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ON THE LOWER TAIL OF GAUSSIAN SEMINORMS

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Let E be an infinite-dimensional vector space carrying a Gaussian measure μ with mean 0 and a measurable norm q. Let $F(t) := \mu(q < t)$. By a result of Borell, F is logarithmically concave. But we show that F' may have infinitely many local maxima for norms $q = \sup_n|f_n|/a_n$ where f_n are independent standard normal variables. We also consider Hilbertian norms $q = (\Sigma b_n f_n^2)^{\frac{1}{2}}$ with $b_n > 0$, $\Sigma b_n < \infty$. Then as $t\downarrow 0$ we can have $F(t)\downarrow 0$ as rapidly as desired, or as slowly as any function which is $o(t^n)$ for all n. For $b_n = 1/n^2$ and in a few closely related cases, we find the exact asymptotic behavior of F at 0. For more general b_n we find inequalities bounding F between limits which are not too far apart.

1. Introduction. Let $\eta = (\eta_j)$ be a sequence of independent Gaussian, mean 0, variance 1, random variables in all of this paper. We shall then study the distribution of

$$q(\eta) \qquad \text{or} \quad q(\eta - a)$$

where $q : \mathbb{R}^\infty \to \overline{\mathbb{R}}_+ = [0, \infty]$ is a seminorm and a $\in \mathbb{R}^\infty$. In particular we shall study the behavior of $P(q(\eta) \leqslant t)$ as $t \to 0$.

In Section 3 we study *supremum norms*, that is, seminorms of the following form:

$$q(x) = \sup_n\{|x_n|/a_n\} \qquad \forall x = (x_n) \in \mathbb{R}^\infty \tag{1.1}$$

where (a_n) is a given sequence of positive numbers.

In Section 4 and Section 5 we study *Hilbertian norms*, that is, a seminorm of the following form:

$$q(x) = \{\Sigma_{n=1}^\infty \tau_n^2 x_n^2\}^{\frac{1}{2}} \qquad \forall x = (x_n) \in \mathbb{R}^\infty \tag{1.2}$$

where (τ_n) is a given sequence of positive numbers.

The setting above actually covers the following general case: Let E be a locally convex space and μ a Gaussian Radon probability on E, with mean 0; that is, μ is a Radon probability on E, whose finite dimensional marginals all are Gaussian and have mean 0. In particular if $x' \in E'$ (= the topological dual of E), then x' has a Gaussian distribution, when x' is considered as a random variable on $(E, \mathfrak{B}, \mu)$. Hence we have $E' \subseteq L^2(\mu)$, and so we may consider the L^2-closure of E', which we denote H'. Then H' is a Hilbert space and its dual H may be identified with a

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subspace of E in the following manner:

$$H = \{x \in E | \varphi \to \varphi(x) \quad \text{is} \quad L^2\text{-continuous on} \quad H'\}$$
$$= \{x \in E | \exists K > 0 : |\langle x', x\rangle|^2 \leqslant K \textstyle\int_E \langle x', y\rangle^2 \mu(dy) \forall x' \in E'\}$$

H is the reproducing kernel Hilbert space (RKHS) of μ, and we define the Hilbert norm, $\|\cdot\|$, in H by

$$\|x\| = \sup\{|\langle x', x\rangle| : x' \in E', \textstyle\int_E \langle x', y\rangle^2 \mu(dy) \leqslant 1\}.$$

From [5] we have that $L^2(\mu)$ and H are separable, and we can find biorthonormal bases $\{f_j\} \subseteq E'$ and $\{e_j\} \subseteq H$ for H' and H, satisfying the following Karhunen-Loéve expansion:

(1.3) $f_1, f_2, \cdots$ are independent Gaussian, mean 0, variance 1, random variables on the probability space $(E, \mathfrak{B}, \mu)$.

(1.4) $$\langle f_j, e_i\rangle = \delta_{ij}$$

(1.5) $$x = \Sigma_{j=1}^{\infty} \langle f_j, x\rangle e_j \qquad \text{for} \quad \mu - \text{a.a.} \quad x \in E^{\infty}$$

(1.6) $$H = \{x \in E | \Sigma_{j=1}^{\infty} \langle f_j, x\rangle^2 < \infty\}$$

(1.7) $$\|x\| = \{\Sigma_{j=1}^{\infty} \langle f_j, x\rangle^2\}^{\frac{1}{2}}, \qquad x \in H$$

So the study of a seminorm $r : E \to \overline{\mathbb{R}}_+$ reduces to the study of the seminorm

$$q(t) = r(\Sigma_1^{\infty} t_j e_j)$$

on $\mathbb{R}^{\infty}$ (we put $q = \infty$ if the sum diverges).

Our original case is a special example taking $E = \mathbb{R}^{\infty}$ and μ equal to the infinite product of $N(0, 1)$. In this case we may take f_j to be the projection on the jth coordinate and e_i to be the ith unit vector, and we have

$$H = l^2 = \{x \in \mathbb{R}^{\infty} | \Sigma_{n=1}^{\infty} |x_n|^2 < \infty\},$$
$$\|x\| = \{\Sigma_{n=1}^{\infty} |x_n|^2\}^{\frac{1}{2}},$$

and the series in (1.5) converges for all $x \in \mathbb{R}^{\infty}$.

If q is a Borel measurable seminorm on $\mathbb{R}^{\infty}$ we define (μ is a given measure, and H and $\|\cdot\|$ are defined as above)

$$\|q\| = \sup\{q(x) | x \in H \|x\| \leqslant 1\}.$$

Then we have (Kallianpur [8], Borell [3], Marcus and Shepp [9])

THEOREM 1.1. *The two probabilities* $P(q(\eta) < \infty)$ *and* $P(q(\eta) = 0)$ *are* 0 *or* 1, *and* $q(\eta) < \infty$ a.s. *implies* $\|q\| < \infty$.

Moreover if $q(\eta) < \infty$ a.s. *then*

$$\lim_{t\to\infty} t^{-2} \log P(q(\eta) > t) = -\tfrac{1}{2}\|q\|^{-2}.$$

If $\|q\| = 0$, *then* $q(\eta) = $ *constant* a.s.

This theorem settles the behavior of the upper tail of the distribution function of q. Notice that q may be constant a.s. without being 0 a.s., e.g.

$$q(x) = \lim \sup_{n\to\infty} |x_n|(2 \log n)^{-\frac{1}{2}}.$$

Then $q(\eta) = 1$ a.s., q is a seminorm and $\|q\| = 0$. If

$$q(x) = \lim \sup_{n\to\infty} |x_n|$$

then q is a seminorm with $\|q\| = 0$ and $q(\eta) = \infty$ a.s.

We shall mainly be concerned with seminorms of the form (1.1) or (1.2). These seminorms satisfy the following:

$$(1.8) \qquad q(x) = \sup_n q(x_1, \cdots, x_n, 0, 0, \cdots) \qquad \forall x,$$

in contrast to the examples above. Note that (1.8) implies

(1.9) q is lower semicontinuous on $\mathbb{R}^\infty$, and so in particular Borel measurable.

$$(1.10) \quad q(x_1, \cdots, x_n, 0, 0, \cdots) \leqslant q(x_1, \cdots, x_m, 0, 0, \cdots) \qquad \forall m \geqslant n.$$

In [2] Borell introduces the class of 0-*convex measures*. A Radon probability μ on the locally convex space E is 0-convex if μ satisfies

$$(1.11) \qquad \mu_*(\lambda A + (1-\lambda)B) \geqslant \mu(A)^\lambda \mu(B)^{1-\lambda}$$

for all $0 \leqslant \lambda \leqslant 1$ and all Borel sets A and B. Here μ_* denotes the inner measure generated by μ. Borell proves in [2] that μ is 0-convex if and only if all finite dimensional marginals are 0-convex and in [3] he proves that a probability μ on $\mathbb{R}^n$ is 0-convex, if and only if μ is concentrated on some affine subspace L of $\mathbb{R}^n$ with

$$\mu \ll \lambda_L \qquad \text{and} \quad \log\left(\frac{d\mu}{d\lambda_L}\right) \qquad \text{concave}$$

where λ_L is Lebesgue measure on L. In particular,

(1.12) any Gaussian measure is 0-convex.

Using this one easily establishes a conjecture of Marcus and Shepp (see [9], page 435) on the number of jumps of the distribution of q (see also Cirel'son [6]). Suppose that $q : \mathbb{R}^\infty \to \overline{\mathbb{R}}_+$ is a Borel measurable seminorm with $q(\eta) < \infty$ a.s. and put

$$F(t) = P(q(\eta) \leqslant t) \qquad t \geqslant 0,$$
$$C(q) = \inf\{t \geqslant 0 | F(t) > 0\}.$$

Then from (1.11) (with $A = \{q(x) \leqslant t\}$, $B = \{q(x) \leqslant s\}$) we find that

$$(1.13) \qquad \log F(t) \qquad \text{is concave:} \quad \mathbb{R}_+ \to \overline{\mathbb{R}}_- = [-\infty, 0].$$

Now a concave function is absolutely continuous on the interior of the set where it is finite. So we have

THEOREM 1.2. *If* $q(\eta) < \infty$ a.s., *then* F *admits right and left derivatives everywhere except possibly at* $t = C(q)$.

Moreover $F'(t)$ exists except possibly at countably many points t where F' has a jump downwards. Moreover $F''(t)$ exists Lebesgue-a.e. *and*

$$F(t) = p + \int_0^t F'(s)\,ds \qquad \text{for} \quad t \geqslant C(q)$$
$$= 0 \qquad \text{for} \quad t < C(q)$$

where $p = P(q(\eta) = C(q))$.

Hence, apart from a possible jump at $C(q)$, F is absolutely continuous. If $C(q) > 0$, then we shall see in Example 3.2 that $p = P(q(\eta) = C(q))$ may take any value in $[0, 1]$.

If $C(q) = 0$, then by Theorem 1.1 the jump p is either 0 or 1, the latter case occurs if and only if $q = 0$ a.s.

Finally let $B_q(a, r)$ denote the closed q-ball with center at a and radius r:

$$B_q(a, r) = \{x \in \mathbb{R}^\infty | q(x - a) \leqslant r\}.$$

2. The measure of a translated ball. Let q be a Borel measurable seminorm: $\mathbb{R}^\infty \to \overline{\mathbb{R}}_+$, then we put $\pi_n x = (x_1, \cdots, x_n, 0, 0, \cdots)$ and

$$q_n(x) = q(\pi_n x) \qquad \forall x \in \mathbb{R}^\infty,$$
$$q_n^*(x) = \sup\{|\textstyle\sum_{j=1}^n x_j y_j| : q_n(y) \leqslant 1\} \qquad \forall x \in \mathbb{R}^\infty.$$

Note that $q_n^*(x)$ is everywhere finite if and only if $q_n(x) = 0$ implies $\pi_n x = 0$. Obviously we have

$$|\textstyle\sum_{j=1}^n x_j y_j| \leqslant q_n(x) q_n^*(y) \qquad \text{if} \quad q_n(x) < \infty \tag{2.1}$$

(with the usual convention: $0 \cdot \infty = 0$). Note that if q satisfies (1.8) then by (1.10) we have

$$q_n(x) \uparrow q(x) \qquad \forall x. \tag{2.2}$$

THEOREM 2.1. *Let q be a Borel measurable seminorm on $\mathbb{R}^\infty$ with $q(\eta) < \infty$* a.s. *Then we have*

$$P(q(\eta - a) \leqslant t) \leqslant P(q(\eta) \leqslant t) \qquad \forall t \geqslant 0, \forall a \in \mathbb{R}^\infty. \tag{2.1.1}$$

Moreover if q satisfies (1.8) *and $a \in l^2$, then*

$$\exp\left(-\tfrac{1}{2}\|a\|^2\right)F(t) \leqslant F(t, a) \leqslant \exp\left(-\tfrac{1}{2}\|\pi_n a\|^2 + t q_n^*(a)\right)F(t) \tag{2.1.2}$$

for all $n \geqslant 1$ and all $t \geqslant 0$. Here $\|\cdot\|$ is the usual norm on l^2 and

$$F(t) = P(q(\eta) \leqslant t), \qquad F(t, a) = P(q(\eta - a) \leqslant t).$$

REMARK. (2.1.1) and (2.1.2) show that $F(t)$ and $F(t, a)$ are of the same order of magnitude as $t \to 0$ for $a \in l^2$. If $q_n(x) = 0$ implies $\pi_n x = 0$ then we have

$$F(t, a) \sim \exp\left(-\tfrac{1}{2}\|a\|^2\right)F(t) \qquad \text{as} \quad t \to 0$$

for $a \in l^2$.

PROOF. (2.1.1): Let $a \in \mathbb{R}^\infty$ and $t \geqslant 0$ be given, then we put $K = B_q(0, t)$ and $\alpha = F(t, a)$. Then K is convex closed and symmetric and

$$F(t, a) = \mu(K + a) = \alpha$$

where μ is the probability law of η on $\mathbb{R}^\infty$. Let

$$M = \{x \in \mathbb{R}^\infty | \mu(K + x) \geqslant \alpha\}.$$

Then M is symmetric, by symmetry of K and μ, and M is convex by (1.11). Moreover $a \in M$ hence $0 = \frac{1}{2}a + \frac{1}{2}(-a) \in M$. That is

$$F(t) = \mu(K) \geqslant \alpha = \mu(K + a) = F(t, a)$$

and (2.1.1) is proved.

(2.1.2): Let μ be the probability law of η, that is μ is the infinite product of $N(0, 1)$ and let μ_a be the probability law of $\eta - a$, that is μ shifted by a. If $a \in l^2$, then μ_a is absolutely continuous with respect to μ and

$$\mu_a(dx) = e^{-\frac{1}{2}\|a\|^2 - \langle x, a\rangle}\mu(dx)$$

where $\langle x, a\rangle = \Sigma_j x_j a_j$, whenever the sum converges, which it does μ-a.s. if $a \in l^2$. Hence

$$F(t, a) = \mu_a(q \leqslant t) = \int_{\{q \leqslant t\}} e^{-\frac{1}{2}\|a\|^2 - \langle x, a\rangle}\mu(dx)$$

and since μ and q are symmetric, the Cauchy-Schwarz inequality gives:

$$\begin{aligned} F(t) = \mu(q \leqslant t) &\leqslant \left\{\int_{\{q \leqslant t\}} e^{-\langle x, a\rangle}\mu(dx)\right\}^{\frac{1}{2}}\left\{\int_{\{q \leqslant t\}} e^{\langle x, a\rangle}\mu(dx)\right\}^{\frac{1}{2}} \\ &= \int_{\{q \leqslant t\}} e^{-\langle x, a\rangle}\mu(dx). \end{aligned}$$

So we have $F(t, a) \geqslant e^{-\frac{1}{2}\|a\|^2}F(t)$.

Let $a \in l^2$ and let $n \geqslant 1$ be given and fixed. Then we put $b = \pi_n a$ and $c = a - b$, and we find

$$\begin{aligned} F(t, a) &= \int_{\{q \leqslant t\}} e^{-\frac{1}{2}\|a\|^2 - \langle x, a\rangle}\mu(dx) \\ &= e^{-\frac{1}{2}\|b\|^2}\int_{\{q \leqslant t\}} e^{-\langle x, b\rangle - \langle x, c\rangle - \frac{1}{2}\|c\|^2}\mu(dx) \end{aligned}$$

and since $q_n(x) \leqslant q(x)$ we have (cf. (2.1))

$$|\langle x, b\rangle| = |\Sigma_1^n x_j a_j| \leqslant q_n(x) q_n^*(a) \leqslant t q_n^*(a)$$

for $x \in \{q \leqslant t\}$. Hence we find

$$\begin{aligned} F(t, a) &\leqslant \exp\left(-\tfrac{1}{2}\|b\|^2 + t q_n^*(a)\right)\int_{\{q \leqslant t\}} e^{-\frac{1}{2}\|c\|^2 - \langle x, c\rangle}\mu(dx) \\ &= \exp\left(-\tfrac{1}{2}\|b\|^2 + t q_n^*(a)\right)F(t, c) \\ &\leqslant \exp\left(-\tfrac{1}{2}\|b\|^2 + t q_n^*(a)\right)F(t) \end{aligned}$$

since $F(t, c) \leqslant F(t)$ by (2.1.1).

COROLLARY 2.2. *Let q be a Borel measurable seminorm with* $q(\eta) < \infty$ a.s. *If* $F = \{x \in \mathbb{R}^\infty | q(x) < \infty\}$ *is q-separable, that is, if*

$$\forall \varepsilon > 0, \exists (x_j) \subseteq F \quad \text{so that} \quad F \subseteq \bigcup_{j=1}^{\infty} B_q(x_j, \varepsilon),$$

then $C(q) = 0$, *that is*, $F(t) > 0 \qquad \forall t > 0$.

PROOF. If $C(q) > 0$, then by (2.1.1) we have

$$P(\eta \in B_q(x, \varepsilon)) = 0 \qquad \forall x \in \mathbb{R}^\infty \qquad \forall \varepsilon < C(q).$$

But then separability of F implies $P(\eta \in F) = 0$, which contradicts $q(\eta) < \infty$ a.s.

EXAMPLE 2.3. Let q be defined by

$$q(x) = \{\Sigma_{j=1}^{\infty} \sigma_j^2 x_j^2\}^{\frac{1}{2}}$$

where $\Sigma_j \sigma_j^2 < \infty$. Then $q(\eta) < \infty$ a.s. and

$$q_n^*(x) = \{\Sigma_{j=1}^{n} \sigma_j^{-2} x_j^2\}^{\frac{1}{2}}.$$

Taking $a_n = \sigma_n^{-1} e_n$ (e_n is the nth unit vector) gives

$$\|a_n\| = \|\pi_n a_n\| = \sigma_n^{-1};\ q_n^*(a_n) = \sigma_n^{-2}.$$

So we have

$$\exp\left(-\tfrac{1}{2}\sigma_n^{-2}\right)F(t) \leqslant F(t, a_n) \leqslant \exp\left(-\left(\tfrac{1}{2} - t\right)\sigma_n^{-2}\right)F(t). \tag{2.3}$$

In particular we have

$$\Sigma_{n=1}^{\infty}\{-\log F(t, a_n)\}^{-1} < \infty \qquad \forall 0 \leqslant t < \tfrac{1}{2}. \tag{2.4}$$

Note that if $0 \leqslant t < (\frac{1}{2})^{\frac{1}{2}}$, then the balls $B_q(a_n, t)$, $n = 1, 2, \cdots$ are mutually disjoint and so

$$\Sigma_n F(t, a_n) < \infty \qquad \forall 0 \leqslant t < \left(\tfrac{1}{2}\right)^{\frac{1}{2}}, \tag{2.5}$$

which is much weaker than (2.4).

3. Sup-norms. We shall now consider seminorms of the form (1.1), so let $a_n > 0$ for all $n \geqslant 1$, and let

$$q(x) = \sup_n |x_n| / a_n. \tag{3.1}$$

Let Φ be the standard normal distribution function on $\mathbb{R}$, and put

$$R(t) = 2(1 - \Phi(t)) = \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \int_t^{\infty} e^{-\frac{1}{2}x^2} dx.$$

Then we have the following elementary inequality:

$$\left(\frac{2}{\pi}\right)^{\frac{1}{2}}(1 + t)^{-1} e^{-\frac{1}{2}t^2} \leqslant R(t) \leqslant \frac{4}{3}\left(\frac{2}{\pi}\right)^{\frac{1}{2}}(1 + t)^{-1} e^{-\frac{1}{2}t^2} \qquad \forall t \geqslant 0 \tag{3.2}$$

and if $F(t) = P(q(\eta) \leqslant t)$, then

$$F(t) = \Pi_{j=1}^{\infty}(1 - R(ta_j)). \tag{3.3}$$

Hence $F(t) > 0$ if and only if $\Sigma_{j=1}^{\infty} R(ta_j) < \infty$, that is if and only if

$$\Sigma_{n=1}^{\infty} \frac{\exp\left(-\frac{1}{2}t^2a_n^2\right)}{1 + ta_n} < \infty.$$

Now we note that this sum converges for all $t > t_0$ if and only if

$$\Sigma_{n=1}^{\infty} \exp\left(-\tfrac{1}{2}t^2a_n^2\right) < \infty \qquad \forall t > t_0.$$

So if we define

$$C_0(a) = \inf\left\{t > 0 | \Sigma_{n=1}^{\infty} \exp\left(-\tfrac{1}{2}t^2a_n^2\right) < \infty\right\}, \tag{3.4}$$

then $C_0(a) = C(q)$, in other words:

$$C_0(a) = \inf\{t > 0 | F(t) > 0\}; \tag{3.5}$$

and from (3.2) and (3.3) we deduce:

$$F(t) \leqslant \exp\left\{-\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \Sigma_{n=1}^{\infty} \frac{\exp\left(-\frac{1}{2}t^2a_n^2\right)}{1 + ta_n}\right\} \tag{3.6}$$

Now suppose that $a_n \geqslant a > 0 \;\; \forall n \geqslant 1$, then by use of the inequality:

$$1 - x \geqslant \exp\left(\frac{x}{y}\log(1 - y)\right) \qquad \forall 0 \leqslant x \leqslant y \leqslant 1$$

we find (put $x = R(ta_n)$ and $y = R(ta)$):

$$F(t) \geqslant \exp\left\{-\psi(t) \Sigma_{n=1}^{\infty} \frac{4\exp\left(-\frac{1}{2}t^2a_n^2\right)}{3(1 + ta_n)}\right\} \tag{3.7}$$

where ψ is given by

$$\psi(t) = -\left(\frac{2}{\pi}\right)^{\frac{1}{2}} R(ta)^{-1} \log(1 - R(ta)). \tag{3.8}$$

It is easily checked that

$$\psi(t) \sim \left(\frac{2}{\pi}\right)^{\frac{1}{2}} \log\frac{1}{t} \qquad \text{as} \quad t \to 0. \tag{3.9}$$

So the estimates in (3.6) and (3.7) are fairly close together. Summarizing these observations we have proved:

THEOREM 3.1. *Let q be given by* (3.1), *and let C_0 be given by* (3.4). *Then we have*

$$q(\eta) < \infty \quad \text{a.s.} \quad \textit{if and only if} \quad C_0(a) < \infty. \tag{3.1.1}$$

$$C_0(a) = \inf\{t | F(t) > 0\}. \tag{3.1.2}$$

$$F(t) \leqslant \exp\left\{-\left(\frac{2}{\pi}\right)^{\frac{1}{2}} \Sigma_{n=1}^{\infty} \frac{\exp\left(-\frac{1}{2}t^2a_n^2\right)}{1 + a_nt}\right\}. \tag{3.1.3}$$

If $a_n \geqslant a > 0$ *and* ψ *is given by* (3.8), *then* $\psi(t) \sim -(\frac{2}{\pi})^{\frac{1}{2}} \log t$ *as* $t \to 0$, *and*

$$F(t) \geqslant \exp\left\{ -\psi(t)\Sigma_{n=1}^{\infty} \frac{4\exp\left(-\frac{1}{2}t^2 a_n\right)}{3(1+a_n t)} \right\} \tag{3.1.4}$$

where F *is the distribution function of* $q(\eta)$.

EXAMPLE 3.2. Let $\alpha \geqslant 0$ and $\beta > 0$; then we consider the sequence $a_1 = a_2 = \beta$ and

$$a_n = (2\beta^2(\log n + \alpha \log\log n))^{\frac{1}{2}} \qquad n \geqslant 3. \tag{3.10}$$

Then we have

$$\Sigma_{n=3}^{\infty} \exp\left(-\tfrac{1}{2}t^2 a_n^2\right) = \Sigma_{n=3}^{\infty} n^{-(\beta t)^2} (\log n)^{-\alpha(\beta t)^2}.$$

So we have $C_0(a) = 1/\beta$. Put $\gamma = 1/\beta$, then

$$(\log n)^{\frac{1}{2}} \leqslant 1 + \gamma a_n \leqslant \left(3 + 2\alpha^{\frac{1}{2}}\right)(\log n)^{\frac{1}{2}} \qquad \forall n \geqslant 3;$$

hence we find

$$\Sigma_{n=1}^{\infty} \frac{\exp\left(-\frac{1}{2}a_n^2\gamma^2\right)}{1+\gamma a_n} \leqslant e^{-\frac{1}{2}} + \Sigma_{n=3}^{\infty} n^{-1}(\log n)^{-\alpha-\frac{1}{2}}$$

$$\geqslant e^{-\frac{1}{2}} + \left(3 + 2\alpha^{\frac{1}{2}}\right)^{-1} \Sigma_{n=3}^{\infty} n^{-1} (\log n)^{-\alpha - \frac{1}{2}}.$$

If

$$s(\alpha) = \Sigma_{n=3}^{\infty} n^{-1}(\log n)^{-\alpha-\frac{1}{2}}$$

then (3.1.3) and (3.1.4) give

$$k\exp(-ms(\alpha)) \leqslant F(\gamma) \leqslant K\exp\left(-M\left(3 + 2\alpha^{\frac{1}{2}}\right)^{-1}s(\alpha)\right)$$

where k, m, K and M are positive finite constants not depending on α. So if $\alpha \leqslant \frac{1}{2}$ then $F(\gamma) = 0$, and if $\alpha > \frac{1}{2}$, then $F(\gamma) > 0$.

If $\alpha \downarrow \frac{1}{2}$, then $s(\alpha) \to \infty$ so $F(\gamma) \to 0$, and if $\alpha \to \infty$ then $s(\alpha) \to 0$ so $F(\gamma) \to 1$. Hence F can have a jump of any size $p \in [0, 1[$ at any point $\gamma \in]0, \infty[$.

However, from (3.3) it follows that $F(t) < 1$ for all $t > 0$. So F cannot have a jump of size 1, when q is a sup-norm. Also since q is a norm (i.e., $q(x) = 0$ implies $x = 0$), $C(q) > 0$, so F cannot have a jump at 0.

Now let

$$q_N(x) = \max_{1 \leqslant j \leqslant N}\{|x_j|/a_j\}$$

with a_n defined by (3.10). If $0 < b_2 < \gamma < b_1$ then

$$F(b_1) - F(b_2) \geqslant F(\gamma)$$

and since $q_N \to q$ and b_1 and b_2 are continuity points of F we can achieve the following lemma by taking α sufficiently large:

LEMMA 3.3. *Let* $0 < b_2 < b_1$ *and* $\varepsilon > 0$, *then there exist* $a_1, \cdots, a_N > 0$ *so that*

$$F_N(b_1) - F_N(b_2) > 1 - \varepsilon,$$

where F_N *is the distribution function of*

$$\max_{1 \leqslant j \leqslant N}\{|\eta_j|/a_j\}.$$

THEOREM 3.4. *Let* $\{b_j\}$ *be any strictly decreasing sequence of positive numbers. Then there exist sequences* $\{a_j\}$ *and* $\{m_j\}$ *so that*

$$b_{j+1} < m_j < b_j \qquad \forall j \geqslant 1, \tag{3.4.1}$$

$$F(b_j) - F(m_j) \geqslant 2(b_j - m_j) \qquad \forall j \geqslant 1, \tag{3.4.2}$$

$$F(m_j) \leqslant m_j - b_{j+1} \qquad \forall j \geqslant 1 \tag{3.4.3}$$

where F *is the distribution function of*

$$q(\eta) = \sup_n\{|\eta_n|/a_n\}.$$

In particular F *has a mode in each of the intervals*: $]b_{j+1}, b_{j-1}[$, $j = 2, 3, \cdots$, *in spite of the* log-*concavity of* F.

PROOF. Let $\mathfrak{F}$ denote the set of distribution functions of random variables of the form

$$Q = \max_{1 \leqslant j \leqslant N}\{|\eta_j|/a_j\}$$

with $N \geqslant 1$ and $a_1, \cdots, a_N$ positive. Then any infinite product of distributions from $\mathfrak{F}$ is the distribution function of $q(\eta)$ for some sup-norm q of the form (3.1).

The distribution F will be an infinite product

$$F(x) = \Pi_{j=1}^{\infty} F_j(x)$$

where $F_j \in \mathfrak{F}$. The F_j and m_j are defined inductively by:

(i) $$F_j(b_j) - F_j(m_j) \geqslant 4(b_j - m_j)\Pi_{i=1}^{j-1} F_i(b_j)^{-1} \qquad \forall j \geqslant 1$$

(ii) $$F_j(b_j) - F_j(m_j) \geqslant p_j \qquad \forall j \geqslant 1$$

(iii) $$F_j(b_j) - F_j(m_j) \geqslant 1 - (m_j - b_{j+1}) \qquad \forall j \geqslant 1$$

where (p_j) is any fixed sequence with $0 < p_j < 1$ and

$$\Pi_1^{\infty} p_j = \tfrac{1}{2}.$$

First m_1 is chosen so that $m_1 \in]b_2, b_1[$ and $4(b_1 - m_1) < 1$, then we choose $F_1 \in \mathfrak{F}$ by Lemma 3.3, such that

$$F_1(b_1) - F_1(m_1) > \max\{4(b_1 - m_1), p_1, 1 - (m_1 - b_2)\}.$$

Then (i)—(iii) are satisfied for $j = 1$.

If $F_1, \cdots, F_n$ and $m_1, \cdots, m_n$ are constructed, then we choose $m_{n+1} \in]b_{n+2}, b_{n+1}[$, so that

$$4(b_{n+1} - m_{n+1}) < \Pi_{i=1}^{n} F_i(b_{n+1})$$

(note that $F(t) > 0 \quad \forall t > 0 \quad \forall F \in \mathcal{F}$). Then by Lemma 3.3 we can choose $F_{n+1} \in \mathcal{F}$, so that (i)—(iii) holds for $j = n + 1$.

Now we note that (i)—(iii) imply

$$\text{(iv)} \qquad \Pi_{i=j+1}^{n} F_i(b_j) \geq \Pi_{i=j+1}^{n} F_i(b_i) \geq \Pi_{i=1}^{\infty} p_i = \tfrac{1}{2} \qquad \forall n \geq j+1,$$

$$\text{(v)} \qquad F_j(m_j) \leq 1 - \big(F_j(b_j) - F_j(m_j)\big) \leq m_j - b_{j+1}.$$

Now we put

$$F(x) = \Pi_{j=1}^{\infty} F_j(x), \qquad G_n(x) = \Pi_{j=1}^{n} F_j(x)$$

Then by (i) and (iv) we have for $j \leq n$:

$$\begin{aligned} G_n(b_j) - G_n(m_j) &\geq \big(G_j(b_j) - G_j(m_j)\big)\Pi_{i=j+1}^{n} F_i(b_j) \\ &\geq \tfrac{1}{2}\big(G_j(b_j) - G_j(m_j)\big) \\ &\geq \tfrac{1}{2} G_{j-1}(b_j)\big(F_j(b_j) - F_j(m_j)\big) \\ &\geq 2(b_j - m_j). \end{aligned}$$

So we see that F satisfies (3.4.2), and since $F \leq F_j$, it follows from (v) that F satisfies (3.4.3).

Since $F(b_j) > 0$ it follows from Theorem 1.2 that F is absolutely continuous on $]b, \infty[$, where $b = \lim_{n\to\infty} b_n$. Now (3.4.2) implies that $F'(x) \geq 2$ for some $x \in]m_j, b_j[$ and (3.4.3) implies that $F'(x) \leq 1$ for some $x \in]b_{j+1}, m_j[$. That is, F has at least one mode in each of the intervals $]b_{j+1}, b_{j-1}[$ for $j \geq 2$.

THEOREM 3.5. *Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be increasing, then there exist positive numbers $\{a_j\}$, so that*

$$0 < F(t) \leq f(t) \qquad \forall 0 < t \leq 1$$

where F is the distribution function of

$$Q = \sup_n \{|\eta_n|/a_n\}.$$

PROOF. Let $\{p_n\}$ be defined by

$$p_0 = f(\tfrac{1}{2}), \qquad p_n = f(2^{-n-1})/f(2^{-n}) \qquad \text{for} \quad n \geq 1$$

and let $\mathcal{F}$ be defined as in the proof of Theorem 3.4, then we can find $F_n \in \mathcal{F}$ so that

$$F_n(2^{-n+1}) - F_n(2^{-n}) \geq \max\{1 - 2^{-n}, 1 - p_n\}.$$

Let $F = \Pi_0^{\infty} F_n$, then F is the distribution of some $Q = q(\eta)$; where q is a sup-norm. Moreover if $2^{-n-1} \leq t \leq 2^{-n} (n \geq 0)$, then

$$\begin{aligned} F(t) &\leq F(2^{-n}) \leq \Pi_{j=0}^{n} F_j(2^{-n}) \leq \Pi_{j=0}^{n} F_j(2^{-j}) \\ &\leq \Pi_{j=0}^{n}\big(1 - \big(F_j(2^{-j+1}) - F_j(2^{-j})\big)\big) \leq \Pi_{j=0}^{n} p_j \\ &= f(2^{-n-1}) \leq f(t) \end{aligned}$$

and

$$F(t) \geqslant F(2^{-n-1}) \geqslant \Pi_{j=0}^{n+1} F_j(2^{-n-1}) \Pi_{j=n+2}^{\infty} F_j(2^{-j+1})$$
$$\geqslant \Pi_{j=0}^{n+1} F_j(2^{-n-1}) \Pi_{j=n+2}^{\infty}(1 - 2^{-j}) > 0$$

and the theorem is proved.

4. Hilbertian norms. We shall in this section study Hilbertian norms, that is, norms of the form (1.2). Before proceeding we shall assume that τ_n is given by $\tau_n = \tau(n)$, where $\tau : [1, \infty[\to \mathbb{R}_+$ satisfies

(4.1) $$\tau \quad \text{is decreasing,} \quad \tau(t) > 0 \quad \forall t \geqslant 1$$

(4.2) $$\tau(t) \leqslant t^{-\frac{1}{2}} \quad \forall t \geqslant 1$$

(4.3) $$\int_1^\infty \tau(t)^2 \, dt < \infty.$$

And we shall consider the norm

(4.4) $$q(x) = \left\{\Sigma_{n=1}^{\infty} \tau(n)^2 x_n^2\right\}^{\frac{1}{2}} \quad \forall x = (x_n) \in \mathbb{R}^\infty.$$

Note that $\Sigma \tau_n^2 < \infty$ by (4.1) and (4.3), so $Q = q(\eta)$ is finite a.s. Now let F_n and F^n be the two marginals:

$$F_n(t) = P\left(\Sigma_1^n \tau(j)^2 \eta_j^2 \leqslant t^2\right)$$
$$F^n(t) = P\left(\Sigma_{n+1}^{\infty} \tau(j)^2 \eta_j^2 \leqslant t^2\right).$$

If $B_n(t)$ denotes the euclidean ball of radius t centered at the origin, then we have

(4.5) $$F_n(t) = (2\pi)^{-n/2} \Pi_{j=1}^n \tau(j)^{-1} \int_{B_n(t)} \exp\left(-\tfrac{1}{2} \Sigma_{j=1}^n \tau(j)^{-2} x_j^2\right) dx$$

(4.6) $$F_n(s) F^n\left((t^2 - s^2)^{\frac{1}{2}}\right) \leqslant F(t) \leqslant F_n(t) \quad \forall 0 \leqslant s \leqslant t,$$

since $Q^2 = Q_n^2 + R_n^2$ where

$$Q_n^2 = \Sigma_{j=1}^n \tau(j)^2 \eta_j^2, \qquad R_n^2 = \Sigma_{n+1}^{\infty} \tau(j)^2 \eta_j^2$$

and Q_n and R_n are independent.

THEOREM 4.1. *Let q be the seminorm given by* (4.4), *where τ satisfies* (4.1)—(4.3). *Let*

(4.1.1) $$\varphi(t) = t^{-\frac{1}{2}} \tau(t)^{-1} \quad \text{for} \quad t \geqslant 1.$$

If F is the distribution function of $Q = q(\eta)$, then there exists $A_1 > 0$, so that

(4.1.2) $$F(t) \leqslant A_1 \exp\left\{\int_1^x \log \varphi(y) \, dy + \log \varphi(x) + (x - 1) \log t\right\}$$

for all $x \geqslant 1$ and all $t \in [0, 1]$.

REMARK. In applications of (4.1.2) one should try to minimize the right-hand side in x for t fixed. That is, take $x \geqslant 1$ to be a suitable solution to

$$\log \varphi(x) + \varphi'(x)/\varphi(x) + \log t = 0.$$

Ignoring the middle term one reasonable choice is $\varphi(x) = 1/t$ or $x = \varphi^{-1}(1/t)$.

PROOF. Let V_n be the volume of the n-dimensional unit ball. Then by Stirling's formula we have

$$V_n = \Gamma(\tfrac{1}{2})^n \Gamma(1 + \tfrac{1}{2}n)^{-1} = a_1(2\pi)^{\frac{1}{2}n} n^{-\frac{1}{2}(n+1)} e^{n/2} e^{-\theta/n} \tag{4.7}$$

where $a_1 = \pi^{-\frac{1}{2}} \doteq 0.56$ and $0 < \theta < \frac{1}{6}$. Hence by (4.5) and (4.6) we have

$$F(t) \leqslant a_1 \exp\{-\Sigma_1^n \log \tau(j) - \tfrac{1}{2}(n+1)\log n + \tfrac{1}{2}n + n \log t\} \tag{4.8}$$

for all $n \geqslant 1$ and all $t \geqslant 0$. Since $f = -\log \tau$ is increasing we have

$$f(1) + \textstyle\int_1^n f(y)\, dy \leqslant \Sigma_1^n f(j) \leqslant f(n) + \int_1^n f(y)\, dy. \tag{4.9}$$

So we have for the exponent in (4.8):

$$\begin{aligned} &-\Sigma_1^n \log \tau(j) - \tfrac{1}{2}(n+1)\log n + \tfrac{1}{2}n + n \log t \\ &\leqslant -\textstyle\int_1^n \log \tau(y)\, dy - \log \tau(n) - \tfrac{1}{2}\int_1^n \log y\, dy + \tfrac{1}{2} - \tfrac{1}{2}\log n + n \log t \\ &= \textstyle\int_1^n \log \varphi(y)\, dy - \log \tau(n) - \tfrac{1}{2}\log n + n \log t + \tfrac{1}{2}. \end{aligned}$$

Now we note that $\varphi(y) \geqslant 1$ for $y \geqslant 1$ by (4.3). So if $n \leqslant x \leqslant n + 1$ we have $x \leqslant 2n$ and we find

$$\begin{aligned} &\textstyle\int_1^n \log \varphi(y)\, dy \leqslant \int_1^x \log \varphi(y)\, dy, \\ &-\log \tau(n) - \tfrac{1}{2}\log n \leqslant -\log \tau(x) - \tfrac{1}{2}\log x + \tfrac{1}{2}\log 2 = \log \varphi(x) + \tfrac{1}{2}\log 2, \\ &n \log t \leqslant (x-1)\log t \qquad \text{for} \quad 0 < t \leqslant 1. \end{aligned}$$

Inserting this in (4.8) gives

$$F(t) \leqslant A_1 \exp\{\textstyle\int_1^x \log \varphi(y)\, dy + \log \varphi(x) + (x-1)\log t\}$$

where

$$A_1 = a_1(2e)^{\frac{1}{2}} = (2e/\pi)^{\frac{1}{2}} \doteq 1.32. \tag{4.10}$$

THEOREM 4.2. *Let q be the seminorm given by* (4.4), *where τ satisfies* (4.1)–(4.3). *Suppose in addition that τ satisfies*

(4.2.1) $\log \tau(x)$ *is convex*

(4.2.2) $\varphi(x) = x^{-\frac{1}{2}}\tau(x)^{-1}$ *increases to* $+\infty$ *on* $[1, \infty]$.

If F is the distribution of $Q = q(\eta)$, then for some constant A_2 we have

(4.2.3) $F(t) \leqslant A_2 x^{-\frac{1}{2}} \tau(x)^{-\frac{1}{2}} e^{-H(x-1)}$ *if* $x \geqslant 1$ *and* $\varphi(x) \leqslant t^{-1}$

where H is defined by

$$H(x) = \int_1^x \frac{t\varphi'(t)}{\varphi(t)}\, dt \qquad \textit{for} \quad x \geqslant 1. \tag{4.2.4}$$

REMARK. Again in applications of (4.2.3) we have to choose an appropiate x. One possible choice is $\varphi(x) = 1/t$ or $x = \varphi^{-1}(1/t)$.

PROOF. When $f = -\log \tau$ is increasing and concave one may improve (4.9) to (4.11)

$$\tfrac{1}{2}(f(n+1) - f(2)) + f(1) + \textstyle\int_1^n f(y)\,dy \leqslant \Sigma_1^n f(j) \leqslant \tfrac{1}{2}\big(f(1) + \tfrac{1}{2}f(n)\big) + \int_1^n f(y)\,dy$$

by estimating the integral over $[j-1, j]$ by the area of two trapezoids:

$$\tfrac{1}{2}(f(j) + f(j-1)) \leqslant \textstyle\int_{j-1}^{j} f(y)\,dy \leqslant \tfrac{1}{2}(f(j) - f'(j) + f(j))$$

and noting that since f' is decreasing we have

$$\Sigma_{j=2}^n f'(j) \geqslant \textstyle\int_2^{n+1} f'(x)\,dx = f(n+1) - f(2).$$

Using this we can estimate the exponent in (4.8) by

$$\begin{aligned} &-\Sigma_{j=1}^n \log \tau(j) - \tfrac{1}{2}(n+1)\log n + \tfrac{1}{2}n + n\log t \\ &\leqslant \textstyle\int_1^n \log \tau(y)\,dy - \tfrac{1}{2}\{\log \tau(n) + \log \tau(1) + \int_1^n \log y\,dy + \log n - 1\} + n\log t \\ &= \textstyle\int_1^n \log \varphi(y)\,dy - \tfrac{1}{2}\log(n\tau(n)) + n\log t + \tfrac{1}{2} - \tfrac{1}{2}\log \tau(1). \end{aligned}$$

By partial integration we find

$$\begin{aligned} \textstyle\int_1^n \log \varphi(y)\,dy &= n\log \varphi(n) - \log \varphi(1) - \int_1^n \frac{t\varphi'(t)}{\varphi(t)}\,dt \\ &= n\log \varphi(n) + \log \tau(1) - H(n) \end{aligned}$$

since $\varphi(1) = \tau(1)^{-1}$. So we have

$$F(t) \leqslant a_2 \exp\{-H(n) + n\log(t\varphi(n)) - \tfrac{1}{2}\log(n\tau(n))\} \tag{4.12}$$

where

$$a_2 = a_1 \exp\{\tfrac{1}{2} + \tfrac{1}{2}\log \tau(1)\} \leqslant (e/\pi)^{\frac{1}{2}}$$

since $\log \tau(1) \leqslant 0$. Now suppose that $n \leqslant x \leqslant n + 1 (n \geqslant 1)$. Then $H(n) \geqslant H(x-1)$ since H is increasing by (4.2.2) and if $\varphi(x) \leqslant 1/t$, then

$$\log(t\varphi(n)) \leqslant \log(t\varphi(x)) \leqslant 0$$

and finally since $n \geqslant \frac{1}{2}x$

$$\begin{aligned} \log n + \log \tau(n) &\geqslant \log x + \log \tau(x) + \log(n/x) \\ &\geqslant \log(x\tau(x)) - \log 2. \end{aligned}$$

Inserting this in (4.12) gives

$$F(t) \leqslant A_2 x^{-\frac{1}{2}} \tau(x)^{-\frac{1}{2}} e^{-H(x-1)}$$

where

$$A_2 = a_2 2^{\frac{1}{2}} \leqslant (2e/\pi)^{\frac{1}{2}} \doteq 1.32, \tag{4.13}$$

proving the theorem.

THEOREM 4.3. *Let q be given by* (4.4) *where τ satisfies* (4.1)–(4.3), *and let*

$$\psi(x) = \textstyle\int_x^\infty \tau(t)^2\,dt \qquad \text{for} \quad x \geqslant 1. \tag{4.3.1}$$

If F is the distribution of $Q = q(\eta)$, then for some $B_1 > 0$ we have
(4.3.2)

$$F(t) \geqslant B_1\left\{1 - \frac{\psi(x)}{t^2 - s^2}\right\}\exp\left\{\int_1^x \log \varphi(y)\,dy - \tfrac{1}{2}\log x + (x+1)\log s - \frac{s^2}{2\tau(x)^2}\right\}$$

for all $x \geqslant 1$ and all $0 \leqslant s \leqslant t \leqslant 1$. Here φ is defined as above in (4.1.1).

PROOF. From (4.5) and (4.7) we deduce the following lower bound of F_n:

$$(4.14)\quad F_n(t) \geqslant b_1 \exp\left\{-\Sigma_1^n \log \tau(j) - \tfrac{1}{2}(n+1)\log n + \tfrac{1}{2}n + n\log t - \frac{t^2}{2\tau(n)^2}\right\}$$

since we have

$$\tfrac{1}{2}\Sigma_{j=1}^n \tau(j)^{-2}x_j^2 \leqslant \frac{t^2}{2\tau(n)^2} \qquad \text{for all} \quad x \in B_n(t).$$

Here $b_1 = \pi^{-\frac{1}{2}}e^{-1/6}$.
Since $\log \tau(1) \leqslant 0$ we find from (4.9)

$$\begin{aligned}
&-\Sigma_1^n \log \tau(n) - \tfrac{1}{2}(n+1)\log n + \tfrac{1}{2}n + n\log t - \tfrac{1}{2}t^2\tau(n)^{-2}\\
&\quad\geqslant -\int_1^n \log \tau(y)\,dy - \tfrac{1}{2}\int_1^n \log y\,dy + \tfrac{1}{2} - \tfrac{1}{2}\log n + n\log t - \tfrac{1}{2}t^2\tau(n)^{-2}\\
&\quad= \int_1^n \log \varphi(y)\,dy - \tfrac{1}{2}\log n - \tfrac{1}{2}t^2\tau(n)^{-2} + \tfrac{1}{2} + n\log t\\
&\quad\geqslant \int_1^x \log \varphi(y)\,dy - \tfrac{1}{2}\log x - \tfrac{1}{2}t^2\tau(x)^{-2} + \tfrac{1}{2} - \tfrac{1}{2}\log 2 + (x+1)\log t
\end{aligned}$$

for $n - 1 < x \leqslant n$, $n \geqslant 2$ and $0 \leqslant t \leqslant 1$. So we have

$$(4.15)\quad F_n(t) \geqslant b_2 \exp\left\{\int_1^x \log \varphi(y)\,dy - \tfrac{1}{2}\log x - \tfrac{1}{2}t^2\tau(x)^{-2} + (x+1)\log t\right\}$$

for $n \geqslant 2$, $0 \leqslant t \leqslant 1$ and $n - 1 < x \leqslant n$, where $b_2 = b_1(e/2)^{\frac{1}{2}}$.
Let $R_n^2 = \Sigma_{n+1}^\infty \eta_j^2\tau(j)^2$. Then by Chebyshev's inequality we have

$$\begin{aligned}
F^n(u) &= 1 - P(R_n^2 > u^2) \geqslant 1 - u^{-2}ER_n^2\\
&= 1 - u^{-2}\Sigma_{n+1}^\infty \tau(j)^2 \geqslant 1 - u^{-2}\int_x^\infty \tau(y)^2\,dy\\
&= 1 - u^{-2}\psi(x)
\end{aligned}$$

whenever $n - 1 < x \leqslant n$, $n \geqslant 2$. So by (4.6) and (4.15) we have

$$F(t) \geqslant B_1\left\{1 - \frac{\psi(x)}{t^2 - s^2}\right\}\exp\left\{\int_1^x \log \varphi(y)\,dy - \tfrac{1}{2}\log x + (x+1)\log s - \frac{s^2}{2\tau(x)^2}\right\}$$

for $0 \leqslant s \leqslant t \leqslant 1$ and $x \geqslant 1$, where B_1 and b_2 are given by the equation

$$(4.16)\qquad B_1 = b_2 = (2\pi)^{-\frac{1}{2}}e^{\frac{1}{3}} = 0.56.$$

THEOREM 4.4. *Let q be given by* (4.4) *where τ satisfies* (4.1)–(4.3) *and* (4.2.1)–(4.2.2). *Let φ, ψ and H be given as above*:

$$\varphi(x) = x^{-\frac{1}{2}}\tau(x)^{-1} \qquad \text{for} \quad x \geqslant 1$$

$$\psi(x) = \int_x^\infty \tau(y)^2\,dy \qquad \text{for} \quad x \geqslant 1$$

$$H(x) = \int_1^x \frac{t\varphi'(t)}{\varphi(t)}\,dt \qquad \text{for} \quad x \geqslant 1$$

If F is the distribution of $Q = q(\eta)$, then for some $B_2 > 0$, we have

$$\text{(4.4.1)} \qquad F(t) \geqslant B_2 x^{-\frac{1}{2}}\tau(x)^{-\frac{1}{2}}\left\{1 - \frac{\psi(x)}{t^2 - s^2}\right\}\exp\left\{-H(x+1) - \frac{s^2}{2\tau(x)^2}\right\}$$

whenever $x \geqslant 1$ and $1/\varphi(x) \leqslant s \leqslant t \leqslant 1$.

PROOF. Using (4.11) we have for the exponent in (4.14):

$$\begin{aligned}
-\Sigma_1^n &\log\tau(j) - \tfrac{1}{2}(n+1)\log n + \tfrac{1}{2}n + n\log t - \tfrac{1}{2}t^2\tau(n)^{-2} \\
&\geqslant -\int_1^n \log\tau(y)\,dy - \tfrac{1}{2}\log\tau(n+1) + \tfrac{1}{2}\log\tau(2) - \tfrac{1}{2}\int_1^n \log y\,dy \\
&\quad + \tfrac{1}{2} - \tfrac{1}{2}\log n + n\log t - \tfrac{1}{2}t^2\tau(n)^{-2} \\
&= \int_1^n \log\varphi(y)\,dy - \tfrac{1}{2}\log(n\tau(n+1)) + n\log t - \tfrac{1}{2}t^2\tau(n)^{-2} + \tfrac{1}{2} + \tfrac{1}{2}\log\tau(2) \\
&= -H(n) + n\log(t\varphi(n)) - \tfrac{1}{2}\log(n\tau(n+1)) - \tfrac{1}{2}t^2\tau(n)^{-2} + \alpha
\end{aligned}$$

where we have used the equality:

$$\int_1^n \log\varphi(y)\,dy = -H(n) + n\log\varphi(n) + \log\tau(1)$$

and where $\alpha = \frac{1}{2} + \frac{1}{2}\log\tau(2) + \log\tau(1)$.

If $n - 1 \leqslant x \leqslant n$, $n \geqslant 2$ and $\varphi(x) \geqslant 1/t$, then

$$\begin{aligned}
-H(n) &\geqslant -H(x+1), \\
\log(t\varphi(n)) &\geqslant \log(t\varphi(x)) \geqslant 0, \\
-\tfrac{1}{2}\log(n\tau(n+1)) &= -\tfrac{1}{2}\log x + \tfrac{1}{2}\log(x/n) - \tfrac{1}{2}\log\tau(n+1) \\
&\geqslant -\tfrac{1}{2}\log(x\tau(x)) - \tfrac{1}{2}\log 2,
\end{aligned}$$

so we find as before

$$F(t) \geqslant B_2 x^{-\frac{1}{2}}\tau(x)^{-\frac{1}{2}}\left\{1 - \frac{\psi(x)}{t^2 - s^2}\right\}\exp\left\{-H(x+1) - \frac{s^2}{2\tau(x)^2}\right\}$$

where

$$\text{(4.17)} \qquad B_2 = b_1\tau(1)^{-1}\left(\tfrac{1}{2}e\tau(2)\right)^{\frac{1}{2}}.$$

EXAMPLE 4.5. If $\tau(x) = x^{-\alpha}(\alpha > \frac{1}{2})$, then we have

$$F(t) \leqslant At^{\rho(1-\alpha)}\exp\left(-\left(\alpha - \tfrac{1}{2}\right)t^{-2\rho}\right) \tag{4.5.1}$$

$$F(t) \geqslant Bt^{\rho(3-\alpha)}\exp\left(-\alpha(1+\rho)^{\rho}t^{-2\rho}\right) \tag{4.5.2}$$

where $\rho = (2\alpha - 1)^{-1}$ and A and B are positive constants.

In this case the functions φ, ψ and H take the form:

$$\varphi(x) = x^{\alpha - \frac{1}{2}}, \qquad \psi(x) = \rho x^{1-2\alpha}, \qquad H(x) = \left(\alpha - \tfrac{1}{2}\right)(x-1).$$

Then $\varphi(t^{-2\rho}) = t^{-1}$, so putting $x = t^{-2\rho}$ in (4.2.3) gives (4.5.1).

Putting $x = s^{-2\rho}$ in (4.4.1) gives

$$F(t) \geqslant B_2 s^{\rho(1-\alpha)}\left\{1 - \frac{\rho s^2}{t^2 - s^2}\right\}\exp(-\alpha s^{-2\rho}) \tag{4.18}$$

for $0 \leqslant s \leqslant t \leqslant 1$. Now we take

$$s = t\left(\frac{1 - \frac{1}{2}t^{2\rho}}{\rho + 1}\right)^{\frac{1}{2}}.$$

Then we have

$$1 - \frac{\rho s^2}{t^2 - s^2} = \frac{t^2 - (\rho+1)s^2}{t^2 - s^2} = \frac{t^{2\rho}(1+\rho)}{2\rho + t^{2\rho}} \geqslant \tfrac{1}{2}t^{2\rho}$$

$$\begin{aligned} s^{-2\rho} &= (\rho+1)^{\rho}\left(1 - \tfrac{1}{2}t^{2\rho}\right)^{-\rho}t^{-2\rho} \\ &\leqslant (p+1)^{\rho}\left(1 + \tfrac{1}{2}t^{2\rho}\rho 2^{\rho+1}\right)t^{-2\rho} \\ &= (\rho+1)^{\rho}t^{-2\rho} + \quad \text{constant} \end{aligned}$$

where we in the last inequality used:

$$\left(1 - \tfrac{1}{2}t^{2\rho}\right)^{-\rho} = 1 + \tfrac{1}{2}t^{2\rho}\rho\xi^{-\rho-1} \leqslant 1 + \tfrac{1}{2}t^{2\rho}\rho 2^{\rho+1}$$

where $\frac{1}{2} \leqslant 1 - \frac{1}{2}t^{2\rho} \leqslant \xi \leqslant 1$. Inserting all this in (4.18) gives (4.5.2).

EXAMPLE 4.6. Let $\tau(x) = x^{-\frac{1}{2}}(1 + \log x)^{-1}$; then we have

$$F(t) \leqslant A\exp(-te^{t^{-1}-1}), \tag{4.6.1}$$

$$F(t) \geqslant B\exp\left(-\left(\tfrac{1}{2} + 3t^2\right)e^{t^{-2}+1}\right) \tag{4.6.2}$$

where A and B are positive constants.

In this case we have

$$\begin{aligned} \varphi(x) &= 1 + \log x, \\ \psi(x) &= (1 + \log x)^{-1}, \\ H(x) &= \int_1^x \frac{dt}{1 + \log t}, \end{aligned}$$

and since

$$\frac{d}{dt}\frac{t}{1+\log t} = \frac{\log t}{(1+\log t)^2} \leqslant \frac{1}{1+\log t} \qquad \forall t \geqslant 0,$$

$$\frac{d}{dt}\frac{t}{\log t - 1} = \frac{\log t - 2}{(\log t - 1)^2} \geqslant \frac{1}{1+\log t} \qquad \forall t \geqslant e^3,$$

we have

$$H(x) \geqslant \frac{x}{1+\log x} - 1$$

$$H(x) \leqslant H(e^3) + \frac{x}{\log x - 1} - \frac{e^3}{2} \qquad \forall x \geqslant e^3.$$

Let us choose $x = e^{1/t-1}$ in (4.2.3); then we find

$$x^{-\frac{1}{2}}\tau(x)^{-\frac{1}{2}} = \exp\left(-1/(4t) + \tfrac{1}{4} - \tfrac{1}{2}\log t\right),$$

$$H(x-1) \geqslant \frac{x-1}{1+\log x} - 1 = te^{t^{-1}-1} - t - 1,$$

and since $t \leqslant 1/(4t) + \frac{1}{2}\log t$ for t sufficiently small (4.6.1) follows.

Let us then choose $x = e^{1/s} - 1$ in (4.4.1); then it is easily checked that $\varphi(x) \geqslant s^{-1}$, and we have

$$x^{-\frac{1}{2}}\tau(x)^{-\frac{1}{2}} = (e^{1/s}-1)^{-\frac{1}{4}}\varphi(x)^{\frac{1}{2}} \geqslant e^{-1/(4s)},$$

$$\psi(x) = \varphi(x)^{-1} \leqslant s,$$

$$H(x+1) \leqslant K + se^{1/s}/(1-s)$$

$$\tfrac{1}{2}s^2\tau(x)^{-2} \leqslant \tfrac{1}{2}s^2\tau(x+1)^{-2} = \tfrac{1}{2}(1+s)^2e^{1/s}.$$

Then we choose $s = t^2/(1+t^2)$, and we find

$$x^{-\frac{1}{2}}\tau(x)^{-\frac{1}{2}} \geqslant \exp\left(-\frac{1+t^2}{4t^2}\right),$$

$$1 - \frac{\psi(x)}{t^2 - s^2} \geqslant \frac{t^2 - s^2 - s}{t^2 - s^2} = \frac{t^4}{1+t^4+t^2} \geqslant \frac{t^4}{3} \qquad t \leqslant 1,$$

$$-H(x+1) \geqslant -K - t^2e^{t^{-2}+1},$$

$$-\tfrac{1}{2}s^2\tau(x)^{-2} \geqslant -\tfrac{1}{2}(1+t^2)^2e^{t^{-2}+1} = -\left(\tfrac{1}{2} + \tfrac{1}{2}t^4 + t^2\right)e^{t^{-2}+1}.$$

And since

$$-4\log t + \tfrac{1}{2}t^4e^{t^{-2}+1} + \frac{1+t^2}{4t^2} \leqslant t^2e^{t^{-2}+1}$$

for t sufficiently small, (4.6.2) follows by inserting the inequalities above in (4.4.1).

EXAMPLE 4.7. Let $\tau(x) = x^{-\frac{1}{2}}e^{-x}$; then we have

$$F(t) \leqslant At^{-\frac{3}{2}}\left(\log\frac{1}{t}\right)^{-\frac{1}{4}}\exp\left(-\tfrac{1}{2}\left(\log\frac{1}{t}\right)^2\right), \tag{4.7.1}$$

$$F(t) \geqslant Bt^{1+\log 2}\left(\log\frac{1}{t}\right)^{-\frac{1}{4}}\exp\left(-\tfrac{1}{2}\left(\log\frac{1}{t}\right)^2\right), \tag{4.7.2}$$

where A and B are positive constants.

In this case we have:

$$\varphi(x) = e^x, \qquad H(x) = \tfrac{1}{2}(x+1)(x-1),$$

$$\psi(x) = \int_x^\infty \frac{e^{-2t}}{t}\,dt \leqslant \frac{e^{-2x}}{2}.$$

Choosing $x = \log(1/t)$ in (4.2.3) gives

$$x^{-\frac{1}{2}}\tau(x)^{-\frac{1}{2}} = \left(\log\frac{1}{t}\right)^{-\frac{1}{4}}t^{-\frac{1}{2}},$$

$$-H(x-1) = -\tfrac{1}{2}x(x-2) = -\tfrac{1}{2}\left(\log\frac{1}{t}\right)^2 - \log t.$$

Now (4.7.1) follows by use of (4.2.3).

Choosing $x = \log(1/s)$ and $s = \frac{1}{2}t$ gives

$$x^{-\frac{1}{2}}\tau(x)^{-\frac{1}{2}} = \left(\log\frac{1}{t} + \log 2\right)^{-\frac{1}{4}}2^{\frac{1}{2}}t^{-\frac{1}{2}} \geqslant kt^{-\frac{1}{2}}\left(\log\frac{1}{t}\right)^{-\frac{1}{4}}$$

$$-H(x+1) = -\tfrac{1}{2}x(x+2) = -\tfrac{1}{2}\left(\log\frac{1}{t} + \log 2\right)^2 + \log t$$

$$= -\tfrac{1}{2}\left(\log\frac{1}{t}\right)^2 - \tfrac{1}{2}(\log 2)^2 + (1+\log 2)\log t$$

$$-\tfrac{1}{2}s^2\tau(x)^{-2} = -\tfrac{1}{2}s^2xe^{2x} = -\tfrac{1}{2}\log\frac{1}{t} - \tfrac{1}{2}\log 2$$

$$1 - \frac{\psi(x)}{t^2 - s^2} \geqslant 1 - \frac{e^{-2x}}{2(t^2-s^2)} = 1 - \frac{s^2}{2(t^2-s^2)} = \frac{5}{6}.$$

Now (4.7.2) follows from (4.4.1).

EXAMPLE 4.8. Let $f : \mathbb{R}_+ \to \mathbb{R}_+$ be increasing and with $\lim_{t\to 0} f(t) = 0$. Then there exists a function $\tau : [1, \infty[\to \mathbb{R}_+$ satisfying (4.1)–(4.3) and such that

$$F(t) \leqslant f(t) \qquad \forall 0 \leqslant t \leqslant 1/e \tag{4.8.1}$$

where F is the distribution of $Q = \{\sum_1^\infty \eta_j^2\tau(j)^2\}$

Let $p(t) = [1/(et)]$ (here $[x]$ denote the integer part of x) and let $n_0 = 1$ and for $p \geqslant 1$:

$$n_p = 1 + \left[-2\log f\left(\frac{1}{e(p+1)}\right) - 2\log(e(p+1))\right].$$

We assume that $0 < f(t) \leqslant 1$ for $0 < t \leqslant 1$, $f(t)/t$ increases and $\lim_{t\to 0} f(t)/t = 0$, which is possible by substituting f by $tf(t)$. Let $\tau(1) = 1$, and

$$\tau(t) = p^{-1} n_p^{-\frac{1}{2}} \qquad \text{for} \quad n_{p-1} < t \leqslant n_p.$$

Then $n_p \leqslant n_{p+1}$ for all $p \geqslant 0$ and $n_p \to \infty$, since $f(x)/x$ increases and tends to 0 as $x \to 0$; (4.1)–(4.3) are easily checked, and

$$\varphi(t) = t^{-\frac{1}{2}} p(n_p)^{\frac{1}{2}} \quad \text{for} \quad n_{p-1} < t \leqslant n_p.$$

So

$$\begin{aligned}\textstyle\int_{n_{p-1}}^{n_p} \log \varphi(t)dt &= (n_p - n_{p-1})\log\left(p(n_p)^{\frac{1}{2}}\right) - \tfrac{1}{2}\left(n_p(\log n_p - 1) - n_{p-1}(\log n_{p-1} - 1)\right) \\ &\leqslant (n_p - n_{p-1})\log\left(p(n_p)^{\frac{1}{2}}\right) - (n_p - n_{p-1})\left(\log(n_p)^{\frac{1}{2}} - \tfrac{1}{2}\right) \\ &= (n_p - n_{p-1})\left(\log p + \tfrac{1}{2}\right),\end{aligned}$$

and

$$\textstyle\int_1^{n_p} \log \varphi(t)dt \leqslant (n_p - 1)\left(\log p + \tfrac{1}{2}\right).$$

Inserting this in the exponent of (4.1.2) with $x = n_p$ we get

$$F(t) \leqslant A_1 \exp\left(n_p\left(\log p + \tfrac{1}{2} + \log t\right) - \tfrac{1}{2} - \log t\right).$$

Taking $p = p(t)$ gives $p \leqslant (et)^{-1}$, so

$$\log p + \tfrac{1}{2} + \log t \leqslant -\tfrac{1}{2}$$

and $(1/e(p + 1)) \leqslant t$ gives

$$-\tfrac{1}{2} n_p \leqslant \log f(t) + \log t$$

so

$$F(t) \leqslant e^{-\frac{1}{2}} A_1 f(t) \leqslant f(t)$$

since $e^{-\frac{1}{2}} A_1 = (2/\pi)^{\frac{1}{2}} \leqslant 1$ (cf. (4.10)).

EXAMPLE 4.9. Let $g :]0, 1] \to \mathbb{R}_+$ be an increasing function with $g(1) < 1$ and

$$g(t) = 0(t^n) \qquad \text{as} \quad t \to 0 \qquad \forall n \geqslant 1.$$

Then there exists a function $\tau : [1, \infty[\to \mathbb{R}_+$ satisfying (4.1)–(4, 3), and such that

$$F(t) \geqslant g(t) \qquad \forall t \in [0, 1] \tag{4.9.1}$$

where F is the distribution of $Q = \{\Sigma_1^\infty \eta_j^2 \tau(j)^2\}^{\frac{1}{2}}$.

There exist constants $A_n > 0$, so that

$$g(t) \leqslant A_n t^{n+3} \qquad \forall 0 \leqslant t \leqslant 1, \forall n \geqslant 1.$$

Hence if $B_n = \log A_n$ we have

$$\log g(t) \leqslant B_n + (n + 3)\log t \qquad \forall t \in [0, 1] \forall n \geqslant 1.$$

Let $\tau_0^2 = 1$ and put

$$\alpha_n = e^{-B_n - 1} 2^{-n-7} (n + 1)^{-\frac{1}{2}} \quad \forall n \geqslant 1.$$

Then we define τ_n^2 inductively by

$$\tau_n^2 = \min\{\alpha_n, 2^{-n}\tau_0^2, 2^{-n+1}\tau_1^2, \cdots, 2^{-1}\tau_{n-1}^2\}$$

for $n \geqslant 1$, and we put

$$\tau(t) = \tau_n \quad \text{for} \quad n < t \leqslant n+1 \quad \text{and} \quad n \geqslant 0.$$

Then τ satisfies (4.1)–(4.3), and we have

$$\psi(n) = \int_n^\infty \tau(t)^2 dt = \Sigma_{j=n}^\infty \tau_j^2 \leqslant \Sigma_{j=n}^\infty 2^{-(j-n)}\tau_n^2 = 2\tau_n^2.$$

Let $n \geqslant 1$ and $4\psi(n) \leqslant t^2 \leqslant 4\psi(n-1)$, then we shall apply (4.3.2) with $x = n$ and $s = \frac{1}{2}t$. The exponent in (4.3.2) gives ($s^2 \leqslant \psi(n-1) \leqslant 2\tau_{n-1}^2$)

$$\begin{aligned}
&\int_1^n \log \varphi(y)dy - \tfrac{1}{2}\log n + (n+1)\log s - \tfrac{1}{2}s^2\tau_{n-1}^{-2} \\
&\quad \geqslant -\tfrac{1}{2}\log n + (n+1)\log t - (n+1)\log 2 - 1 \\
&\quad = (B_{n-1} + (n+3)\log t) - 2\log t - B_{n-1} - \tfrac{1}{2}\log n - \log 2^{n+1} - 1 \\
&\quad \geqslant \log g(t) - \log t^2 + \log \alpha_{n-1} + 5\log 2 \\
&\quad \geqslant \log g(t) + 2\log 2
\end{aligned}$$

since $t^2 \leqslant 4\psi(n-1) \leqslant 8\tau_{n-1}^2 \leqslant 8\alpha_{n-1}$. The factor in (4.3.2) gives

$$1 - \frac{\psi(n)}{t^2 - s^2} = 1 - \frac{\psi(n)}{3s^2} \geqslant \frac{2}{3}$$

since $\psi(n) \leqslant s^2$. Now since $B_1 = 0$, $56 \geqslant \frac{1}{2}$ we have

$$F(t) \geqslant B_1(8/3)g(t) \geqslant g(t)$$

for $t \in [0, 1]$ and $t \leqslant 2(\psi(0))^{\frac{1}{2}}$. However $\psi(0) \geqslant \tau_0^2 = 1$, so (4.9.1) holds.

5. Exact distributions. We shall now give some cases where the distribution of Q can be given in an exact form for certain Hilbertian norms q. Note that (3.3) gives the exact distribution for sup-norms. Let q and Q be given by

$$q(x) = \{\Sigma_{j=1}(x_{2j-1}^2 + x_{2j}^2)/(2\lambda_j)\}^{\frac{1}{2}}, \tag{5.1}$$

$$Q^2 = q(\eta)^2 = \Sigma_{j=1}^\infty(\eta_{2j-1}^2 + \eta_{2j}^2)/(2\lambda_j). \tag{5.2}$$

Let Q_n^2 denote the nth partial sum in (5.2). Since $(\eta_{2j-1}^2 + \eta_{2j}^2)/(2\lambda_j)$ is exponentially distributed with parameter λ_j, we have (see, e.g., [7], page 40)

$$P(Q_n \leqslant x) = 1 - \Sigma_{j=1}^n A_j^n e^{-\lambda_j x^2} \qquad \forall x \geqslant 0 \tag{5.3}$$

if $\lambda_i \neq \lambda_j \forall i \neq j$, and where A_j^n is defined by

$$A_j^n = \Pi_{k=1;\, k\neq j}^n (1 - \lambda_j/\lambda_k)^{-1} \qquad \text{for } j = 1, \cdots, n.$$

Now let

$$A_j = \Pi_{k\neq j}(1 - \lambda_j/\lambda_k)^{-1} = \Pi_{k\neq j}\frac{\lambda_k}{\lambda_k - \lambda_j} \tag{5.4}$$

and assume that (λ_j) satisfies:

(5.5) $$0 < \lambda_1 < \lambda_2 < \cdots,$$

(5.6) $$\Sigma_1^\infty \lambda_j^{-1} < \infty,$$

(5.7) $$\Sigma_1^\infty |A_j| e^{-\lambda_j x} < \infty \quad \forall x > 0.$$

Then we have for $j \leqslant n$

$$|A_j^n| = |A_j| \pi_{k=n+1}^\infty (1 - \lambda_j/\lambda_k) \leqslant |A_j|.$$

So by (5.3), (5.7) and the dominated convergence theorem we deduce:

(5.8) $$P(Q \leqslant x) = 1 - \Sigma_{j=1}^\infty A_j e^{-\lambda_j x^2} \qquad \forall x > 0.$$

And if we assume, in addition to (5.5)–(5.7), that we have

(5.9) $$\Sigma_{j=1}^\infty \lambda_j |A_j| e^{-\lambda_j x} < \infty \qquad \forall x > 0,$$

then the density, f, of Q is given by

(5.10) $$f(t) = 2t \Sigma_{j=1}^\infty \lambda_j A_j e^{-\lambda_j t^2}.$$

So under (5.5)–(5.7) the formulae (5.8) gives the distribution of Q defined by (5.2), and under (5.9) the density of Q is given by (5.10). Note that sign $A_j = (-1)^{j-1}$, so the series in (5.8) and (5.10) are alternating.

In order to use (5.8) and (5.10) we should be able to find A_j. One way is the following: suppose that we have given a product formula

$$\varphi(x) = \Pi_{k=1}^\infty (1 - \psi(x)/\lambda_k)$$

where φ and ψ are differentiable. Let x_j be a solution to $\psi(x_j) = \lambda_j$; then $\varphi(x_j) = 0$ and for $x \neq x_j$

$$\Pi_{k \neq j}(1 - \psi(x)/\lambda_k) = -\lambda_j \frac{\varphi(x) - \varphi(x_j)}{x - x_j} \left\{ \frac{\psi(x) - \psi(x_j)}{x - x_j} \right\}^{-1}.$$

Letting $x \to x_j$ gives

(5.11) $$A_j = \Pi_{k \neq j}(1 - \lambda_j/\lambda_k)^{-1} = -\frac{\psi'(x_j)}{\lambda_j \varphi'(x_j)}.$$

The series (5.8) and (5.10) will in general be divergent or at least slowly convergent at $x = 0$. But the Poisson summation formulae

(5.12) $$\Sigma_{n=-\infty}^\infty \cos(yn) \hat{f}(xn) = \frac{2\pi}{x} \Sigma_{k=-\infty}^\infty f\left(\frac{y + 2k\pi}{x}\right)$$

(f is an even density, $\hat{f}$ its Fourier transform) may in certain cases be used to transform the sums (5.8) and (5.10) into sums which are rapidly convergent for small x (see, e.g., [7], page 630 for the validity of (5.12)).

From the product formula (see [1], page 255)

$$\frac{\sin \pi x}{\pi x} = \Pi_{k=1}^\infty (1 - x^2 k^{-2}),$$

we find by (5.11):

$$\Pi_{k\neq j}(1 - k^2/j^2)^{-1} = 2(-1)^{j-1}.$$

So if $\lambda_j = j^2$ we have

$$(5.13) \qquad P(Q \leqslant x) = 1 - \Sigma_{j=1}^{\infty} 2(-1)^{j-1} e^{-j^2x^2} = \Sigma_{j=-\infty}^{\infty}(-1)^j e^{-j^2x^2}.$$

If we put $f =$ the normal density in (5.12) we get

$$(5.14) \qquad \Sigma_{n=-\infty}^{\infty} \cos(ny) e^{-\frac{1}{2}x^2n^2} = \frac{(2\pi)^{\frac{1}{2}}}{x} \Sigma_{k=-\infty}^{\infty} \exp\left(-\frac{(y+2k\pi)^2}{2x^2}\right).$$

Putting $y = \pi$ and $x = t2^{\frac{1}{2}}$ gives

$$\Sigma_{n=-\infty}^{\infty}(-1)^n e^{-t^2n^2} = \frac{2\pi^{\frac{1}{2}}}{t} \Sigma_{k=0}^{\infty} \exp\left(-\frac{(2k+1)^2\pi^2}{4t^2}\right).$$

The term for $k = 0$ is clearly dominant for small t, so we have

THEOREM 5.1. *Let*

$$Q = \{\Sigma_{j=1}^{\infty}(\eta_{2j-1}^2 + \eta_{2j}^2)/(2j^2)\}^{\frac{1}{2}}.$$

Then we have

$$(5.1.1) \qquad P(Q \leqslant t) = \frac{2\pi^{\frac{1}{2}}}{t} \Sigma_{k=0}^{\infty} \exp\left(-\frac{(2k+1)^2\pi^2}{4t^2}\right) \qquad \forall t > 0,$$

$$(5.1.2) \qquad P(Q \leqslant t) \sim \frac{2\pi^{\frac{1}{2}}}{t} \exp\left(-\frac{\pi^2}{4t^2}\right) \qquad as \quad t \to 0.$$

From (5.11) and the product formulae (see [1], page 255):

$$\cos\left(\tfrac{1}{2}\pi x\right) = \Pi_{k=1}^{\infty}(1 - x^2/(2k-1)^2),$$

we find

$$\Pi_{k\neq j}(1 - (2j-1)^2/(2k-1)^2)^{-1} = \frac{4}{\pi}(-1)^{j-1}(2j-1)^{-1}.$$

So if $\lambda_j = (2j-1)^2$, then the density of Q is given by (cf. (5.10)):

$$f(t) = \frac{8t}{\pi} \Sigma_{j=1}^{\infty}(2j-1)(-1)^{j-1} \exp(-(2j-1)^2t^2)$$

$$= \frac{4t}{\pi} \Sigma_{j=-\infty}^{\infty}(2j-1)(-1)^{j-1} \exp(-(2j-1)^2t^2).$$

Differentiating (5.14) with respect to y gives

$$\Sigma_{n=-\infty}^{\infty} n \sin(ny) e^{-\frac{1}{2}x^2n^2} = \frac{(2\pi)^{\frac{1}{2}}}{x} \Sigma_{k=-\infty}^{\infty} \frac{y+2k\pi}{x^2} \exp\left(-\frac{(y+2k\pi)^2}{2x^2}\right),$$

so putting $y = \pi/2$ and $x = t2^{\frac{1}{2}}$ gives

$$\begin{aligned} f(t) &= \frac{4t}{\pi}\Sigma_{n=-\infty}^{\infty} n \sin\left(\tfrac{1}{2}n\pi\right)e^{-t^2n^2} \\ &= \frac{4}{\pi^{\frac{1}{2}}}\Sigma_{k=-\infty}^{\infty}\frac{(4k+1)\pi}{4t^2}\exp\left(-\frac{(4k+1)^2\pi^2}{16t^2}\right) \\ &= \frac{\pi^{\frac{1}{2}}}{t^2}\left\{\Sigma_{k=0}^{\infty}(4k+1)\exp\left(-\frac{(4k+1)^2\pi^2}{16t^2}\right)\right. \\ &\quad \left. -\Sigma_{k=1}^{\infty}(4k-1)\exp\left(-\frac{(4k-1)^2\pi^2}{16t^2}\right)\right\} \\ &= \frac{\pi^{\frac{1}{2}}}{t^2}\Sigma_{j=0}^{\infty}(2j+1)(-1)^j\exp\left(-\frac{(2j+1)^2\pi^2}{16t^2}\right). \end{aligned}$$

For small t the term for $j = 0$ dominates the others. So we have

$$f(t) \sim \frac{\pi^{\frac{1}{2}}}{t^2}\exp\left(-\left(\frac{\pi}{4t}\right)^2\right) \qquad \text{as} \quad t \to 0.$$

Hence $F(t) = \int_0^t f(s)ds$, and satisfies

$$F(t) \sim \int_0^t \frac{\pi^{\frac{1}{2}}}{s^2}\exp\left(-\left(\frac{\pi}{4s}\right)^2\right)ds \qquad \text{as} \quad t \to 0$$

by l'Hospital's rule; but

$$\int_0^t \frac{\pi^{\frac{1}{2}}}{s^2}\exp\left(-\left(\frac{\pi}{4s}\right)^2\right)ds = 4\left(1 - \Phi\left(\frac{\pi}{t8^{\frac{1}{2}}}\right)\right)$$

and as $t \downarrow 0$,

$$1 - \Phi\left(\frac{\pi}{t8^{\frac{1}{2}}}\right) \sim \left(t\pi^{-1}8^{\frac{1}{2}}\right)(2\pi)^{-\frac{1}{2}}\exp\left(-\frac{\pi^2}{16t^2}\right),$$

that is,

$$F(t) \sim 8t\pi^{-3/2}\exp\left(-\frac{\pi^2}{16t^2}\right).$$

And we have proved:

THEOREM 5.2. *Let*

$$Q = \left\{\Sigma_{j=1}^{\infty}\frac{\eta_{2j-1}^2 + \eta_{2j}^2}{2(2j-1)^2}\right\}^{\frac{1}{2}}.$$

and let f denote the density of Q. Then we have

$$f(t) = \frac{\pi^{\frac{1}{2}}}{t^2}\Sigma_{j=0}^{\infty}(2j+1)(-1)^j\exp\left(-\frac{(2j+1)^2\pi^2}{16t^2}\right), \tag{5.2.1}$$

$$P(Q \leqslant t) \sim 8t\pi^{-3/2}\exp\left(-\frac{\pi^2}{16t^2}\right) \qquad \textit{as} \quad t \to 0. \tag{5.2.2}$$

THEOREM 5.3. *Let*

$$Q = \{\Sigma_{j=1}^{\infty} j^{-2}\eta_j^2\}^{\frac{1}{2}}.$$

Then we have

$$P(Q \leqslant t) \leqslant t^{-1}(2\pi)^{\frac{1}{2}}(1 + \varepsilon_1(t))\exp\left(-\frac{\pi^2}{8t^2}\right), \tag{5.3.1}$$

$$P(Q \leqslant t) \geqslant 4t\pi^{-3/2}2^{\frac{1}{2}}(1 + \varepsilon_2(t))\exp\left(-\frac{\pi^2}{8t^2}\right) \tag{5.3.2}$$

where $\varepsilon_j(t) \to_{t\to 0} 0$ *for* $j = 1, 2$.

PROOF. Let Q_1 and Q_2 be the random variables defined in Theorem 5.1 and Theorem 5.2 respectively. Then

$$Q_1/2^{\frac{1}{2}} \leqslant Q \leqslant 2^{\frac{1}{2}}Q_2,$$

so

$$P(Q \leqslant t) \leqslant P\left(Q_1 \leqslant 2^{\frac{1}{2}}t\right),$$

$$P(Q \leqslant t) \geqslant P\left(Q_2 \geqslant t/2^{\frac{1}{2}}\right),$$

and the theorem follows from (5.1.2) and (5.2.2).

Added in proof. We thank David Siegmund for calling our attention to a paper of T. W. Anderson and D. A. Darling (*Ann. Math. Statist.* **23** 191–212), where they give an exact series for the distribution of Q from Theorem 5.3, from which the exact behavior at $t = 0$ can be read off (Anderson and Darling, page 202).

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