The Annals of Probability
1979, Vol. 7, No. 2, 304–318

EXTENDED RENEWAL THEORY AND MOMENT CONVERGENCE IN ANSCOMBE'S THEOREM

BY Y. S. CHOW[1], CHAO A. HSIUNG[2] AND T. L. LAI[3]

Columbia University

In this paper, an L_p analogue of Anscombe's theorem is shown to hold and is then applied to obtain the variance and other central moments of the first passage time $T_c = \inf\{n \geq 1 : S_n > cn^\alpha\}$, where $0 \leq \alpha < 1$, $S_n = X_1 + \cdots + X_n$ and $X_1, X_2, \cdots$ are i.i.d. random variables with $EX_1 > 0$. The variance of T_c in the special case $\alpha = 0$ has been studied by various authors in classical renewal theory, and our approach in this paper provides a simple treatment and a natural extension (to the case of a general α) of this classical result. The related problem concerning the asymptotic behavior of $\max_{j \leq n} j^{-\alpha} S_j$ is also studied, and in this connection, certain maximal inequalities are obtained and they are applied to prove the corresponding moment convergence results of the theorems of Erdös and Kac, and of Teicher.

1. Introduction and summary. Let $X, X_1, X_2, \cdots$ be i.i.d. random variables with $\infty > EX = \mu > 0$ and let $0 \leq \alpha < 1$. Setting $S_n = X_1 + \cdots + X_n$, define

$$T_c = \inf\{n \geq 1 : S_n > cn^\alpha\}, \tag{1.1}$$

where c is a positive constant. While in the special case $\alpha = 0$ the first passage time T_c has been extensively studied in classical renewal theory, the case of general α has been recently considered by a number of authors in the literature (cf. [2], [7], [11], [12], [16]), and it is well known (cf. [7], page 281) that as $c \to \infty$,

$$(c/\mu)^{-1/(1-\alpha)} T_c \to 1 \quad \text{a.s.}, \tag{1.2}$$

$$ET_c \sim (c/\mu)^{1/(1-\alpha)}, \tag{1.3}$$

$$ET_c^p \sim (c/\mu)^{p/(1-\alpha)} \qquad \text{if} \quad E(X^-)^p < \infty \qquad p \geq 1. \tag{1.4}$$

Under the additional assumption that X has a finite variance $\sigma^2 > 0$, the asymptotic normality of T_c was first established by Siegmund [12] who showed that as $c \to \infty$,

$$(1-\alpha)(c/\mu)^{-1/(2(1-\alpha))}\{T_c - (c/\mu)^{1/(1-\alpha)}\} \to_{\mathcal{L}} N(0, (\sigma/\mu)^2) \tag{1.5}$$

where $\to_{\mathcal{L}}$ denotes convergence in distribution. While (1.3) gives the asymptotic behavior of the mean of T_c, (1.5) suggests that the following asymptotic formula for

Received July 29, 1977; revised March 14, 1978.

[1]Research supported by the National Science Foundation under Grant NSF-MCS-76-09179.

[2]Currently at National Central University, Chung-Li, Taiwan.

[3]Research supported by the National Science Foundation under Grant NSF-MCS-75-20905.

AMS 1970 *subject classifications.* Primary 60F05, 60G40, 60G50, 60K05.

Key words and phrases. Extended renewal theory, variance of stopping times, Anscombe's theorem, moment convergence, uniform integrability, maximal inequalities.

the variance of T_c may be true:

(1.6) $$\operatorname{Var} T_c \sim (1-\alpha)^{-2}(\sigma/\mu)^2(c/\mu)^{1/(1-\alpha)} \quad \text{as} \quad c \to \infty.$$

When X is nonnegative, Feller [6] proved (1.6) for $\alpha = 0$ in the lattice case and Smith [14] extended it to the nonlattice case. Later Heyde [8] proved it without the restriction to nonnegative random variables. Their methods make use of Blackwell's renewal theorem and the fluctuation theory of random walks to find sufficiently detailed expansions for ET_c and ET_c^2. (Clearly the simple first-order asymptotic expressions (1.3) and (1.4) are too crude to give (1.6)). Recently Siegmund [13] used Wald's lemma for squared sums to derive expansions for ET_c and ET_c^2 with which he proved (1.6) for $\alpha = 0$. All these proofs only deal with the case $\alpha = 0$, and in [7], page 299, Gut pointed out that the validity of (1.6) for $\alpha > 0$ had remained an open question.

In this paper we shall establish (1.6) in the general case $0 \leqslant \alpha < 1$ under minimal moment conditions on X by using (1.5) and showing uniform integrability. More specifically, we obtain the following theorem:

THEOREM 1. *Let $X, X_1, X_2, \cdots$ be* i.i.d. *random variables with $EX = \mu > 0$ and $\operatorname{Var} X = \sigma^2 > 0$ and let $S_n = X_1 + \cdots + X_n$. Let $0 \leqslant \alpha < 1$ and $p \geqslant 2$. For $c > 0$, define T_c as in* (1.1). *Assume that $E|X|^p < \infty$.*

(i) *The family $\{c^{-p/(2(1-\alpha))}|T_c - (c/\mu)T_c^\alpha|^p, c \geqslant 1\}$ is uniformly integrable.*

(ii) *If $\alpha \leqslant \frac{1}{2}$, then* (i) *implies that*

(1.7) $$\left\{c^{-p/(2(1-\alpha))}|T_c - (c/\mu)^{1/(1-\alpha)}|^p, c \geqslant 1\right\} \text{ is uniformly integrable.}$$

Consequently as $c \to \infty$,

(1.8) $$E|T_c - (c/\mu)^{1/(1-\alpha)}|^r \sim (1-\alpha)^{-r}(c/\mu)^{r/(2(1-\alpha))}(\sigma/\mu)^r m_r \text{ for } 0 < r < p,$$

where $m_r = (2\pi)^{-\frac{1}{2}}\int_{-\infty}^{\infty}|x|^r \exp(-\frac{1}{2}x^2)\,dx$ is the rth *absolute moment of the standard normal distribution. This further implies that*

(1.9) $$E\left\{T_c - (c/\mu)^{1/(1-\alpha)}\right\}^r = o(c^{r/(2(1-\alpha))}) \text{ if } r \text{ is a positive odd integer } \leqslant p.$$

(iii) *If $\frac{1}{2} < \alpha < 1$, then* (1.7), (1.8) *and* (1.9) *still hold under the further assumption*

(1.10) $$P[X \geqslant x] = o(x^{-p/(2(1-\alpha))}) \quad \text{as} \quad x \to \infty.$$

The moment condition $E|X|^p < \infty$ is clearly a very natural condition for the uniform integrability result (1.7). Theorem 1 says that this minimal condition suffices for the case $\alpha \leqslant \frac{1}{2}$. We now explain why additional conditions on the tail distribution of X^+ have to be assumed for (1.7) to hold in the case $\alpha > \frac{1}{2}$. Obviously (1.7) implies that given $\varepsilon > 0$ and $0 < \gamma < 1$, there exists c_0 sufficiently large such that for all $c \geqslant c_0$,

(1.11) $$\begin{aligned} \varepsilon &\geqslant c^{-p/(2(1-\alpha))}E\left\{(c/\mu)^{1/(1-\alpha)} - T_c\right\}^p I_{[T_c \leqslant \gamma(c/\mu)^{1/(1-\alpha)}]} \\ &\geqslant (1-\gamma)^p \mu^{-p/(1-\alpha)} c^{p/(2(1-\alpha))} P\left[T_c \leqslant \gamma(c/\mu)^{1/(1-\alpha)}\right]. \end{aligned}$$

Hence in particular (1.7) implies that

$$c^{p/(2(1-\alpha))}P[X > c] = c^{p/(2(1-\alpha))}P[T_c = 1] \to 0 \quad \text{as} \quad c \to \infty.$$

Therefore (1.7) implies (1.10). For $\alpha \leqslant \frac{1}{2}$, (1.10) is automatically satisfied under the assumption $E|X|^p < \infty$ (since $p/(2(1-\alpha)) \leqslant p$). However, for the case $\alpha > \frac{1}{2}$, (1.10) is a necessary additional condition for the uniform integrability result (1.7).

In the special case $p = 2$, the uniform integrability result (1.7) has been obtained by other methods by Woodroofe ([16], Theorem 5.1) under the more restrictive moment condition $E|X|^q < \infty$ for some $q > \max\{4, 1/(1-\alpha)\}$. Woodroofe ([16], pages 74–75) gave some applications of this result in his development of an asymptotic expansion for ET_c under further conditions on X.

Putting $r = 2$ in (1.8) and $r = 1$ in (1.9), we obtain the asymptotic formula (1.6) for Var T_c as an immediate corollary.

COROLLARY 1. *Let $X, X_1, X_2, \cdots$ be* i.i.d. *random variables with $EX = \mu > 0$ and $\infty >$* Var $X = \sigma^2 > 0$. *Let $0 \leqslant \alpha < 1$ and define T_c for $c > 0$ as in* (1.1). *Assume furthermore that* (1.10) *with $p = 2$ holds in the case $\frac{1}{2} < \alpha < 1$. Then* (1.6) *holds.*

The proof of Theorem 1 will be given in Section 4. As shown in [7], page 298, the asymptotic normality (1.5) of T_c is a consequence of the following theorem of Anscombe [1]: If $Y, Y_1, Y_2, \cdots$ are i.i.d. with $EY = 0$ and $EY^2 = \sigma^2 < \infty$ and if $\{M(b), b \geqslant b_0\}$ is a family of positive integer-valued random variables such that as $b \to \infty$,

$$b^{-1}M(b) \to_P \lambda \quad \text{for some positive constant } \lambda, \tag{1.12}$$

then as $b \to \infty$,

$$b^{-\frac{1}{2}}\Sigma_1^{M(b)} Y_i \to_{\mathfrak{L}} N(0, \lambda\sigma^2). \tag{1.13}$$

It turns out that Theorem 1 can likewise be proved by using the following L_p analogue of Anscombe's theorem.

THEOREM 2. *Let $Y, Y_1, Y_2, \cdots$ be* i.i.d. *with $EY = 0$, and let $\{M(b), b \geqslant b_0\}$ be a family of positive integer-valued random variables.*

(i) *For $p \geqslant 1$, if $E|Y|^{p+1} < \infty$ and*

$$\{(b^{-1}M(b))^p, b \geqslant b_0\} \text{ is uniformly integrable,} \tag{1.14}$$

then

$$\left\{\left(b^{-\frac{1}{2}}|\Sigma_1^{M(b)} Y_i|\right)^p, b \geqslant b_0\right\} \text{ is uniformly integrable.} \tag{1.15}$$

(ii) *Suppose for $b \geqslant b_0$, $M(b)$ is a stopping time with respect to the σ-fields $\mathfrak{F}_n$, $n \geqslant 1$, where $\mathfrak{F}_n$ is the σ-field generated by $\{Y_1, \cdots, Y_n\}$. For $p \geqslant 2$, if $E|Y|^p < \infty$ and* (1.14) *holds, then* (1.15) *still holds.*

The proof of Theorem 2 will be given in Section 3. While Anscombe's theorem says that the condition (1.12) on the convergence in probability of $b^{-1}M(b)$

guarantees the asymptotic normality of $b^{-\frac{1}{2}}\Sigma_1^{M(b)}Y_i$ when $EY^2 < \infty$, Theorem 2 says that likewise the uniform integrability condition (1.14) on $b^{-1}M(b)$ guarantees the conclusion (1.15) on the uniform integrability of randomly stopped partial sums under natural moment conditions on Y. Since L_p convergence is equivalent to convergence in probability and uniform integrability, Theorem 2 and Anscombe's theorem together yield the following moment convergence result associated with (1.13).

COROLLARY 2. *Let $Y, Y_1, Y_2, \cdots$ be* i.i.d. *with $EY = 0$ and $EY^2 = \sigma^2$. Let $\{M(b), b \geqslant b_0\}$ be a family of positive integer-valued random variables.*

(i) *For $p \geqslant 1$, if $E|Y|^{p+1} < \infty$ and*

$$b^{-1}M(b) \to_{L_p} \lambda \text{ for some positive constant } \lambda, \tag{1.16}$$

then for every continuous function $f : (-\infty, \infty) \to (-\infty, \infty)$ satisfying $f(x) = 0(|x|^p)$ as $|x| \to \infty$,

$$\lim_{b\to\infty} Ef\left(b^{-\frac{1}{2}}\Sigma_1^{M(b)} Y_i\right) = Ef(Z), \tag{1.17}$$

where Z is a normal random variable with mean 0 and variance $\lambda\sigma^2$.

(ii) *Suppose $p \geqslant 2$ and $M(b)$ is a stopping time (with respect to $\mathcal{F}_n$ as in Theorem 2) for $b \geqslant b_0$. Then we can replace in* (i) *the moment condition $E|Y|^{p+1} < \infty$ by the weaker condition $E|Y|^p < \infty$.*

Closely related to Siegmund's central limit theorem (1.5) for T_c is the following central limit theorem of Teicher [15] for $\max_{j\leqslant n} j^{-\alpha}S_j$: If $X, X_1, X_2, \cdots$ are i.i.d. with $EX = \mu > 0$ and $\infty > \operatorname{Var} X = \sigma^2 > 0$, then for $0 \leqslant \alpha < 1$,

$$\left(\max_{j\leqslant n} j^{-\alpha}S_j - \mu n^{1-\alpha}\right)/n^{\frac{1}{2}-\alpha} \to_{\mathcal{L}} N(0, \sigma^2). \tag{1.18}$$

In Section 5 we shall obtain the moment convergence theorems corresponding to this result and to an analogous result of Erdös and Kac [5] for zero-mean random variables under minimal moment assumptions on X.

2. A maximal inequality for driftless random walks. A very useful tool in the analysis of the first passage time T_c and the maximal function $\max_{j\leqslant n} j^{-\alpha}S_j$ in extended renewal theory is the following inequality which was established in [4].

LEMMA 1. *Let $1 \leqslant r \leqslant 2$ and $\alpha \geqslant 0$ such that $r\alpha < 1$. There exists an absolute constant $C_{r,\alpha}$ such that if $Y, Y_1, Y_2, \cdots$ are* i.i.d. *with $EY = 0$ and $U_n = Y_1 + \cdots + Y_n$, then for $x > 0$ and $k, n = 1, 2, \cdots$,*

$$P\left[\max_{j\leqslant n} j^{-\alpha}U_j \geqslant x\right] \leqslant P\left[\max_{j\leqslant n} j^{-\alpha}Y_j \geqslant x/(2k)\right] + \left\{C_{r,\alpha} n^{1-r\alpha}(k/x)^r E|Y|^r\right\}^k. \tag{2.1}$$

PROOF. See Lemma 5 of [4].

The right-hand side of (2.1) involves $\max_{j\leqslant n} j^{-\alpha}Y_j$, and this can often be easily handled by using the following elementary inequality (again assuming

$Y, Y_1, Y_2, \cdots$ to be i.i.d.):

(2.2) $$P\left[\max_{j \leqslant n} j^{-\alpha} Y_j \geqslant u\right] \leqslant \Sigma_{j=1}^{n} P\left[Y \geqslant u j^{\alpha}\right].$$

3. Proof of Theorem 2. The following proof makes use of similar ideas as in [9], Section 4. Let $U_n = Y_1 + \cdots + Y_n$. For $x > 0$,

(3.1) $$P\left[|U_{M(b)}| \geqslant b^{\frac{1}{2}} x\right] \leqslant P\left[M(b) > bx\right] + P\left[\max_{j \leqslant bx} |U_j| \geqslant b^{\frac{1}{2}} x\right].$$

Let $p \geqslant 1$ and $k = [p] + 1$. We note that

(3.2) $$P\left[\max_{j \leqslant bx} |U_j| \geqslant b^{\frac{1}{2}} x\right] \leqslant P\left[\max_{j \leqslant bx} |Y_j| > b^{\frac{1}{2}} x/(2k)\right] + P\left[\max_{j \leqslant bx} |U_j| \geqslant b^{\frac{1}{2}} x, \max_{j \leqslant bx} |Y_j| \leqslant b^{\frac{1}{2}} x/(2k)\right].$$

By an argument developed in [3] (see the proof of (3.3) on page 55 of [3]),

(3.3) $$\begin{aligned} P\left[\max_{j \leqslant bx} |U_j| \geqslant b^{\frac{1}{2}} x, \max_{j \leqslant bx} |Y_j| \leqslant b^{\frac{1}{2}} x/(2k)\right] &\leqslant P^k\left[\max_{j \leqslant bx} |U_j| \geqslant b^{\frac{1}{2}} x/(2k)\right] \\ &\leqslant (4k^2 EY^2/x)^k, \text{ by Kolmogorov's inequality.} \end{aligned}$$

Assume that $E|Y|^{p+1} < \infty$ and (1.14) holds. Then

(3.4) $$\begin{aligned} \int_t^\infty x^{p-1} P\left[\max_{j \leqslant bx} |Y_j| > b^{\frac{1}{2}} x/(2k)\right] dx &\leqslant b \int_t^\infty x^p P\left[|Y| > b^{\frac{1}{2}} x/(2k)\right] dx \\ &\leqslant \int_{b^{1/2} t}^{\infty} u^p P\left[|Y| > u/(2k)\right] du \quad \text{for } b \geqslant 1 \text{ (since } p + 1 \geqslant 2). \end{aligned}$$

The last term in the above inequality converges to 0 uniformly in $b \geqslant 1$ as $t \to \infty$ since $E|Y|^{p+1} < \infty$. Since $k = [p] + 1$,

(3.5) $$\int_t^\infty x^{p-1-k}\, dx \to 0 \quad \text{as} \quad t \to \infty.$$

From (1.14), (3.1), (3.2), (3.3), (3.4) and (3.5), we obtain that

$$\int_t^\infty x^{p-1} P\left[|U_{M(b)}| \geqslant b^{\frac{1}{2}} x\right] dx \to 0 \qquad \text{uniformly in } b \geqslant 1 \qquad \text{as } t \to \infty.$$

This completes the proof of part (i) of the theorem.

Now assume that $p \geqslant 2$, $E|Y|^p < \infty$ and $M(b)$ is a stopping time for each $b \geqslant 1$. Take a small positive number ε and choose $K > 0$ such that

(3.6) $$E|Y|^p I_{[|Y| > K]} < \varepsilon.$$

Define

$$Y_n' = Y_n I_{[|Y_n| < K]} - EYI_{[|Y| < K]}, \qquad Y_n'' = Y_n - Y_n',$$
$$U_n' = Y_1' + \cdots + Y_n', \qquad U_n'' = Y_1'' + \cdots + Y_n''.$$

Since $M(b)$ is a stopping time and $p \geqslant 2$, by Lemma 2.3 of [7],

(3.7) $$E|U''_{M(b)}|^p \leqslant C(p, E|Y_1''|^p) E(M(b))^{p/2},$$

where $C(p, x)$ is an absolute constant depending only on p and x. As the proof in [7], pages 280–281, shows, the constant $C(p, x)$ can in fact be chosen such that $\lim_{x \downarrow 0} C(p, x) = 0$. Therefore given $\eta > 0$, we can choose $\varepsilon > 0$ such that for all $b \geqslant 1$,

$$(3.8) \qquad E|b^{-\frac{1}{2}} U''_{M(b)}|^p \leqslant \eta E(b^{-1}M(b))^{p/2},$$

in view of (3.6) and (3.7). Since Y_1' is a bounded random variable, we obtain by part (i) of the theorem that

$$(3.9) \qquad \left\{ |b^{-\frac{1}{2}} U'_{M(b)}|^p, b \geqslant 1 \right\} \quad \text{is uniformly integrable.}$$

From (3.8) and (3.9), the desired conclusion (1.15) follows.

4. Proof of Theorem 1.

PROOF OF THEOREM 1 (i). Since $E|X|^p < \infty$, it follows from (1.2) and (1.4) that as $c \to \infty$,

$$(4.1) \qquad (c/\mu)^{-1/(1-\alpha)} T_c \to_{L_p} 1$$

(cf. [10], page 140), and therefore by Theorem 2(ii),

$$(4.2) \qquad \left\{ \left(c^{-1/(2(1-\alpha))} |S_{T_c} - \mu T_c| \right)^p, c \geqslant 1 \right\} \quad \text{is uniformly integrable.}$$

By Lemma 3.2 of [7], $\lim_{c \to \infty} c^{-1/(1-\alpha)} E X_{T_c}^p = 0$. Since $p \geqslant 2$, this implies that

$$(4.3) \qquad E\left\{ c^{-1/(2(1-\alpha))} X_{T_c} \right\}^p \to 0 \quad \text{as} \quad c \to \infty.$$

We note that

$$(4.4) \qquad 0 \leqslant S_{T_c} - cT_c^{\alpha} \leqslant S_{T_c - 1} - c(T_c - 1)^{\alpha} + X_{T_c} \leqslant X_{T_c}.$$

Moreover,

$$(4.5) \qquad T_c - (c/\mu) T_c^{\alpha} = \mu^{-1} \{ (\mu T_c - S_{T_c}) + (S_{T_c} - cT_c^{\alpha}) \}.$$

From (4.2), (4.3), (4.4) and (4.5), the desired uniform integrability of $\{c^{-p/(2(1-\alpha))} |T_c - (c/\mu) T_c^{\alpha}|^p, c \geqslant 1\}$ follows.

REMARK. When $\alpha = 0$, Theorem 1(i) says that the family $\{c^{-p/2}|T_c - c/\mu|^p, c \geqslant 1\}$ is uniformly integrable. Thus for classical renewal theory ($\alpha = 0$), the L_p analogue of Anscombe's theorem has provided us with a simple proof of the desired uniform integrability (1.7). To obtain (1.7) for extended renewal theory ($0 \leqslant \alpha < 1$), we shall make use of Theorem 1(i) and the following lemmas.

LEMMA 2. *Let* $0 \leqslant \alpha < 1$. *Then for all* $c > 0$ *and* $x \geqslant 0$,

$$(4.6) \qquad t \geqslant c^{1/(1-\alpha)} + c^{1/(2(1-\alpha))} x \Rightarrow t - ct^{\alpha} \geqslant (1-\alpha) c^{1/(2(1-\alpha))} x.$$

PROOF. Define $f(t) = t - ct^{\alpha}$, $t \geqslant 0$. Then $f'(t) > 0$ iff $t > (c\alpha)^{1/(1-\alpha)}$. Hence the implication (4.6) holds if

$$(4.7) \qquad \{c^{1/(1-\alpha)} + c^{1/(2(1-\alpha))} x\} - c\{c^{1/(1-\alpha)} + c^{1/(2(1-\alpha))} x\}^{\alpha} \geqslant (1-\alpha) c^{1/(2(1-\alpha))} x.$$

To prove (4.7), set $A = c^{1/(1-\alpha)}$, $\rho A = c^{1/(2(1-\alpha))}x$. Then (4.7) can be rewritten as

$$A + \alpha\rho A - A^{1-\alpha}(A + \rho A)^{\alpha} \geqslant 0. \tag{4.8}$$

Since $0 \leqslant \alpha < 1$, it is easy to see that $1 + \alpha\rho \geqslant (1 + \rho)^{\alpha}$ for all $\rho \geqslant 0$. Hence (4.8) holds.

LEMMA 3. *Given $0 \leqslant \alpha < 1$ and $0 < \gamma < 1 - \alpha$, there exists $0 < \theta < 1$ such that for all $c > 0$ and $0 \leqslant x \leqslant \theta c^{1/(2(1-\alpha))}$,*

$$(1-\theta)c^{1/(1-\alpha)} \leqslant t \leqslant c^{1/(1-\alpha)} - c^{1/(2(1-\alpha))}x \Rightarrow t - ct^{\alpha} \leqslant -c^{1/(2(1-\alpha))}\gamma x. \tag{4.9}$$

PROOF. Choose $0 < \theta < 1$ such that $1 - \theta > \alpha^{1/(1-\alpha)}$ and

$$(1-u)^{\alpha} > 1 - (1-\gamma)u \quad \text{for} \quad 0 \leqslant u \leqslant \theta. \tag{4.10}$$

Since $f'(t) > 0$ if $t \geqslant (1-\theta)c^{1/(1-\alpha)}(> (c\alpha)^{1/(1-\alpha)})$, where $f(t) = t - ct^{\alpha}$, it therefore suffices to show that for $0 \leqslant x \leqslant \theta c^{1/(2(1-\alpha))}$,

$$\{c^{1/(1-\alpha)} - c^{1/(2(1-\alpha))}x\} - c\{c^{1/(1-\alpha)} - c^{1/(2(1-\alpha))}x\}^{\alpha} \leqslant -c^{1/(2(1-\alpha))}\gamma x. \tag{4.11}$$

But this follows easily from (4.10).

LEMMA 4. *Let $X, X_1, X_2, \cdots$ be* i.i.d. *such that $EX = \mu > 0$ and $E|X|^r < \infty$ for some $1 < r \leqslant 2$. For $0 \leqslant \alpha < 1$ and $c > 0$, define T_c as in* (1.1). *Let $q \geqslant r$.*

(i) *Assume that*

$$P[X \geqslant x] = o(x^{-q}) \quad \text{as} \quad x \to \infty. \tag{4.12}$$

Then for every $0 < \theta < 1$, as $c \to \infty$,

$$\begin{aligned} P[T_c \leqslant (1-\theta)(c/\mu)^{1/(1-\alpha)}] &= o(c^{-q}) && \text{if} \quad \alpha q > 1, \\ &= o(c^{-q}\log c) && \text{if} \quad \alpha q = 1, \\ &= o(c^{-(q-1)/(1-\alpha)}) && \text{if} \quad \alpha q < 1. \end{aligned} \tag{4.13}$$

(ii) *For the case $\alpha q = 1$ above, if we replace the condition* (4.12) *by the stronger condition*

$$\lim_{x\to\infty}\int_x^{x^{\rho}} u^{q-1}P[X \geqslant u]\,du = 0 \quad \text{for some} \quad \rho > 1, \tag{4.14}$$

then we can sharpen (4.13) (*for the case $\alpha q = 1$*) *as*:

$$P[T_c \leqslant (1-\theta)(c/\mu)^{1/(1-\alpha)}] = o(c^{-q}). \tag{4.15}$$

REMARK. One of our subsequent applications of Lemma 4 is when $q \geqslant 2 = r$ and $E|X|^q < \infty$. In this case we note that (4.12) and (4.14) are automatically satisfied.

PROOF. Without loss of generality, we can assume that $1 < r < 1/\alpha$. Set $n = [(1-\theta)(c/\mu)^{1/(1-\alpha)}]$ and $\varepsilon = (1-\theta)^{-(1-\alpha)} - 1$ (> 0). Then $c \geqslant (1 + \varepsilon)\mu n^{1-\alpha}$ and

$$(4.16)\quad P\big[T_c \leqslant (1-\theta)(c/\mu)^{1/(1-\alpha)}\big]$$
$$= P\big[\max_{j\leqslant n} j^{-\alpha}S_j > c\big]$$
$$\leqslant P\big[\max_{j\leqslant n} j^{-\alpha}(S_j - \mu j) + \mu n^{1-\alpha} > (1+\varepsilon)\mu n^{1-\alpha}\big]$$
$$= P\big[\max_{j\leqslant n} j^{-\alpha}(S_j - \mu j) > \varepsilon\mu n^{1-\alpha}\big].$$

To handle the last term in (4.16), we apply Lemma 1 with $x = \varepsilon\mu n^{1-\alpha} \sim \varepsilon(1-\theta)^{1-\alpha}c$. By choosing the integer k in Lemma 1 large enough, it then suffices to show that $P[\max_{j\leqslant n} j^{-\alpha}(X_j - \mu) > x/(2k)]$ satisfies the right-hand side of (4.13) (respectively (4.15)) under the condition (4.12) (respectively (4.14)).

Let $Y_j = X_j - \mu$ and let $y = x/(2k)$. Assume (4.12). Then by (2.2),

$$(4.17)\quad P\big[\max_{j\leqslant n} j^{-\alpha}Y_j \geqslant y\big] \leqslant \Sigma_{j=1}^n P\big[Y \geqslant yj^\alpha\big] = o\big(y^{-q}\Sigma_{j=1}^n j^{-\alpha q}\big).$$

Since $y \sim \varepsilon(1-\theta)^{1-\alpha}c/(2k)$ and $n \sim (1-\theta)(c/\mu)^{1/(1-\alpha)}$, (4.17) implies that $P[\max_{j\leqslant n} j^{-\alpha}Y_j \geqslant y]$ satisfies the right-hand side of (4.13).

Now assume that $\alpha q = 1$ and that (4.14) holds. Obviously (4.14) implies that for $k = 2, 3, \cdots$,

$$\int_x^{x(\rho^k)} u^{q-1}P\big[X \geqslant u\big]\,du = \int_{x(\rho^{k-1})}^{x(\rho^k)} + \cdots + \int_{x^\rho}^{x(\rho^2)} + \int_x^{x^\rho} = o(1).$$

Hence (4.14) implies that

$$(4.18)\qquad \lim_{x\to\infty}\int_x^{x\rho} u^{q-1}P\big[X \geqslant u\big]\,du = 0 \qquad \text{for all}\quad \rho > 1.$$

By (2.2) and (4.18),

$$P\big[\max_{j\leqslant n} j^{-\alpha}Y_j \geqslant y\big] \leqslant \Sigma_{j=1}^n P\big[Y \geqslant yj^\alpha\big] \leqslant \int_1^{n+1} P\big[Y \geqslant \tfrac{1}{2}yt^\alpha\big]\,dt$$
$$= 0\big(y^{-1/\alpha}\int_{y/2}^{yn^\alpha} u^{(1/\alpha)-1}P\big[Y \geqslant u\big]\,du\big)$$
$$= o(c^{-q}) \qquad \text{as}\quad c \to \infty,\ \text{since}\ q = 1/\alpha.$$

PROOF OF THEOREM 1(ii)–(iii). To prove (1.7), we make use of Theorem 1(i). Without loss of generality, we shall assume that $\mu = 1$. By Lemma 2, for all $c > 0$ and $x \geqslant 0$,

$$(4.19)\qquad P\big[T_c - c^{1/(1-\alpha)} \geqslant c^{1/(2(1-\alpha))}x\big] \leqslant P\big[T_c - cT_c^\alpha \geqslant (1-\alpha)c^{1/(2(1-\alpha))}x\big].$$

Take $0 < \gamma < 1 - \alpha$ and choose $0 < \theta < 1$ as in Lemma 3. Then by (4.9), for all $c > 0$ and $0 \leqslant x \leqslant \theta c^{1/(2(1-\alpha))}$,

$$(4.20)\qquad P\big[T_c - c^{1/(1-\alpha)} \leqslant -c^{1/(2(1-\alpha))}x\big] \leqslant P\big[T_c \leqslant (1-\theta)c^{1/(1-\alpha)}\big]$$
$$+ P\big[T_c - cT_c^\alpha \leqslant -c^{1/(2(1-\alpha))}\gamma x\big].$$

Obviously,

$$(4.21)\qquad P\big[T_c - c^{1/(1-\alpha)} \leqslant -c^{1/(2(1-\alpha))}x\big] = 0 \qquad \text{if } x > c^{1/(2(1-\alpha))},$$
$$\leqslant P\big[T_c \leqslant (1-\theta)c^{1/(1-\alpha)}\big] \qquad \text{if } x \geqslant \theta c^{1/(2(1-\alpha))}.$$

In view of (4.19), (4.20), (4.21) and the uniform integrability of $\{c^{-p/(2(1-\alpha))}|T_c - cT_c^{\alpha}|^p, c \geqslant 1\}$ by Theorem 1(i), (1.7) would hold if it can be shown that as $c \to \infty$,

$$c^{p/(2(1-\alpha))}P\left[T_c \leqslant (1-\theta)c^{1/(1-\alpha)}\right] \to 0. \tag{4.22}$$

(For then given $\eta > 0$, there exists A large enough such that the left-hand side of (4.22) is $\leqslant \eta$ if $c \geqslant A$. On the other hand, if $c < A$, then $P[T_c - c^{1/(1-\alpha)} \leqslant -c^{1/(2(1-\alpha))}x] = 0$ for $x \geqslant A^{1/(2(1-\alpha))}$.)

To prove (4.22), for the case $\alpha = \frac{1}{2}$ and $p = 2$ in Theorem 1(ii), we apply Lemma 4(ii) with $q = r = 2$. For the other cases, we apply Lemma 4(i) with $r = 2$ and

(a) $q = p/(2(1-\alpha))$ for Theorem 1(iii), noting that here $\alpha q > 1$;

(b) $q = p$ for the case $\alpha < \frac{1}{2}$ in Theorem 1(ii), noting that $2(1-\alpha) > 1$ and $p - 1 \geqslant p/2$;

(c) $q = p$ for the case $\alpha = \frac{1}{2}$ and $p > 2$ in Theorem 1(ii), noting that here $\alpha q > 1$.

5. Moment convergence in the theorems of Teicher and of Erdös and Kac; and some related maximal inequalities. In this section, we first consider moment convergence in Teicher's theorem (1.18). As shown by Teicher [15], (1.18) is actually a consequence of the central limit theorem (1.5) for T_c. In fact,

$$P\left[\max_{j \leqslant n} j^{-\alpha}S_j > \mu n^{1-\alpha} + n^{\frac{1}{2}-\alpha}x\right] = P\left[T_c \leqslant n\right], \tag{5.1}$$

where

$$c = \mu n^{1-\alpha} + n^{\frac{1}{2}-\alpha}x. \tag{5.2}$$

For fixed real x, the relation (5.2) implies that as $c \to \infty$,

$$n = (c/\mu)^{1/(1-\alpha)} - \{(1-\alpha)\mu\}^{-1}(x + o(1))(c/\mu)^{1/(2(1-\alpha))}. \tag{5.3}$$

From (1.5), (5.1) and (5.3), (1.18) follows.

While the asymptotic expansion (5.3) holds for fixed x, it obviously does not hold uniformly in large c as $x \to \infty$. Hence although we have established moment convergence in the central limit theorem (1.5) for T_c, the corresponding moment convergence in Teicher's central limit theorem (1.18) for $\max_{j \leqslant n} j^{-\alpha}S_j$ needs further study. Making use of the basic maximal inequality (2.1), we are able to obtain the following counterpart of Theorem 1 concerning moment convergence for Teicher's theorem.

THEOREM 3. *Let $X, X_1, X_2, \cdots$ be* i.i.d. *random variables with $EX = \mu > 0$ and* Var $X = \sigma^2 > 0$ *and let $S_n = X_1 + \cdots + X_n$. Let $0 \leqslant \alpha < 1$ and $p \geqslant 2$. Assume that $E|X|^p < \infty$.*

(i) *If $\alpha < \frac{1}{2}$, then*

$$\left\{\left|\left(\max_{j \leqslant n} j^{-\alpha}S_j - \mu n^{1-\alpha}\right)/n^{\frac{1}{2}-\alpha}\right|^p, n \geqslant 1\right\} \quad \textit{is uniformly integrable.} \tag{5.4}$$

Consequently, as $n \to \infty$,

$$E(\max_{j \leqslant n} j^{-\alpha} S_j) = \mu n^{1-\alpha} + o(n^{\frac{1}{2}-\alpha}); \tag{5.5}$$

$$\operatorname{Var}(\max_{j \leqslant n} j^{-\alpha} S_j) \sim \sigma^2 n^{1-2\alpha}. \tag{5.6}$$

(ii) *If $\alpha = \frac{1}{2}$ and $p > 2$, then* (5.4), (5.5) *and* (5.6) *again hold. For $\alpha = \frac{1}{2}$ and $p = 2$,* (5.4), (5.5) *and* (5.6) *still hold under the further assumption*

$$\int_x^\infty u\, P[X \geqslant u]\, du = o((\log x)^{-1}) \qquad \text{as } x \to \infty. \tag{5.7}$$

(iii) *If $\frac{1}{2} < \alpha < 1$, then* (5.4), (5.5) *and* (5.6) *still hold under the further assumption* (1.10).

Remark. Setting $n = 1$ in (5.4), it is clear that the uniform integrability result (5.4) implies $E|X|^p < \infty$. We now show that (1.10) is also a necessary condition for (5.4) to hold in the case $\frac{1}{2} < \alpha < 1$. Clearly (5.4) implies that as $n \to \infty$,

$$\begin{aligned} o(1) &= \int_{n^{1/2}}^{\infty} x^{p-1} P\left[\max_{j \leqslant n} j^{-\alpha} S_j \geqslant \mu n^{1-\alpha} + n^{\frac{1}{2}-\alpha} x\right] dx \\ &\geqslant \int_{n^{1/2}}^{2n^{1/2}} x^{p-1} P\left[X_1 \geqslant \mu n^{1-\alpha} + n^{\frac{1}{2}-\alpha} x\right] dx \\ &\geqslant p^{-1}(2^p - 1) n^{p/2} P\left[X \geqslant (\mu + 2) n^{1-\alpha}\right], \end{aligned}$$

and so (1.10) holds. The condition (5.7) for the boundary case $\alpha = \frac{1}{2}$ and $p = 2$ is slightly stronger than the moment condition $E(X^+)^2 < \infty$ but is weaker than $E(X^+)^2 \log X^+ < \infty$. The reason why we need it will be evident from the following proof (see (5.16) below).

Proof. We need only prove (5.4), since (5.5) and (5.6) are immediate consequences of (5.4) and Teicher's theorem. Define

$$\tilde{S}_n = (\max_{j \leqslant n} j^{-\alpha} S_j - \mu n^{1-\alpha}) / n^{\frac{1}{2}-\alpha}. \tag{5.8}$$

Let $Y_i = X_i - \mu$ and $U_n = \Sigma_1^n Y_i$. The proof of the uniform integrability of $(\tilde{S}_n^-)^p$ is easy, since for $x \geqslant 0$,

$$P[\tilde{S}_n \leqslant -x] \leqslant P\left[n^{-\alpha} S_n \leqslant \mu n^{1-\alpha} - x n^{\frac{1}{2}-\alpha}\right] = P\left[n^{-\frac{1}{2}} U_n \leqslant -x\right],$$

and $E|Y_1|^p < \infty$ implies that $\{(n^{-\frac{1}{2}}|U_n|)^p, n \geqslant 1\}$ is uniformly integrable (as can be easily shown by applying Theorem 2(ii) with $M(b) = b = n$).

To prove the uniform integrability of $(\tilde{S}_n^+)^p$, first consider the case $\frac{1}{2} < \alpha < 1$. Let $q = p/(2(1-\alpha))$. Then $\alpha q > 1$. Take $0 < \gamma < 1$. For $x \geqslant 0$,

$$\begin{aligned} &P\left[\max_{\gamma n \leqslant j \leqslant n} j^{-\alpha} S_j \geqslant \mu n^{1-\alpha} + x n^{\frac{1}{2}-\alpha}\right] \\ &\qquad \leqslant P\left[\max_{\gamma n \leqslant j \leqslant n} j^{-\alpha} U_j \geqslant x n^{\frac{1}{2}-\alpha}\right] \\ &\qquad \leqslant P\left[n^{-\frac{1}{2}} \max_{j \leqslant n} U_j \geqslant \gamma^\alpha x\right]. \end{aligned} \tag{5.9}$$

Since $\{(n^{-\frac{1}{2}}|\max_{j \leqslant n} U_j|)^p, n \geqslant 1\}$ is uniformly integrable (see Theorem 4(ii) below), it suffices to show that

$$(5.10)\qquad \lim_{n\to\infty}\int_1^\infty x^{p-1}P\Big[\max_{j\leqslant\gamma n} j^{-\alpha}S_j \geqslant \mu n^{1-\alpha} + xn^{\frac{1}{2}-\alpha}\Big]\,dx = 0.$$

To prove (5.10), we note that for $1 \leqslant x \leqslant n^{\frac{1}{2}}$,

$$(5.11)\qquad \begin{aligned} P\Big[\max_{j\leqslant\gamma n} j^{-\alpha}S_j \geqslant \mu n^{1-\alpha} + xn^{\frac{1}{2}-\alpha}\Big] &\leqslant P\Big[\max_{j\leqslant\gamma n} j^{-\alpha}S_j > \mu n^{1-\alpha}\Big] \\ &= P\big[T_{\mu n^{1-\alpha}} \leqslant \gamma n\big] = o(n^{-q(1-\alpha)}) = o(n^{-p/2}), \end{aligned}$$

by (4.13) (for the case $\alpha q > 1$). Thus

$$\lim_{n\to\infty}\int_1^{n^{\frac{1}{2}}} x^{p-1}P\Big[\max_{j\leqslant\gamma n} j^{-\alpha}S_j \geqslant \mu n^{1-\alpha} + xn^{\frac{1}{2}-\alpha}\Big]\,dx = 0.$$

Also a change of variable $x = n^{\frac{1}{2}}t$ gives

$$(5.12)\qquad \begin{aligned} &\int_{n^{1/2}}^\infty x^{p-1}P\Big[\max_{j\leqslant\gamma n} j^{-\alpha}S_j \geqslant \mu n^{1-\alpha} + xn^{\frac{1}{2}-\alpha}\Big]\,dx \\ &\leqslant n^{p/2}\int_1^\infty t^{p-1}P\Big[\max_{j\leqslant\gamma n} j^{-\alpha}U_j \geqslant n^{1-\alpha}t\Big]\,dt = o(1). \end{aligned}$$

To see the last relation above, by choosing k large enough and taking $1 < r < 1/\alpha$ in the maximal inequality (2.1), we need only show that for every $\varepsilon > 0$,

$$(5.13)\qquad \int_1^\infty t^{p-1}P\Big[\max_{j\leqslant\gamma n} j^{-\alpha}Y_j \geqslant \varepsilon n^{1-\alpha}t\Big]\,dt = o(n^{-p/2}).$$

By (2.2) and (1.10),

$$(5.14)\qquad \begin{aligned} \int_1^\infty t^{p-1}P\Big[\max_{j\leqslant n} j^{-\alpha}Y_j \geqslant \varepsilon n^{1-\alpha}t\Big]\,dt &\leqslant \sum_{j=1}^n\int_1^\infty t^{p-1}P\big[Y_1 \geqslant \varepsilon n^{1-\alpha}tj^{\alpha}\big]\,dt \\ &= o\Big(n^{-q(1-\alpha)}\sum_{j=1}^n j^{-\alpha q}\int_1^\infty t^{p-1-q}\,dt\Big). \end{aligned}$$

Since $\alpha q > 1$ and $q > p$, (5.13) follows from (5.14). Hence we have proved (iii).

To prove the uniform integrability of $(\tilde{S}_n^+)^p$ for part (ii) of the theorem, when $\alpha = \frac{1}{2}$ and $p > 2$, we note that $\alpha p > 1$ and we need only modify (5.14) as follows: by (2.2),

$$(5.15)\qquad \begin{aligned} &\int_1^\infty t^{p-1}P\Big[\max_{j\leqslant n} j^{-\frac{1}{2}}Y_j \geqslant \varepsilon n^{\frac{1}{2}}t\Big]\,dt \\ &\leqslant \sum_{j=1}^n\int_1^\infty t^{p-1}P\big[Y_1 \geqslant \varepsilon n^{\frac{1}{2}}tj^{\frac{1}{2}}\big]\,dt \\ &\leqslant \big(\varepsilon n^{\frac{1}{2}}\big)^{-p}\sum_{j=1}^n j^{-p/2}\int_{\varepsilon n^{1/2}}^\infty u^{p-1}P\big[Y_1 \geqslant u\big]\,du \ \Big(\text{setting } u = \varepsilon n^{\frac{1}{2}}tj^{\frac{1}{2}}\Big) \\ &= o(n^{-p/2}), \text{ since } E|Y_1|^p < \infty \text{ and } p > 2. \end{aligned}$$

When $\alpha = \frac{1}{2}$ and $p = 2$, (5.11) still holds by Lemma 4(ii) (setting $q = 2$) and we can modify (5.15) as follows:

$$(5.16)\qquad \begin{aligned} \int_1^\infty tP\Big[\max_{j\leqslant n} j^{-\frac{1}{2}}Y_j \geqslant \varepsilon n^{\frac{1}{2}}t\Big]\,dt &\leqslant \big(\varepsilon n^{\frac{1}{2}}\big)^{-2}(1 + \log n)\int_{\varepsilon n^{1/2}}^\infty uP\big[Y_1 \geqslant u\big]\,du \\ &= o(n^{-1}), \text{ by (5.7)}. \end{aligned}$$

From (5.15) or (5.16), (5.13) still holds in either case. Hence we have proved (ii).

To prove the uniform integrability of $(\tilde{S}_n^+)^p$ for part (i) of the theorem, we note that $\tilde{S}_n^+ \leqslant (\max_{j\leqslant n} j^{-\alpha}U_j)^+/n^{\frac{1}{2}-\alpha}$. Since $\alpha < \frac{1}{2}$ and $E|Y_1|^p < \infty$, the desired conclusion follows from Theorem 4 below.

THEOREM 4. *Let $Y, Y_1, Y_2, \cdots$ be* i.i.d. *random variables with $EY = 0$ and $EY^2 = \sigma^2$ and let $U_n = Y_1 + \cdots + Y_n$. Let $0 \leqslant \alpha < \frac{1}{2}$ and $p \geqslant 2$.*

(i) *Let $\{W(t), t \geqslant 0\}$ be the standard Wiener process and let $Z_\alpha = \sup_{0<t\leqslant 1} t^{-\alpha}W(t)$. Then as $n \to \infty$,*

$$(5.17) \qquad (\max_{j\leqslant n} j^{-\alpha}U_j)/n^{\frac{1}{2}-\alpha} \to_{\mathcal{L}} \sigma Z_\alpha.$$

(ii) *If $E|Y|^p < \infty$, then $(|\max_{j\leqslant n} j^{-\alpha}U_j|/n^{\frac{1}{2}-\alpha})^p$ is uniformly integrable, so* (i) *can be strengthened as*:

$$(5.18) \quad \lim_{n\to\infty} Ef\left(\max_{j\leqslant n} j^{-\alpha}U_j/n^{\frac{1}{2}-\alpha}\right) = Ef(\sigma Z_\alpha) \text{ for every continuous function } f: (-\infty, \infty) \to (-\infty, \infty) \text{ such that } f(x) = 0(|x|^p) \text{ as } x \to \infty.$$

The result (5.17) is a consequence of Donsker's invariance principle and can be regarded as the counterpart of Teicher's theorem (1.18) for driftless random walks. In this driftless case, however, we require $\alpha < \frac{1}{2}$ since $Z_\alpha = \infty$ a.s. for $\alpha \geqslant \frac{1}{2}$. Putting $\alpha = 0$ in (5.17), we obtain the well-known theorem of Erdös and Kac [5]. Thus (5.18) can be regarded as a refinement and generalization of the Erdös-Kac theorem.

To prove the uniform integrability result in Theorem 4(ii), since $(\max_{j\leqslant n} j^{-\alpha}U_j)^- \leqslant Y_1^-$, it suffices to show that $\{(\max_{j\leqslant n} j^{-\alpha}U_j)^+/n^{\frac{1}{2}-\alpha}\}^p$ is uniformly integrable. The maximal inequality (5.19) with $r = 2$ in the following more general lemma gives the desired result.

LEMMA 5. *Let $Y, Y_1, Y_2, \cdots$ be* i.i.d. *random variables with $EY = 0$ and let $U_n = Y_1 + \cdots + Y_n$. Let $\alpha \geqslant 0$ and $1 \leqslant r \leqslant 2$ such that $r\alpha < 1$. Let $p \geqslant r$ and set $k = [p/r] + 1$. Then there exists an absolute constant $A_{p,r,\alpha}$ (depending only on p, r, and α but not on the distribution of Y) such that for all $x > 0$ and $m \geqslant 1$,*

$$(5.19) \quad \begin{aligned} &\textstyle\int_x^\infty t^{p-1}\left\{\sup_{n\geqslant m} P\left[\max_{j\leqslant n} j^{-\alpha}U_j \geqslant n^{(1/r)-\alpha}t\right]\right\} dt \\ &\leqslant A_{p,r,\alpha}\left\{m^{-(p-r)/r}EY^pI_{[Y\geqslant x/(2k)]} + x^{p-kr}(E|Y|^r)^k\right\} \text{ if } p\alpha < 1, \\ &\leqslant A_{p,r,\alpha}\left\{m^{-p(1-r\alpha)/r}(1+\log m)EY^pI_{[Y\geqslant x/(2k)]} + x^{p-kr}(E|Y|^r)^k\right\} \text{ if } p\alpha = 1, \\ &\leqslant A_{p,r,\alpha}\left\{m^{-p(1-r\alpha)/r}EY^pI_{[Y\geqslant x/(2k)]} + x^{p-kr}(E|Y|^r)^k\right\} \text{ if } p\alpha > 1. \end{aligned}$$

Moreover, there exists an absolute constant $B_{p,r,\alpha}$ such that for all $n \geqslant 1$,

$$(5.20) \quad \begin{aligned} &E\left\{(\max_{j\leqslant n} j^{-\alpha}U_j)^+/n^{(1/r)-\alpha}\right\}^p \\ &\leqslant B_{p,r,\alpha}\left\{n^{-(p-r)/r}E(Y^+)^p + (E|Y|^r)^{p/r}\right\} \text{ if } p\alpha < 1, \\ &\leqslant B_{p,r,\alpha}\left\{n^{-p(1-r\alpha)/r}(1+\log n)E(Y^+)^p + (E|Y|^r)^{p/r}\right\} \text{ if } p\alpha = 1, \\ &\leqslant B_{p,r,\alpha}\left\{n^{-p(1-r\alpha)/r}E(Y^+)^p + (E|Y|^r)^{p/r}\right\} \text{ if } p\alpha > 1. \end{aligned}$$

PROOF. To prove (5.19), we apply the inequalities (2.1) and (2.2) to obtain

$$(5.21)\quad P\left[\max_{j\leqslant n} j^{-\alpha}U_j \geqslant n^{(1/r)-\alpha}t\right] \leqslant \{C_{r,\alpha}k^r t^{-r}E|Y|^r\}^k + \sum_{j=1}^n P\left[Y \geqslant n^{(1/r)-\alpha}tj^\alpha/(2k)\right].$$

Let $2^\nu \leqslant m < 2^{\nu+1}$ and note that

$$(5.22)\quad \begin{aligned} &\int_x^\infty t^{p-1}\{\sup_{n\geqslant m}\sum_{j=1}^n P\left[Y \geqslant n^{(1-r\alpha)/r}tj^\alpha/(2k)\right]\}\,dt \\ &\leqslant \sum_{i=\nu}^\infty \sum_{j=1}^{2^{i+1}} \int_x^\infty t^{p-1}P\left[2kY \geqslant 2^{i(1-r\alpha)/r}tj^\alpha\right]dt \\ &\leqslant \sum_{i=\nu}^\infty 2^{-ip(1-r\alpha)/r}\sum_{j=1}^{2^{i+1}} j^{-p\alpha}\int_x^\infty u^{p-1}P\left[2kY \geqslant u\right]du. \end{aligned}$$

To see the last relation above, use the change of variable $u = 2^{i(1-r\alpha)/r}j^\alpha t$. From (5.21) and (5.22), (5.19) follows.

To prove (5.20), let $\lambda = (E|Y|^r)^{1/r}$. We note that

$$(5.23)\quad \int_0^\lambda t^{p-1}P\left[\max_{j\leqslant n} j^{-\alpha}U_j \geqslant n^{(1/r)-\alpha}t\right]dt \leqslant \int_0^\lambda t^{p-1}\,dt = \lambda^p/p.$$

Since $E|Y|^r = \lambda^r$, we have

$$(5.24)\quad \lambda^{p-kr}(E|Y|^r)^k = \lambda^p.$$

From (5.19) (with $x = \lambda$) together with (5.23) and (5.24), (5.20) follows.

REMARK. Let $X, X_1, \cdots$ be i.i.d. with $EX = \mu > 0$ and let $S_n = X_1 + \cdots + X_n$. Define T_c as in (1.1), $\tilde{S}_n$ as in (5.8) and let

$$(5.25)\quad \tilde{T}_c = (c/\mu)^{-1/(2(1-\alpha))}\{T_c - (c/\mu)^{1/(1-\alpha)}\}.$$

The inequality (5.19) not only gives an immediate proof of the uniform integrability of $(\tilde{S}_n^+)^p$ in Theorem 3(i), but it also gives a simple alternative proof of the uniform integrability of $(\tilde{T}_c^-)^p$ for the case $\alpha < \frac{1}{2}$ in Theorem 1. To see this, we note that for $0 \leqslant t < (c/\mu)^{1/(2(1-\alpha))}$,

$$(5.26)\quad P\left[\tilde{T}_c^- \geqslant t\right] = P\left[T_c \leqslant (c/\mu)^{1/(1-\alpha)} - t(c/\mu)^{1/(2(1-\alpha))}\right] = P\left[T_c \leqslant h\right],$$

where

$$(5.27)\quad h = (c/\mu)^{1/(1-\alpha)} - t(c/\mu)^{1/(2(1-\alpha))}.$$

It is easy to see that for $0 \leqslant t < (c/\mu)^{1/(2(1-\alpha))}$,

$$(5.28)\quad c - \mu h^{1-\alpha} \geqslant (1-\alpha)\mu t h^{\frac{1}{2}-\alpha}.$$

From (5.26) and (5.28), it follows that for $0 \leqslant t < (c/\mu)^{1/(2(1-\alpha))}$,

$$(5.29)\quad \begin{aligned} P\left[\tilde{T}_c^- \geqslant t\right] &= P\left[\max_{j\leqslant h} j^{-\alpha}S_j - \mu h^{1-\alpha} > c - \mu h^{1-\alpha}\right] \\ &\leqslant P\left[\max_{j\leqslant h} j^{-\alpha}(S_j - \mu j) > (1-\alpha)\mu t h^{\frac{1}{2}-\alpha}\right] \\ &\leqslant \sup_{n\geqslant 1} P\left[\max_{j\leqslant n} j^{-\alpha}(S_j - \mu j)/(\mu(1-\alpha)) > t n^{\frac{1}{2}-\alpha}\right]. \end{aligned}$$

Hence setting $r = 2$ (with $\alpha < \frac{1}{2}$ and $p \geqslant 2$) in (5.19)8 we obtain from (5.29) the uniform integrability of $(\tilde{T}_c^-)^p$ for the case $\alpha < \frac{1}{2}$ in Theorem 1.

Another easy corollary of the inequalities (5.19) and (5.29) is the following interesting analogue of (5.20) for $\tilde{T}_c^-$; given $0 \leqslant \alpha < \frac{1}{2}$ and $p \geqslant 2$, there exists an absolute constant $D_{p,\alpha}$ (depending only on p and α but not on the distribution of X) such that for all $c > 0$,

$$E(\tilde{T}_c^-)^p \leqslant D_{p,\alpha}\mu^{-p}\{E((X-\mu)^+)^p + (\operatorname{Var} X)^{p/2}\}. \tag{5.30}$$

To prove (5.30), let $Y = (X-\mu)/(\mu(1-\alpha))$, $\lambda = (EY^2)^{\frac{1}{2}}$, and note that

$$\int_0^\lambda t^{p-1}P[\tilde{T}_c^- \geqslant t]\,dt \leqslant \int_0^\lambda t^{p-1}\,dt = \lambda^p/p.$$

From (5.19) (with $x = \lambda$, $m = 1$ and $r = 2$), (5.24) and (5.29), we obtain that

$$\int_\lambda^\infty t^{p-1}P[\tilde{T}_c^- \geqslant t]\,dt \leqslant A_{p,\alpha}\{E(Y^+)^p + \lambda^p\}.$$

Hence (5.30) follows.

The maximal inequality (5.20) holds for all n and involves a universal constant $B_{p,r,\alpha}$ and only $E(Y^+)^p$ and $(E|Y|^r)^{p/r}$. In the case $r = 2$, it is in some sense the sharpest possible. When $n = 1$, the left-hand side of (5.20) reduces to $E(Y^+)^p$. On the other hand, as $n \to \infty$, if $E(Y^+)^p < \infty$, then Theorem 4(ii) implies that

$$E\left\{(\max_{j\leqslant n} j^{-\alpha}U_j)^+/n^{\frac{1}{2}-\alpha}\right\}^p \to B_{p,\alpha}^*(EY^2)^{p/2},$$

where $B_{p,\alpha}^* = E(\sup_{0<t\leqslant 1} t^{-\alpha}W(t))^p$ and $\{W(t),\ t \geqslant 0\}$ is the standard Wiener process.

REFERENCES

[1] Anscombe, F. J. (1952). Large-sample theory of sequential estimation. *Proc. Cambridge Philos. Soc.* **48** 600–607.

[2] Chow, Y. S. (1966). On the moments of some one-sided stopping rules. *Ann. Math. Statist.* **37** 382–387.

[3] Chow, Y. S. and Lai, T. L. (1975). Some one-sided theorems on the tail distribution of sample sums with applications to the last time and largest excess of boundary crossings. *Trans. Amer. Math. Soc.* **208** 51–72.

[4] Chow, Y. S. and Lai, T. L. (1978). Paley-type inequalities and convergence rates related to the law of large numbers and extended renewal theory. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **45** 1–19.

[5] Erdös, P. and Kac, M. (1946). On certain limit theorems of the theory of probability. *Bull. Amer. Math.* **52** 292–302.

[6] Feller, W. (1949). Fluctuation theory of recurrent events. *Trans. Amer. Math. Soc.* **67** 98–119.

[7] Gut, A. (1974). On the moments and limit distributions of some first passage times. *Ann. Probability* **2** 277–308.

[8] Heyde, C. C. (1967). Asymptotic renewal results for a natural generalization of classical renewal theory. *J. Roy. Statist. Soc. Ser. B* **29** 141–150.

[9] Lai, T. L. (1975). On uniform integrability in renewal theory. *Bull. Inst. Math. Acad. Sinica* **3** 99–105.

[10] Loève, M. (1963). *Probability Theory*, 3rd ed. Van Nostrand, Princeton.

[11] Siegmund, D. (1967). Some one-sided stopping rules. *Ann. Math. Statist.* **38** 1641–1646.

[12] Siegmund, D. (1968). On the asymptotic normality of one-sided stopping rules. *Ann. Math. Statist.* **39** 1493–1497.

[13] Siegmund, D. (1969). The variance of one-sided stopping rules. *Ann. Math. Statist.* **40** 1074–1077.

[14] Smith, W. L. (1954). Asymptotic renewal theorems. *Proc. Roy. Soc. Edinburgh Sect. A* **64** 9–48.

[15] Teicher, H. (1973). A classical limit theorem without invariance or reflection. *Ann. Probability* **1** 702–704.

[16] Woodroofe, M. (1976). A renewal theorem for curved boundaries and moments of first passage times. *Ann. Probability* **4** 67–80.

Department of Mathematical Statistics
Columbia University
New York, New York 10027