} T_i$ is IHRA.

CONDITION F. $\min_i a_i T_i$ is IHRA whenever $\mathbf{a} \geqslant 0$.

Esary and Marshall (1979) show that the following and only the following implications hold for these conditions:



For the joint distribution H derived from the cumulative damage shock model, A-Hameed and Proschan (1973) show that Condition E holds whenever $F_i(\mathbf{z})$ is decreasing in i for all $\mathbf{z}$. This result is generalized in Section 3, where the stronger Condition B is obtained under appropriate stochastic ordering conditions. Section 4 contains a proof that if $F_i = F$ is independent of i, then H satisfies Condition A. Condition A is also obtained under the assumption that $F_i(\mathbf{z})$ is nonincreasing in i for all $\mathbf{z}$ and $n = 2$. In Section 5 it is shown that H does not necessarily satisfy Condition D. In Section 6 it is shown that H is a distribution of associated random variables and in Section 7 some "new better than used" properties of H are discussed. Section 8 contains some examples.

2. Some notation and basic properties of H. Let X_{ij} denote the damage inflicted by the ith shock on the jth device (component), $i = 1, 2, \cdots, j = 1, 2, \cdots, n$. Then $S_j^{(k)} = \Sigma_{i=1}^k X_{ij}$ is the damage accumulated by the jth component as a result of the first k shocks; let

$$F^{[k_1, \cdots, k_n]}(z_1, \cdots, z_n) = P\{S_j^{(k_j)} \leqslant z_j, j = 1, 2, \cdots, n\}.$$

The random vectors $\mathbf{X}_i = (X_{i1}, \cdots, X_{in})$, $i = 1, 2, \cdots$ are always independent. In case they are also identically distributed, the notation $F^{(k_1, \cdots, k_n)}$ is used in place of $F^{[k_1, \cdots, k_n]}$ If $\mathbf{k}$ and $\mathbf{l}$ are *similarly ordered* vectors (i.e., $(k_\alpha - k_\beta)(l_\alpha - l_\beta) \geqslant 0$, $\alpha, \beta = 1, 2, \cdots, n$), then

$$F^{(\mathbf{k}+\mathbf{l})} = F^{(\mathbf{k})} * F^{(\mathbf{l})},$$

where $*$ denotes convolution. However this relation does not hold in general.

Denote the breaking threshold of the jth device by x_j, $j = 1, 2, \cdots, n$, and assume $x_j \geqslant 0$ for all j. If $x_j = 0$, this means that the jth component fails upon the occurrence of any positive damage.

In the bivariate case, the cumulative damage shock model described in Section 1 leads to the survival function

$$\bar{H}(t_1, t_2) = \Sigma_{k=0}^\infty \Sigma_{l=0}^\infty e^{-\lambda t_1} \frac{(\lambda t_1)^k}{k!} e^{-\lambda(t_2 - t_1)} \frac{[\lambda(t_2 - t_1)]^l}{l!} F^{[k, k+l]}(x_1, x_2), \qquad 0 \leqslant t_1 \leqslant t_2 \tag{2.1}$$

$$= \Sigma_{k=0}^\infty \Sigma_{l=0}^\infty e^{-\lambda t_2} \frac{(\lambda t_2)^k}{k!} e^{-\lambda(t_1 - t_2)} \frac{[\lambda(t_1 - t_2)]^l}{l!} F^{[k+l, k]}(x_1, x_2), \qquad t_1 \geqslant t_2 \geqslant 0.$$

In the n-variate case, with $0 = t_0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n$, $t_n > 0$, and $\theta_j = (t_j - t_{j-1})/t_n$, $j = 1, \cdots, n$,

$$\bar{H}(t_1, \cdots, t_n) = \Sigma_{k=0}^\infty e^{-\lambda t_n} \frac{(\lambda t_n)^k}{k!} \bar{P}_k(\mathbf{x}) \tag{2.2}$$

where, with $l_0 = 0$ and $l_n = k$,

$$\bar{P}_k(\mathbf{x}) = \Sigma_{l_1=0}^k \Sigma_{l_2=l_1}^k \cdots \Sigma_{l_{n-1}=l_{n-2}}^k \binom{k}{l_1, l_2 - l_1, \cdots, l_n - l_{n-1}}$$
$$\cdot \Pi_{j=1}^n \theta_j^{l_j - l_{j-1}} F^{[l_1, \cdots, l_n]}(\mathbf{x}).$$

Let N_j be the number of shocks required to cause failure to the jth device, $j = 1, 2, \cdots, n$. With a corrected misprint, the moments

$$ET_1^r = \frac{r!}{\lambda^r} E\binom{N_1 + r - 1}{r}, \qquad r = 0, 1, \cdots,$$

of T_1 were obtained by A-Hameed and Proschan (1973) in terms of the moments of

N_1. A bivariate generalization of this is

$$ET_1^rT_2^s = \frac{1}{\lambda^{r+s}}E\frac{\Gamma(N_{(1)}+r)\Gamma(N_2+s)\Gamma(N_{(2)}+r+s)}{\Gamma(N_{(1)})\Gamma(N_2+r)\Gamma(N_{(2)}+s)}, \quad 0 \leqslant r \leqslant s, \tag{2.3}$$

where $N_{(1)} \leqslant N_{(2)}$ are obtained by ordering N_1 and N_2.

From (2.3) it follows that

$$\operatorname{Cov}(T_1, T_2) = \lambda^{-2}[\operatorname{Cov}(N_1, N_2) + EN_{(1)}].$$

In Section 6 it is shown that T_1 and T_2 are associated, a fact that implies that $\operatorname{Cov}(N_1, N_2) \geqslant -EN_{(1)}$. Moreover, since associated random variables are independent if and only if they are uncorrelated it follows that a necessary and sufficient condition for T_1 and T_2 to be independent is that $\operatorname{Cov}(N_1, N_2) = -EN_{(1)}$.

3. $\bar{H}$ and condition B. Theorem 1.1 has been generalized to a multivariate setting by A-Hameed and Proschan (1973): They show that if $T_1, \ldots, T_n$ have a joint survival function of the form (2.2) and if

$$F_i(\mathbf{x}) \quad \text{is decreasing in } i \quad \text{for all } \mathbf{x} \geqslant 0 \tag{3.1}$$

then the series system life length, $\min(T_1, \cdots, T_n)$ has an IHRA distribution. Here, this result is generalized to an arbitrary coherent system. To do this, the condition (3.1) must be replaced by an appropriate stochastic ordering condition that depends upon the coherent system.

If φ is a coherent structure function with minimal path sets $P_1, P_2, \cdots, P_p$, then the corresponding life function τ [see Esary and Marshall (1970)] has the representation

$$\tau(t_1, \cdots, t_n) = \max_{1\leqslant j\leqslant p} \min_{i\in P_j} t_i. \tag{3.2}$$

It is not difficult to show that the life function τ^D corresponding to the dual structure function φ^D has the similar representation

$$\tau^D(t_1, \cdots, t_n) = \min_{1\leqslant j\leqslant p} \max_{i\in P_j} t_i. \tag{3.3}$$

LEMMA 3.1. *If $T_1, \cdots, T_n$ are random variables with joint survival function* (2.2) *and τ is the life function of a coherent system of order n, then with* $0/0 = 1$,

$$P\{\tau(T_1, \cdots, T_n) > t\} = \sum_{k=0}^{\infty} e^{-\lambda t}\frac{(\lambda t)^k}{k!} P\left\{\tau^D\left(\frac{S_1^{(k)}}{x_1}, \cdots, \frac{S_n^{(k)}}{x_n}\right) \leqslant 1\right\}.$$

PROOF. Conditional on there having been exactly k shocks by time t,

$$T_i > t \Leftrightarrow \frac{S_i^{(k)}}{x_i} \leqslant 1.$$

Consequently,

$$\min_{i\in P_j} T_i > t \Leftrightarrow \max_{i\in P_j}\frac{S_i^{(k)}}{x_i} \leqslant 1,$$

and so

$$\max_{1\leqslant j\leqslant p} \min_{i\in P_j} T_i > t \Leftrightarrow \min_{1\leqslant j\leqslant p} \max_{i\in P_j}\frac{S_i^{(k)}}{x_i} \leqslant 1.$$

Because of (3.2) and (3.3) this can be rewritten in the form

$$\tau(T_1, \cdots, T_n) > t \Leftrightarrow \tau^D\left(\frac{S_1^{(k)}}{x_1}, \cdots, \frac{S_n^{(k)}}{x_n}\right) \leqslant 1.$$

The result follows by unconditioning on the number k of shocks by time t. □

The proof of Theorem 1.1 makes use of Lemma 1.2. A-Hameed and Proschan (1973) generalize the lemma to obtain their result. Here a still more general version is required.

Let A and $\mathcal{C}$ be subsets of R^n which satisfy

$$\text{(3.4)} \quad \mathbf{u} \in A, \mathbf{z} \in \mathcal{C} \Rightarrow \mathbf{u} - \mathbf{z} \in A \qquad (\text{i.e., } \mathbf{u} \in A, \mathbf{u} - \mathbf{v} \in \mathcal{C} \Rightarrow \mathbf{v} \in A),$$

$$\text{(3.5)} \quad \mathbf{u}, \mathbf{v} \in \mathcal{C} \Rightarrow \mathbf{u} + \mathbf{v} \in \mathcal{C}.$$

For purposes of this paper it is sufficient to take $\mathcal{C} = [0, \infty)^n$; then for example, (3.4) and (3.5) are satisfied when $\mathbf{x} \in \mathcal{C}$ and $A = \{\mathbf{z} : z_i \leqslant x_i, i = 1, 2, \cdots, n\}$.

Lemma 3.2. *If A and $\mathcal{C}$ satisfy* (3.4) *and* (3.5), *and if*

$$\text{(3.6)} \quad P\{\mathbf{X}_i \in \mathcal{C}\} = 1, \qquad i = 1, 2, \cdots,$$

$$\text{(3.7)} \quad P\{\mathbf{X}_i \in A - \mathbf{u}\} \text{ is decreasing in } i = 1, 2, \cdots \text{ for all } \mathbf{u} \in A \cap \mathcal{C},$$

then

$$[P\{\mathbf{S}^{(k)} \in A\}]^{1/k} \quad \text{is decreasing in} \quad k = 1, 2, \cdots.$$

Proof. From (3.4), $\mathbf{v} \notin A$, $\mathbf{u} \in A \Rightarrow \mathbf{u} - \mathbf{v} \notin \mathcal{C}$. Thus (3.6) implies

$$\text{(3.8)} \quad P\{\mathbf{X}_i \in A - \mathbf{v}\} = 0 \quad \text{for all} \quad \mathbf{v} \notin A.$$

From (3.5) and (3.6), $P\{\mathbf{S}^{(k)} \in \mathcal{C}\} = 1$ for all k so that similarly,

$$\text{(3.9)} \quad P\{\mathbf{S}^{(k)} \in A - \mathbf{v}\} = 0 \quad \text{for all} \quad \mathbf{v} \notin A.$$

To see that

$$\text{(3.10)} \quad A - \mathbf{v} \subset A \quad \text{for all} \quad \mathbf{v} \in A \cap \mathcal{C},$$

observe that $\mathbf{z} \in A - \mathbf{v} \Rightarrow \mathbf{z} = \mathbf{u} - \mathbf{v}$ for some $\mathbf{u} \in A$. Since also $\mathbf{v} \in \mathcal{C}$, $\mathbf{u} - \mathbf{v} = \mathbf{z} \in A$ by (3.4).

The proof now proceeds by induction:

$$\begin{aligned} P\{\mathbf{S}^{(2)} \in A\} &= \int_{\mathcal{C}} P\{\mathbf{X}_2 \in A - \mathbf{v}\}\, dF_1(\mathbf{v}) \overset{(3.8)}{=} \int_{A \cap \mathcal{C}} P\{\mathbf{X}_2 \in A - \mathbf{v}\}\, dF_1(\mathbf{v}) \\ &\overset{(3.7)}{\leqslant} \int_{A \cap \mathcal{C}} P\{\mathbf{X}_1 \in A - \mathbf{v}\}\, dF_1(\mathbf{v}) \overset{(3.10)}{\leqslant} \int_{A \cap \mathcal{C}} P\{\mathbf{X}_1 \in A\}\, dF_1(\mathbf{v}) \\ &= [P\{\mathbf{S}^{(1)} \in A\}]^2. \end{aligned}$$

Next suppose that

$$\text{(3.11)} \quad [P\{\mathbf{S}^{(k-1)} \in B\}]^{1/(k-1)} \geqslant [P\{\mathbf{S}^{(k)} \in B\}]^{1/k}$$

for all sets B satisfying (3.4). Note that $B = A - \mathbf{v}$ satisfies (3.4) when $\mathbf{v} \in \mathcal{C}$. Then

$$\begin{aligned}
\left[P\{\mathbf{S}^{(k)} \in A\}\right]^{k+1} &= P\{\mathbf{S}^{(k)} \in A\}\left[\int_{\mathcal{C}} P\{\mathbf{S}^{(k-1)} \in A - \mathbf{v}\}\, dF_k(\mathbf{v})\right]^k \\
&\overset{(3.9)}{=} P\{\mathbf{S}^{(k)} \in A\}\left[\int_{A \cap \mathcal{C}} P\{\mathbf{S}^{(k-1)} \in A - \mathbf{v}\}\, dF_k(\mathbf{v})\right]^k \\
&\overset{(3.11)}{\geqslant} P\{\mathbf{S}^{(k)} \in A\}\left[\int_{A \cap \mathcal{C}} (P\{\mathbf{S}^{(k)} \in A - \mathbf{v}\})^{(k-1)/k}\, dF_k(\mathbf{v})\right]^k \\
&= \left[\int_{A \cap \mathcal{C}} (P\{\mathbf{S}^{(k)} \in A\})^{1/k} (P\{\mathbf{S}^{(k)} \in A - \mathbf{v}\})^{(k-1)/k}\, dF_k(\mathbf{v})\right]^k \\
&\overset{(3.10)}{\geqslant} \left[\int_{A \cap \mathcal{C}} P\{\mathbf{S}^{(k)} \in A - \mathbf{v}\}\, dF_k(\mathbf{v})\right]^k \\
&\overset{(3.9)}{=} \left[\int_{\mathcal{C}} P\{\mathbf{S}^{(k)} \in A - \mathbf{v}\}\, dF_k(\mathbf{v})\right]^k \\
&= \left[\int_{\mathcal{C}} P\{\mathbf{X}_k \in A - \mathbf{v}\}\, dF^{[k]}(\mathbf{v})\right]^k \\
&\overset{(3.7)}{\geqslant} \left[\int_{\mathcal{C}} P\{\mathbf{X}_{k+1} \in A - \mathbf{v}\}\, dF^{[k]}(\mathbf{v})\right]^k \\
&= \left[P\{\mathbf{S}^{(k+1)} \in A\}\right]^k. \qquad \square
\end{aligned}$$

THEOREM 3.3. *Let $T_1, \cdots, T_n$ be random variables with joint survival function (2.2), let τ be the life function of a coherent system of order n and with $0/0 = 1$, let*

$$A_\tau = \left\{\mathbf{z} : \tau^D\left(\frac{z_1}{x_1}, \cdots, \frac{z_n}{x_n}\right) \leqslant 1\right\}.$$

If $P\{\mathbf{X}_i \in [0, \infty)^n\} = 1$ for all i and

(3.12) $\qquad P\{\mathbf{X}_i \in A_\tau - \mathbf{u}\}$ *is decreasing in i for all* $\mathbf{u} \in A_\tau \cap [0, \infty)^n$,

then $\tau(T_1, \cdots, T_n)$ has an IHRA *distribution.*

PROOF. Because τ^D is an increasing function, it is easy to see that with $A = A_\tau$ and $\mathcal{C} = [0, \infty)^n$, conditions (3.4) and (3.5) are satisfied. Consequently it follows from Lemma 3.2 that $[P\{\mathbf{S}^{(k)} \in A_\tau\}]^{1/k}$ is decreasing in $k = 1, 2, \cdots$, i.e.,

$$\left[P\left\{\tau^D\left(\frac{S_1^{(k)}}{x_1}, \cdots, \frac{S_n^{(k)}}{x_n}\right) \leqslant 1\right\}\right]^{1/k}$$

is decreasing in $k = 1, 2, \cdots$. This fact, together with Lemma 3.1 and Lemma 1.2 completes the proof. $\square$

In the univariate case, condition (3.12) reduces to

$$P\{X_i \leqslant x - u\} \quad \text{is decreasing in } i \quad \text{for all } x - u \geqslant 0,$$

which is ordinary stochastic ordering. It is not hard to see using an indicator function for f that (3.12) is satisfied when

(3.13) $\qquad Ef(\mathbf{X}_i) \leqslant Ef(\mathbf{X}_{i+1}), \qquad i = 1, 2, \cdots,$

for all increasing real functions f. This is often taken as a definition of stochastic ordering in the multivariate case [see, e.g., Lehmann (1955)].

COROLLARY 3.4. *If* (3.13) *is satisfied, then* $\tau(T_1, \cdots, T_n)$ *has an* IHRA *distribution for all coherent life functions* τ. *That is,* (3.13) *implies that* $T_1, \cdots, T_n$ *satisfy Condition* B.

Notice that if τ is a series system, then $A_\tau = \{\mathbf{z} : \max_i(z_i/x_i) \leqslant 1\}$ and (3.12) becomes the condition that $P\{\mathbf{X}_i \leqslant \mathbf{z}\}$ is decreasing in $i = 1, 2, \cdots$ for all $\mathbf{z} \in [0, \infty)^n$. This is the stochastic ordering condition that A-Hameed and Proschan (1973) also imposed for a series system.

4. $\bar{H}$ and condition A. The purpose of this section is to show that if $F_1 = F_2 = \cdots$, then the survival function $\bar{H}$ given by (2.2) satisfies the condition

$$\left[\bar{H}(\alpha\mathbf{t})\right]^{1/\alpha} \quad \text{is decreasing in} \quad \alpha > 0 \qquad \text{for all} \quad \mathbf{t} \geqslant 0;$$

that is, $\bar{H}$ satisfies Condition A of Section 1. For $n = 2$, the result holds under the more general condition that $F_i(\mathbf{x})$ is decreasing in $i = 1, 2, \cdots$ for all $\mathbf{x} \geqslant 0$. Whether or not this generalization holds for arbitrary n is unknown.

These multivariate versions of Theorem 1.1 are distinct from the one given in Section 3, but again the proofs make use of Lemma 1.2 and an induction argument.

THEOREM 4.1. *If* $F_1 = F_2 = \cdots = F$ *and* $\bar{H}$ *is given by* (2.2) *then*

$$\left[\bar{H}(\alpha\mathbf{t})\right]^{1/\alpha} \quad \textit{is decreasing in} \quad \alpha \qquad \textit{for all} \quad \mathbf{t} \geqslant 0.$$

The proof of this result is notationally complex, but all essential ideas are illustrated by the case $n = 3$.

PROOF FOR $n = 3$. It is sufficient to prove the result for $0 < t_1 \leqslant t_2 \leqslant t_3$: if $t_1 = 0$, the result reduces to one concerning a lower dimension, and the result for other orderings follows by symmetry considerations. With the notation

$$\theta_1 = t_1/t_3, \qquad \theta_2 = (t_2 - t_1)/t_3, \qquad \theta_3 = (t_3 - t_2)/t_3,$$

it follows from (2.2) that

$$\bar{H}(\alpha t_1, \alpha t_2, \alpha t_3) = \Sigma_{k=0}^{\infty} e^{-\lambda\alpha t_3} \frac{(\lambda\alpha t_3)^k}{k!} \bar{P}_k(\mathbf{x})$$

where

$$\bar{P}_k(\mathbf{x}) = \Sigma_{l_1=0}^{k} \Sigma_{l_2=l_1}^{k} \binom{k}{l_1, l_2 - l_1, k - l_2} \theta_1^{l_1} \theta_2^{l_2 - l_1} \theta_3^{k - l_2} \bar{F}^{(l_1, l_2, k)}(\mathbf{x}).$$

By virtue of Lemma 1.2, it is thus sufficient to prove that $[\bar{P}_k(\mathbf{x})]^{1/k}$ is decreasing in $k = 1, 2, \cdots$. For purposes of an inductive argument, it is useful to rewrite $\bar{P}_{k+1}(\mathbf{x})$ in terms of $\bar{P}_k(\mathbf{u})$. The identity

$$\binom{k+1}{l_1, l_2 - l_1, k + 1 - l_2} = \binom{k}{l_1 - 1, l_2 - l_1, k + 1 - l_2} + \binom{k}{l_1, l_2 - l_1 - 1, k + 1 - l_2} + \binom{k}{l_1, l_2 - l_1, k - l_2}$$

easily yields

$$\bar{P}_{k+1}(\mathbf{x}) = \Sigma_{l_1=0}^{k}\Sigma_{l_2=l_1}^{k}\binom{k}{l_1,\, l_2 - l_1,\, k - l_2}\theta_1^{l_1}\theta_2^{l_2-l_1}\theta_3^{k-l_2}\big[\theta_1 F^{(l_1+1,\, l_2+1,\, k+1)}(\mathbf{x}) + \theta_2 F^{(l_1,\, l_2+1,\, k+1)}(\mathbf{x}) + \theta_3 F^{(l_1,\, l_2,\, k+1)}(\mathbf{x})\big].$$

Consequently

$$\begin{aligned}(4.1)\qquad \bar{P}_{k+1}(\mathbf{x}) = {}& \theta_1 \textstyle\int_{u_1=0}^{x_1}\int_{u_2=0}^{x_2}\int_{u_3=0}^{x_3}\bar{P}_k(\mathbf{x}-\mathbf{u})\, dF(\mathbf{u}) \\ &+ \theta_2 \textstyle\int_{u_2=0}^{x_2}\int_{u_3=0}^{x_3}\bar{P}_k(x_1, x_2 - u_2, x_3 - u_3)\, dF(\infty, u_2, u_3) \\ &+ \theta_3 \textstyle\int_{u_3=0}^{x_3}\bar{P}_k(x_1, x_2, x_3 - u_3)\, dF(\infty, \infty, u_3).\end{aligned}$$

Since $\bar{P}_1(\mathbf{x}) = \theta_1 F(\mathbf{x}) + \theta_2 F(\infty, x_2, x_3) + \theta_3 F(\infty, \infty, x_3)$ is increasing in each x_i, it follows that

$$\begin{aligned}\bar{P}_2(\mathbf{x}) \leqslant {}& \theta_1 \textstyle\int_{u_1=0}^{x_1}\int_{u_2=0}^{x_2}\int_{u_3=0}^{x_3}\bar{P}_1(\mathbf{x})\, dF(\mathbf{u}) + \theta_2 \int_{u_2=0}^{x_2}\int_{u_3=0}^{x_3}\bar{P}_1(\mathbf{x})\, dF(\infty, u_2, u_3) \\ &+ \theta_3 \textstyle\int_{u_3=0}^{x_3}\bar{P}_1(\mathbf{x})\, dF(\infty, \infty, u_3) = \big[\bar{P}_1(\mathbf{x})\big]^2.\end{aligned}$$

Now assume that

$$(4.2)\qquad \big[\bar{P}_{k-1}(\mathbf{u})\big]^{1/(k-1)} \geqslant \big[\bar{P}_k(\mathbf{u})\big]^{1/k} \quad \text{for all} \quad \mathbf{u} \geqslant 0.$$

Then

$$\begin{aligned}\big[\bar{P}_k(\mathbf{x})\big]^{k+1} &\overset{(4.1)}{=} \bar{P}_k(\mathbf{x})\Big\{\theta_1 \textstyle\int_{u_1=0}^{x_1}\int_{u_2=0}^{x_2}\int_{u_3=0}^{x_3}\bar{P}_{k-1}(\mathbf{x}-\mathbf{u})\, dF(\mathbf{u}) \\ &\quad + \theta_2 \textstyle\int_{u_2=0}^{x_2}\int_{u_3=0}^{x_3}\bar{P}_{k-1}(x_1, x_2 - u_2, x_3 - u_3)\, dF(\infty, u_2, u_3) \\ &\quad + \theta_3 \textstyle\int_{u_3=0}^{x_3}\bar{P}_{k-1}(x_1, x_2, x_3 - u_3)\, dF(\infty, \infty, u_3)\Big\}^k \\ &\overset{(4.2)}{\geqslant} \Big\{\theta_1 \textstyle\int_{u_1=0}^{x_1}\int_{u_2=0}^{x_2}\int_{u_3=0}^{x_3}\big[\bar{P}_k(\mathbf{x})\big]^{1/k}\big[\bar{P}_k(\mathbf{x}-\mathbf{u})\big]^{(k-1)/k}\, dF(\mathbf{u}) \\ &\quad + \theta_2 \textstyle\int_{u_2=0}^{x_2}\int_{u_3=0}^{x_3}\big[\bar{P}_k(\mathbf{x})\big]^{1/k}\big[\bar{P}_k(x_1, x_2 - u_2, x_3 - u_3)\big]^{(k-1)/k}\cdot dF(\infty, u_2, u_3) \\ &\quad + \theta_3 \textstyle\int_{u_3=0}^{x_3}\big[\bar{P}_k(\mathbf{x})\big]^{1/k}\big[\bar{P}_k(x_1, x_2, x_3 - u_3)\big]^{(k-1)/k}\cdot dF(\infty, \infty, u_3)\Big\}^k \\ &\geqslant \Big\{\theta_1 \textstyle\int_{u_1=0}^{x_1}\int_{u_2=0}^{x_2}\int_{u_3=0}^{x_3}\bar{P}_k(\mathbf{x}-\mathbf{u})\, dF(\mathbf{u}) \\ &\quad + \theta_2 \textstyle\int_{u_2=0}^{x_2}\int_{u_3=0}^{x_3}\bar{P}_k(x_1, x_2 - u_2, x_3 - u_3)\, dF(\infty, u_2, u_3) \\ &\quad + \theta_3 \textstyle\int_{u_3=0}^{x_3}\bar{P}_k(x_1, x_2, x_3 - u_3)\, dF(\infty, \infty, u_3)\Big\}^k \\ &\overset{(4.1)}{=} \big[\bar{P}_{k+1}(\mathbf{x})\big]^k.\end{aligned}$$

□

The following theorem shows that for $n = 2$, the condition of Theorem 4.1 that $F_1 = F_2 = \cdots$ can be replaced by the weaker condition that $F_1 \geqslant F_2 \geqslant \cdots$.

As noted at the end of Section 3, the weaker condition is exactly condition (3.12) for the special case that τ is a series system, and it is implied by (3.13).

THEOREM 4.2. *If $F_i(\mathbf{x})$ is decreasing in $i = 1, 2, \cdots$ for all $\mathbf{x}$, and $n = 2$, then $\bar{H}$ given by* (2.1) *satisfies Condition A.*

OUTLINE OF PROOF. As in Theorem 4.1, it is sufficient to prove the result for $0 < t_1 \leqslant t_2$. Let

$$\theta_1 = t_1/t_2, \qquad \theta_2 = (t_2 - t_1)/t_2.$$

Then if $\bar{H}$ is given by (2.1) and $\alpha > 0$,

$$\bar{H}(\alpha t_1, \alpha t_2) = \Sigma_{k=0}^{\infty} e^{-\lambda\alpha t_2} \frac{(\lambda\alpha t_2)^k}{k!} \bar{P}_k(x_1, x_2),$$

where

(4.3)
$$\bar{P}_k(z_1, z_2) \equiv \bar{P}_k(z_1, z_2; F_1, F_2, \cdots) = \Sigma_{l=0}^{k} \binom{k}{l} \theta_1^l \theta_2^{k-l} P\{S_1^{(l)} \leqslant z_1, S_2^{(k)} \leqslant z_2\}, \qquad k = 1, 2, \cdots .$$

Consequently, by Lemma 1.2, it is sufficient to prove that

(4.4)
$$\left[\bar{P}_k(x_1, x_2)\right]^{1/k} \geqslant \left[\bar{P}_{k+1}(x_1, x_2)\right]^{1/(k+1)} \qquad \text{for all} \quad \mathbf{x} \geqslant 0, \qquad k = 1, 2, \cdots .$$

From the identity

$$\binom{k+1}{l} = \binom{k}{l} + \binom{k}{l-1}$$

and from (4.3) it follows that

(4.5)
$$\bar{P}_{k+1}(z_1, z_2; F_1, F_2, \cdots) = \theta_1 \int_0^{z_1} \int_0^{z_2} \bar{P}_k(z_1 - u_1, z_2 - u_2; F_2, F_3, \cdots)\, dF_1(u_1, u_2) + \theta_2 \int_{u_2=0}^{z_2} \bar{P}_k(z_1, z_2 - u_2; F_1, F_2, \cdots)\, dF_{k+1}(\infty, u_2), \qquad k = 1, 2, \cdots .$$

Below, use is directly made of the fact that $F_1(\mathbf{x}) \geqslant F_2(\mathbf{x})$ for all $\mathbf{x}$; use is also made of the fact that this implies

$$\int \varphi(u)\, dF_2(\infty, u) \leqslant \int \varphi(u)\, dF_1(\infty, u)$$

for all decreasing functions φ such that the integrals are defined. With this in mind, and with the aid of (4.3) and (4.5), (4.4) can be verified for $k = 1$. In a similar way, $\bar{P}_1(z_1, z_2; G_1, G_2, \cdots) \geqslant [\bar{P}_2(z_1, z_2; G_1, G_2, \cdots)]^{1/2}$ for any sequence $G_1, G_2, \cdots$ of distribution functions such that $G_i(\mathbf{x}) = 0$ for all $\mathbf{x} \not\geqslant 0$ and $G_i(\mathbf{x}) \geqslant G_{i+1}(\mathbf{x})$ for all $\mathbf{x} \geqslant 0$, $i = 1, 2, \cdots$. Assume that for all such sequences,

(4.6)
$$\left[\bar{P}_{k-1}(z_1, z_2; G_1, G_2, \cdots)\right]^{1/(k-1)} \geqslant \left[\bar{P}_k(z_1, z_2; G_1, G_2, \cdots)\right]^{1/k} \qquad \text{for all} \quad \mathbf{z} \geqslant 0.$$

Using this and arguments similar to those above, (4.4) can be verified.

5. $\bar{H}$ and conditions C, D. In Section 3 and 4 it is shown that at least when $F_1 = F_2 = \cdots = F$, the survival function $\bar{H}$ given by (2.2) satisfies Conditions A and B of Figure 1. This figure suggests the question of whether or not Condition C is satisfied. Although this question is open, it may not be an important one: Condition C is of primary interest as a model for a kind of dependence often encountered in practice. Beyond that, its usefulness up to now lies primarily in the fact that it implies Conditions A and B.

It is not difficult to show that Condition D need not be satisfied by $\bar{H}$ of (2.2). For example, if $n = 2$, $x_1 = x_2 = 1$ and F places mass $1/2$ at $(1 - \varepsilon, 1 + \varepsilon)$ and $(1 + \varepsilon, 1 - \varepsilon)$, where $0 < \varepsilon < 1$, then H is absolutely continuous but is not a distribution of independent random variables. As was observed by Esary and Marshall (1979), this means Condition D fails.

6. Association of $T_1, \cdots, T_n$. Random variables $Y_1, \cdots, Y_m$ are said to be *associated* if $\text{Cov}[\psi_1(\mathbf{Y}), \psi_2(\mathbf{Y})] \geq 0$ for all pairs of increasing binary functions ψ_1, ψ_2 defined on R^m. Association, a notion of positive dependence introduced by Esary, Proschan and Wallup (1967), leads to various inequalities such as

$$P\{Y_1 > t_1, \cdots, Y_m > t_m\} \geq \Pi_{i=1}^m P\{Y_i > t_i\}.$$

In the following, repeated use is made of the facts that

(6.1) A set consisting of a single random variable is associated;

(6.2) if two sets of associated random variables are independent, their union is a set of associated random variables; and

(6.3) increasing functions of associated random variables are associated.

Additional results concerning association have been obtained by Esary and Proschan (1970, 1972).

THEOREM 6.1. *If $T_1, \cdots, T_n$ have a joint survival function $\bar{H}$ of the form* (2.2) *and if $F_1 = F_2 = \cdots = F$, then $T_1, \cdots, T_n$ are associated.*

This result need not be true without the condition $F_1 = F_2 = \cdots$. The proof of Theorem 6.1 is based on some preliminary lemmas.

LEMMA 6.2. *Decreasing functions of associated random variables are associated.*

The proof of this lemma is similar to the proof [Esary, Proschan and Wallup (1967)] of (6.3).

LEMMA 6.3. *If $\{N(t), t \geq 0\}$ is a Poisson process and $0 \leq z_1 \leq \cdots \leq z_k$, then $N(z_1), \cdots, N(z_k)$ are associated.*

PROOF. This follows from (6.1), (6.2), (6.3) and the fact that $N(z_1), \cdots, N(z_k)$ are increasing functions of the independent increments of the process. □

LEMMA 6.4. *If $\{F_i\}$ is a decreasing sequence of distributions converging in distribution to F, if H is given by* (2.2) *with F being the identical distribution of the damages, and if H_i is given by* (2.2) *with F_i being the identical distribution of the damages, then H_i converges to H in distribution.*

PROOF. Because of symmetry, it is sufficient to prove that $\lim_{i\to\infty} H_i(\mathbf{t}) = H(\mathbf{t})$ when $0 \leqslant t_1 \leqslant t_2 \leqslant \cdots \leqslant t_n$. In this case, with $t_0 = 0$, $k_0 = 0$,

$$\bar{H}(\mathbf{t}; \mathbf{x}) \equiv \bar{H}(\mathbf{t}) = \Sigma_{k_1=0}^{\infty}\Sigma_{k_2=k_1}^{\infty} \cdots \Sigma_{k_n=k_{n-1}}^{\infty}\Pi_{j=1}^{n} e^{-\lambda(t_j - t_{j-1})}\frac{[\lambda(t_j - t_{j-1})]^{k_j - k_{j-1}}}{(k_j - k_{j-1})!} \bar{F}^{(k_1, \cdots, k_n)}(\mathbf{x})$$

and $\bar{H}_i$ has a similar form with F_i in place of F. Since $\{F_i\}$ converges to F in distribution, it follows [Feller (1971), page 257] that $F_i^{(k_1, \cdots, k_n)}$ converges to $F^{(k_1, \cdots, k_n)}$ in distribution for all $k_1, \cdots, k_n$. Since

$$\Sigma_{k_1=0}^{\infty}\Sigma_{k_2=k_1}^{\infty} \cdots \Sigma_{k_n=k_{n-1}}^{\infty}\Pi_{j=1}^{n} e^{-\lambda(t_j - t_{j-1})}\frac{[\lambda(t_j - t_{j-1})]^{k_j - k_{j-1}}}{(k_j - k_{j-1})!} = 1,$$

it follows from the dominated convergence theorem that if $\mathbf{x}$ is a continuity point of $F^{(k_1, \cdots, k_n)}$ for all $k_1, \cdots, k_n$, then $\bar{H}_i(\mathbf{t})$ converges to $\bar{H}(\mathbf{t})$ for all $\mathbf{t}$. If $\mathbf{x}$ is not a continuity point of $F^{(k_1, \cdots, k_n)}$ for some $k_1, \cdots, k_n$, let $\mathbf{x}^{(j)}$ be a decreasing sequence of continuity points converging to $\mathbf{x}$. By assumption the sequence $\bar{F}_i(\mathbf{x}^{(j)})$ is decreasing in i for fixed j. This implies that for every $(k_1, \cdots, k_n)$, $\bar{F}_i^{(k_1, \cdots, k_n)}(\mathbf{x}^{(j)})$ is decreasing in i. Hence $\bar{H}_i(\mathbf{t}; \mathbf{x}^{(j)})$ is decreasing in i [respectively, j] for fixed j [respectively, i]. Thus, for every j,

$$\lim_{i\to\infty} \bar{H}_i(\mathbf{t}; \mathbf{x}^{(j)}) \geqslant \lim_{i\to\infty} \bar{H}_i(\mathbf{t}; \mathbf{x}^{(i)}) \geqslant \lim_{i\to\infty} \lim_{j\to\infty} \bar{H}_i(\mathbf{t}; \mathbf{x}^{(j)}),$$

so $\lim_{j\to\infty} \lim_{i\to\infty} \bar{H}_i(\mathbf{t}; \mathbf{x}^{(j)}) \geqslant \lim_{i\to\infty} \lim_{j\to\infty} H_i\,(\mathbf{t}; \mathbf{x}^{(j)})$. Similarly,

$$\lim_{i\to\infty} \lim_{j\to\infty} \bar{H}_i(\mathbf{t}; \mathbf{x}^{(j)}) \geqslant \lim_{j\to\infty} \lim_{i\to\infty} \bar{H}_i(\mathbf{t}; \mathbf{x}^{(j)}),$$

hence

$$\lim_{j\to\infty} \lim_{i\to\infty} \bar{H}_i(\mathbf{t}; \mathbf{x}^{(j)}) = \lim_{i\to\infty} \lim_{j\to\infty} \bar{H}_i(\mathbf{t}; \mathbf{x}^{(j)}).$$

Applying the dominated convergence theorem twice gives

$$\lim_{i\to\infty} \bar{H}_i(\mathbf{t}; \mathbf{x}) = \lim_{i\to\infty} \lim_{j\to\infty} \bar{H}_i(t; x^{(j)}) = \lim_{j\to\infty} \lim_{i\to\infty} \bar{H}_i(\mathbf{t}; \mathbf{x}^{(j)})$$
$$= \lim_{j\to\infty} \bar{H}(\mathbf{t}, \mathbf{x}^{(j)}) = \bar{H}(\mathbf{t}; \mathbf{x}). \qquad \square$$

PROOF OF THEOREM 6.1. Suppose first that F places mass p_i at $\mathbf{a}_i = (a_{i1}, \cdots, a_{in})$, $i = 1, 2, \cdots, m$, $\Sigma_1^m p_i = 1$, where each $\mathbf{a}_i \geqslant 0$. In this case the Poisson process $\{N(t), t \geqslant 0\}$ governing the occurrence of shocks decomposes naturally into m independent Poisson processes $\{N_i(t), t \geqslant 0\}$, $i = 1, 2, \cdots, m$, where $N_i(t)$ is the number of damages of magnitude $\mathbf{a}_i$ suffered by time t. The damage to the jth component accumulated by time t can be written in the form

$$M_j(t) = \Sigma_{i=1}^m a_{ij} N_i(t).$$

If

$$X_j(t) = 0 \quad \text{if} \quad M_j(t) > x_j$$
$$= 1 \quad \text{if} \quad M_j(t) \leqslant x_j,$$

then $T_j = \sup\{t : X_j(t) = 1\}, j = 1, 2, \cdots, n$. Choose an integer k and real numbers $0 \leq z_1 \leq \cdots \leq z_k$. By Lemma 6.3 and (6.2), the random variables $N_j(z_i)$, $i = 1, 2, \cdots, k, j = 1, 2, \cdots, n$ are associated. Together with (6.3), this implies that $M_j(z_i)$, $i = 1, 2, \cdots, k$, $j = 1, 2, \cdots, n$ are associated. Since $X_j(t)$ is a decreasing function of $M_j(t)$, it follows with the aid of Lemma 6.2 that $X_j(z_i)$, $i = 1, 2, \cdots, k$, $j = 1, 2, \cdots, n$ are associated. But this fact implies that $T_1, \cdots, T_n$ are associated (Esary and Proschan (1970), page 333).

If F has more than a finite number of points of increase then F can be written as a limit in distribution of a decreasing sequence of distributions F_i, $i = 1, 2 \cdots$ where, for each i, F_i has a finite number of points of increase. Define H_i by (2.2) with F_i being the identical distributions of the damages and note that by the previous paragraph H_i is a distribution of associated random variables. From Lemma 6.4 H_i converges in distribution to H. The proof is complete by recalling that limits in distribution of associated random variables are associated (Esary, Proschan and Walkup (1967)). □

7. An NBU property of $\bar{H}$. If the (univariate) survival function $\bar{G}$ of a device satisfies

$$\bar{G}(t + u)/\bar{G}(t) \leq \bar{G}(u) \qquad \text{for all} \quad t, \quad u \geq 0 \tag{7.1}$$

then the device, when aged but unfailed, has a stochastically shorter remaining life than it did when new. Consequently the device, or its survival function, is said to be "new better than used" (NBU). This property was encountered in a replacement setting by Marshall and Proschan (1970), and has been discussed by various authors (see Barlow and Proschan (1975)).

There are a number of potentially interesting multivariate extensions of the NBU property, two of which are:

CONDITION (i). $\bar{G}(t_1 + \Delta, \cdots, t_n + \Delta) \leq \bar{G}(\mathbf{t})\bar{G}(\Delta, \cdots, \Delta)$ for all Δ, $\mathbf{t} \geq 0$, together with the same condition on all marginal survival functions.

CONDITION (ii). $\bar{G}(\mathbf{u} + \mathbf{t}) \leq \bar{G}(\mathbf{u})\bar{G}(\mathbf{t})$ whenever the vectors $\mathbf{t}$ and $\mathbf{u}$ are similarly ordered (i.e., $(u_i - u_j)(t_i - t_j) \geq 0 \quad i, j = 1, 2, \cdots, n$).

Condition (i) was introduced and discussed by Buchanan and Singpurwalla (1977). Clearly (ii) implies (i). It can be verified that the multivariate exponential distribution of Marshall and Olkin (1967) is characterized by Condition (ii) with equality in place of the inequality.

THEOREM 7.1. *If $F_1 = F_2 = \cdots = F$, then $\bar{H}$ of* (2.2) *satisfies Condition* (ii).

This result can be proved with the aid of

LEMMA 7.2. *If the vectors* $\mathbf{a}$ *and* $\mathbf{b}$ *with nonnegative integer components are similarly ordered and* $F(\mathbf{x}) = 0$ *whenever some* $x_i < 0$, *then*

$$F^{(\mathbf{a}+\mathbf{b})}(\mathbf{x}) \leq F^{(\mathbf{a})}(\mathbf{x})F^{(\mathbf{b})}(\mathbf{x}) \qquad \text{for all} \quad \mathbf{x}. \tag{7.2}$$

PROOF. As noted in Section 2, $F^{(\mathbf{a}+\mathbf{b})}(\mathbf{x}) = (F^{(\mathbf{a})} * F^{(\mathbf{b})})(\mathbf{x})$ because **a** and **b** are similarly ordered. Consequently,

$$F^{(\mathbf{a}+\mathbf{b})}(\mathbf{x}) = \int_{\mathbf{u}\leqslant\mathbf{x}} F^{(\mathbf{a})}(\mathbf{x}-\mathbf{u})\, dF^{(\mathbf{b})}(\mathbf{u}) \leqslant \int_{\mathbf{u}\leqslant\mathbf{x}} F^{(\mathbf{a})}(\mathbf{x})\, dF^{(\mathbf{b})}(\mathbf{u}) = F^{(\mathbf{a})}(\mathbf{x})F^{(\mathbf{b})}(\mathbf{x}). \ \square$$

PROOF OF THEOREM 7.1. For notational simplicity take $\lambda = 1$ in (2.2), and suppose without loss of generality that $t_1 \leqslant \cdots \leqslant t_n$. Let $t_0 = u_0 = 0$. Then

$$\bar{H}(\mathbf{t}+\mathbf{u})$$

$$= \Sigma_{k_1=0}^{\infty} \cdots \Sigma_{k_n=0}^{\infty} e^{-(t_n+u_n)} \Pi_{j=1}^{n} \frac{(t_j - t_{j-1} + u_j - u_{j-1})^{k_j}}{k_j!} F^{(k_1, k_1+k_2, \cdots, \Sigma_{i=1}^{n} k_i)}(\mathbf{x})$$

$$= \Sigma_{k_1=0}^{\infty} \cdots \Sigma_{k_n=0}^{\infty} e^{-(t_n+u_n)} \Pi_{j=1}^{n} \frac{1}{k_j!} \Sigma_{l_j=0}^{k_j} \binom{k_j}{l_j} (t_j - t_{j-1})^{l_j} (u_j - u_{j-1})^{k_j - l_j}$$

$$\cdot F^{(k_1, k_1+k_2, \cdots, \Sigma_{i=1}^{n} k_i)}(\mathbf{x})$$

$$= \Sigma_{l_1=0}^{\infty} \cdots \Sigma_{l_n=0}^{\infty} e^{-(t_n+u_n)} \Pi_{j=1}^{n} \frac{(t_j - t_{j-1})^{l_j}}{l_j!} \Sigma_{m_1=0}^{\infty} \cdots \Sigma_{m_n=0}^{\infty} \Pi_{\alpha=1}^{n} \frac{(u_\alpha - u_{\alpha-1})^{m_\alpha}}{m_\alpha!}$$

$$\cdot F^{(m_1+l_1, m_1+m_2+l_1+l_2, \cdots, \Sigma_{i=1}^{n}(m_i+l_i))}(\mathbf{x}).$$

An application of (7.2) with

$$\mathbf{a} = (m_1, m_1 + m_2, \cdots, \Sigma_{i=1}^{n} m_i), \qquad \mathbf{b} = (l_1, l_1 + l_2, \cdots, \Sigma_{i=1}^{n} l_i)$$

completes the proof. $\square$

8. Examples. Special cases in which $F_1 = F_2 = \cdots = F$ and $\bar{H}$ of (2.1) has exponential marginals or gamma marginals are exhibited below. In addition, a special case of a distribution of Freund is obtained from (2.1).

8.1. *Exponential marginals.* If $\bar{H}$ is given by (2.1), and F_1, F_2 denote the marginal distributions of F, then the marginals $\bar{H}_1(t) = \bar{H}(t, 0)$, $\bar{H}_2(t) = \bar{H}(0, t)$ are given by

$$\bar{H}_i(t) = \Sigma_{k=0}^{\infty} e^{-\lambda t} \frac{(\lambda t)^k}{k!} F_i^{(k)}(x_i), \qquad i = 1, 2.$$

According to Corollary 4.5 of Esary, Marshall and Proschan (1973) these marginals are exponential if and only if F_i has no mass in $(0, x_i]$, $i = 1, 2$. This means that F can have no mass in $\{(x,y) : 0 < x \leqslant x_1\} \cup \{(x,y) : 0 < y \leqslant x_2\}$. Thus, a shock produces no damage (with probability $F\{0, 0\}$), or "kills" one component without affecting the other, or "kills" both components. This means that if H has exponential marginals, it coincides with the bivariate exponential distribution of Marshall and Olkin (1967).

8.2. *Gamma marginals.* Suppose that the damage distribution F gives all its mass to the four corners of the unit square. Then, without loss of generality, the thresholds x_1 and x_2 can be assumed to be integers. In this case, for $i = 1, 2$,

$N_i - x_i - 1$ has a negative binomial (Pascal) distribution, where N_i is the number of shocks to failure of the ith component. Because $T_i = \Sigma_{k=1}^{N_i} Z_k$ where Z_k are i.i.d. exponential random variables (with mean λ^{-1}) that are independent of N_i, it follows that T_i has a gamma distribution. In the bivariate case the distribution obtained in this way is given by

$$(8.1)\quad \bar{H}(t_1, t_2) = e^{-P_{10}\lambda t_1 - P_{.1}\lambda t_2} \Sigma_{j=0}^{x_1} \Sigma_{n=0}^{x_2} \Sigma_{i=\max(j,n)}^{j+n} \frac{P_{01}^{i-j} P_{10}^{i-n} P_{11}^{j+n-i} (\lambda t_1)^i}{(i-j)!\,(i-n)!\,(j+n-i)!}$$
$$\times \Sigma_{m=0}^{x_2-n} \frac{[P_{.1}\lambda(t_2 - t_1)]^m}{m!}, \qquad t_2 \geqslant t_1 \geqslant 0.$$

The case $t_1 \geqslant t_2 \geqslant 0$ can be obtained from this expression by interchanging P_{01} and P_{10}, $P_{1.}$ and $P_{.1}$, t_1 and t_2, x_1 and x_2, and j and m.

If $x_1 = x_2 = 0$ (or more generally $0 \leqslant x_1 < 1$, $0 \leqslant x_2 < 1$), then as expected, $\bar{H}$ is the bivariate exponential of Marshall and Olkin (1967).

In general, the (integer) value $x_i + 1$ is the shape parameter of the gamma marginal distribution of T_i, $i = 1, 2$. Other simple explicit special cases can be obtained with $x_1 = 0$, $x_2 = 1$, with $x_1 = x_2 = 1$, and with $x_1 = 0$, $x_2 = 2$.

Random variables T_1, T_2 with the joint distribution of (8.1) have a representation of the form

$$(8.2)\qquad T_1 = \Sigma_{j=1}^{N_1} Z_j, \qquad T_2 = \Sigma_{j=1}^{N_2} Z_j,$$

where N_1 and N_2 are correlated but independent of the independent exponentially distributed random variables $Z_1, Z_2, \cdots$, and where $N_i - x_i - 1$ has a negative binomial distribution, $i = 1, 2$. A similar model has been studied by Gaver (1973). For his distribution,

$$T_1 = \Sigma_{j=1}^{M+x_1+1} Z_j, \qquad T_2 = \Sigma_{j=1}^{M+x_2+1} W_j$$

where $Z_1, Z_2, \cdots, W_1, W_2, \cdots$ are i.i.d. exponential random variables independent of the negative binomial random variable M. It is easy to verify that if T_1 and T_2 have Gaver's representation and, if T_1 and T_2 are exponentially distributed, then T_1, T_2 have a joint distribution which is a special case of the bivariate exponential distribution of Downton (1970). This is to be contrasted with the conclusion of Section 8.1 above.

Another model similar to (8.2) has been considered by Arnold (1975). Explicitly, Arnold considers the case that $X_1, X_2, \cdots$ are i.i.d. (not necessarily exponential) and

$$T_i = \Sigma_{j=1}^{M_i} X_j, \qquad i = 1, 2, \cdots, n.$$

Moreover, $M_1, \cdots, M_n$ have the same distribution (multivariate geometric) that the random variables $N_1, \cdots, N_n$ of Section 2 have when $F_1 = F_2 = \cdots = F$, F has mass only on the coordinate axes, and the thresholds are all zero. Arnold proves that the T_i are independent if $X_1, X_2, \cdots$ are exponentially distributed by using generating functions. This independence also follows from the remark concerning independence in Section 1.

8.3. *Example.* To obtain a special case of the density discussed by Freund (1961), let $x_1 = x_2 = 2$ and suppose $F_1 = F_2 = \cdots = F$ where F gives mass α/λ to (1, 3) and β/λ to (3, 1), $\lambda = \alpha + \beta$. If Z_1, Z_2 denote the first two waiting times in the underlying shock process then $T_1 = Z_1$ and $T_2 = Z_1 + Z_2$ with probability α/λ and $T_1 = Z_1 + Z_2$, $T_2 = Z_1$ otherwise. In this case, T_1, T_2 have joint density

$$\begin{aligned} h(t_1, t_2) &= \alpha(\alpha + \beta)e^{-(\alpha+\beta)t_2}, \qquad & 0 \leqslant t_1 \leqslant t_2 \\ &= \beta(\alpha + \beta)e^{-(\alpha+\beta)t_1}, \qquad & 0 \leqslant t_2 \leqslant t_1. \end{aligned}$$

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