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A COMPARISON OF STOCHASTIC INTEGRALS[1]

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Two different stochastic integrals have been developed during the last ten years. One is largely associated with the work of E. J. McShane (the star integral), and the other has grown out of the work of H. Kunita and S. Watanabe (the dot integral). Assuming the customary conditions that guarantee the existence of the star integral, we give a formula relating the two integrals. We show that the star integral is equal to the dot integral provided one takes a projection of the integrand onto the space of predictable processes before evaluating the dot integral. This essentially embeds the theory of the star integral into that of the dot integral.

1. Introduction. During the last ten years two stochastic integrals have developed separately. McShane has developed the Itô-belated integral in a series of papers culminating in his book [8]. See especially the account given in [9]. For convenience we denote this integral as the *star integral* and we write $H * Z$ for the Itô-belated integral of H with respect to Z. A second integral known as "the stochastic integral" has been developed from the work of Kunita and Watanabe [6], and we call this the *dot integral*. We write $H \cdot Z$ for the dot integral of H with respect to Z. A brief history of the development of this integral is given in the comprehensive treatment by Meyer [11] where references to the many contributors may be found. In this paper we establish the relationship between the two integrals.

Let H be a locally bounded jointly measurable process and let Z satisfy the $K\Delta t$-condition after small amendments, which implies that Z is a semimartingale (see Section 2 for definitions of these terms; $K\Delta t$ and $K\Delta t$ after small amendments are taken from McShane [8 or 9] and are given in Section 3, with one small change). Let ${}^3H$ denote the previsible projection of H. In Theorem (4.9) and Corollary (5.19) we show

$$H * Z = {}^3H \cdot Z \tag{1.1}$$

$$H * ZX = {}^3H \cdot [Z, X] \tag{1.2}$$

where X is also $K\Delta t$ after small amendments, $H * ZX$ is the second-order Itô-belated integral, $[Z, X]$ is the quadratic variation process of Z and X, and where equality of processes means indistinguishability.

Because of the separate development of the star and dot integrals there has been a certain duplication of effort; for example, the relations (1.1) and (1.2) which we

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establish here combined with the previous work in [13, 14] and the results of Doléans-Dade [2] and Doléans-Dade and Meyer [4] show that Theorem (11.2) of [9] actually holds under weaker hypothesis.

At first glance the star and dot integrals do not seem comparable, since the dot integral is defined only for semimartingales, while the star integral is defined for any process for which a limit of sums exists. However, we show in Section 3 that the sufficient conditions given by McShane for the existence of the star integral also imply that the differentials in the star integral are semimartingales. Hence we extend a result of Pop-Stojanovic [12] to the case where the paths of the differentials need not be continuous. The principal results, which establish (1.1) and (1.2), are given in Theorem (4.9) and Corollary (5.19).

2. Preliminaries. We assume that the reader is familiar with both the star (Itô-belated) integral as given in [9] and the dot integral as given in, for example, [11]. We shall make use of the "general theory of processes" as set forth in Dellacherie [1].

We assume throughout the paper that $(\Omega, \mathcal{F}, p)$ is a complete probability space and $(\mathcal{F}_t)_{t \geqslant 0}$ is a complete, right-continuous increasing family of σ-algebras with $\mathcal{F} = \mathcal{F}_\infty = V\mathcal{F}_t$. All stochastic processes are assumed to be adapted to $(\mathcal{F}_t)$. A process Z which satisfies a $K\Delta t$-condition or satisfies a $K\Delta t$-condition after small amendments (precise definitions are given in (3.1) and (3.2)) a.s. has paths with right and left limits. By the assumptions on $(\mathcal{F}_t)$ Z then has a version which is *càdlàg* (right continuous with left limits). *We always assume that all differential processes Z, X are càdlàg and moreover that $Z_0 = X_0 = 0$.*

A jointly measurable process H is said to be *locally bounded* if there exist stopping times (T^n) increasing to ∞ a.s. such that $(H_t^{T^n})_{t \geqslant 0}$ is adapted to $(\mathcal{F}_{t \wedge T^n})_{t \geqslant 0}$ and $\sup_t |H_t^{T^n}| \leqslant K_n$ a.s., where K_n is a constant. For locally bounded jointly measurable H we let $H * Z$ be the *first order Itô-belated* integral, which is usually denoted $\int_0^t H(s)dZ(s)$. We let $H * ZX$ be the *second order Itô-belated integral*, usually denoted $\int_0^t H(s)dZ(s)dX(s)$. A process Z is a *semimartingale* if it can be written $Z = M + A$ where M is a local martingale and A is an adapted càdlàg process which has paths a.s. of bounded variation on compact time intervals. If Z is a semimartingale, the sums $\Sigma_{t_i \in \mathcal{P}^n}(Z_{t_{i+1}} - Z_{t_i})^2$ converge in probability as the mesh of the partitions $\mathcal{P}^n$ of $[0, t]$ tends to 0. The limit is denoted $[Z, Z]_t$, and a càdlàg version can be chosen. If $[Z, Z]_t \in L^1$ for each t, it has a dual previsible projection (cf. [1], page 107) and we denote it $(\langle Z, Z \rangle_t)_{t \geqslant 0}$. We let $\mathfrak{M}$ denote the class of square integrable martingales with $M_0 = 0$ and $\mathfrak{M}_{\text{loc}}$ the class of locally square integrable martingales with $M_0 = 0$. Let $\mathcal{Q}$ be the family of adapted càdlàg processes (A_t) which a.s. have paths of bounded variation on compact intervals with $A_0 = 0$. If H is a bounded jointly measurable process, ${}^3H$ denotes its previsible projection and, if $(A_t) \in \mathcal{Q}$ which satisfies $A_t \in L^1$ for each t then we denote by A^3 the dual previsible projection of A. Following Dellacherie [1], we let $((0, T)) = \{(t, \omega) : 0 < t < T(\omega)\}$ denote the stochastic interval. For a càdlàg process X we set $\Delta X_s = X_s - X_{s-}$, where X_{s-} is the left limit. For fixed ω the function $t \to Z_t(\omega)$ will be denoted $Z.(\omega)$. If Z is a semimartingale, it has a unique continuous local martingale part which we denote by Z^c. We follow the convention that $\Delta Z_s^2 = (\Delta Z_s)^2 = (Z_s - Z_{s-})^2$.

3. The $K\Delta t$ hypotheses. The existence of the star integral for general integrands was established by McShane ([8], [9]) under certain hypotheses on the differential processes. In this section we describe these conditions and we show in Theorem (3.9) that a process satisfying the $K\Delta t$-condition after small amendments is a quasi-left-continuous semimartingale. This will permit us to compare the star and the dot integral in Sections 4 and 5 and to establish (1.1) and (1.2). Definition (3.1) below of a $K\Delta t$-condition is taken from [8, 9].

(3.1) DEFINITION. An (adapted) process Z *satisfies a $K\Delta t$-condition on* $[a, b]$ if (i) $E(Z_a^2) < \infty$ and (ii) there exists a constant K such that for all u, v with $a \leqslant u \leqslant v \leqslant b$, then a.s.

(a) $|E\{Z_v - Z_u | \mathcal{F}_u\}| \leqslant K(v - u)$

(b) $E\{(Z_v - Z_u)^2 | \mathcal{F}_u\} \leqslant K(v - u)$.

A process satisfying (3.1) is called a $K\Delta t$ *process*.

(3.2) DEFINITION. An (adapted) process Z satisfies a *$K\Delta t$-condition after small amendments on* $[0, \infty)$ if there exist stopping times T^n increasing a.s. to ∞ and $K\Delta t$ processes Z^n such that for each n the processes $(Z_{t \wedge T^n})_{t \geqslant 0}$ and $(Z^n_{t \wedge T^n})_{t \geqslant 0}$ are indistinguishable.

Definition (3.2) above is slightly different than the one given by McShane in [8, 9] but achieves the same purpose and is more "natural" from the standpoint of the "general theory of processes."

Pop-Stojanovic [12] has shown that if a $K\Delta t$ process Z has continuous sample paths, then it is a semimartingale. The assumption of continuity of the paths is not necessary, however, as Lemma (3.4) shows.

(3.3) LEMMA. *If Z satisfies a $K\Delta t$-condition, then $(Z_t + Kt)$ is a submartingale and thus, without loss of generality, Z can be assumed to have right continuous paths with left limits ("càdlàg" paths).*

PROOF. The $K\Delta t$ condition implies that $E(|Z_t|) < \infty$ for each t. Part (a) of the $K\Delta t$-condition implies that $-K(t - s) \leqslant E\{Z_t - Z_s | \mathcal{F}_s\}$ for $s < t$ and hence $Z_t + Kt$ is a submartingale. Since we assume that the filtration $(\mathcal{F}_t)$ is complete and right continuous, and since $Z_t + Kt$ is a submartingale it is well known that a version $Z_t' + Kt$ can be chosen such that $Z_t' = Z_t$ a.s. each t and Z_t' has càdlàg paths.

We now establish two lemmas needed in the proof of Theorem (3.6), a theorem which shows that if a process Z satisfies a $K\Delta t$-condition, then it is a special semimartingale which is decomposable into the sum of two $K\Delta t$ processes, one of which is a locally square integrable martingale and the other is an adapted process with Lipschitz paths.

(3.4) LEMMA. *If Z satisfies a $K\Delta t$-condition then it is a special semimartingale.* (*See* [11], *page* 310, *for a discussion of special semimartingales.*)

PROOF. By Lemma (3.3), $Z_t + Kt$ is a submartingale. It is well known that a right continuous submartingale X has a unique decomposition $X = M + A$ where M is a local martingale and A is a previsible, increasing process which is locally integrable. Thus the submartingale $Z_t + Kt$ is a semimartingale, hence also Z_t is a semimartingale.

Since $Z_t + Kt = M_t + A_t$ with A previsible and locally integrable, $Z_t = M_t + (A_t - Kt)$ where $(A_t - Kt)$ is previsible and locally integrable. Thus Z_t is a special semimartingale and the lemma is proved.

(3.5) LEMMA. *If Z is $K\Delta t$, then $[Z, Z]_t \in L^1$, the paths of $\langle Z, Z\rangle$ are Lipschitz continuous, and Z is quasi-left-continuous.* (*See* [1], *page* 85 *for the definition of quasi-left-continuity.*)

PROOF. Fix t and let $\mathcal{P}^n$ be partitions of $[0, t]$ with $\lim_{n\to\infty}$ mesh $\mathcal{P}^n = 0$. Then $\Sigma_{t_j\in\mathcal{P}^n}(Z_{t_{j+1}} - Z_{t_j})^2$ converges in probability to $[Z, Z]_t$ (cf. [11], page 358). Let $\{n'\}$ be a sequence of integers such that the convergence is a.s. Then

$$\begin{aligned} E\{[Z, Z]_t\} &= E\left\{\lim_{n'}\Sigma\left(Z_{t_{j+1}} - Z_{t_j}\right)^2\right\} \\ &\leqslant \liminf \Sigma E\left\{E\left\{\left(Z_{t_{j+1}} - Z_{t_j}\right)^2 | \mathcal{F}_{t_j}\right\}\right\} \\ &\leqslant Kt. \end{aligned}$$

Thus $[Z, Z]_t \in L^1(dP)$ for each t. Let $\langle Z, Z\rangle_t$ denote its dual previsible projection. Since $[Z, Z]_t$ charges precisely the same stopping times as Z does, as is shown in [1], page 111, it suffices to show that $\langle Z, Z\rangle_t$ is continuous in order to establish the quasi-left-continuity of Z. Thus it remains to show only that $\langle Z, Z\rangle.$ is a.s. Lipschitz continuous. It is elementary that $\Sigma_{t_j\in\mathcal{P}^n}E\{(Z_{t_{j+1}} - Z_{t_j})^2|\mathcal{F}_{t_j}\}$ converges in $\sigma(L^1, L^\infty)$ to $\langle Z, Z\rangle_t$. We omit the details of the proof. However, Lepingle [7], page 314, has established the convergence for $1 < p \leqslant 2$ when $Z^c = 0$. For $p = 2$, one need not assume $Z^c = 0$. Fix s and t and let $\Lambda_{s,t} = \{\langle Z, Z\rangle_t - \langle Z, Z\rangle_s > K(t - s)\}$. If $P(\Lambda_{s,t}) > 0$, then $E\{1_{\Lambda_{s,t}}(\langle Z, Z\rangle_t - \langle Z, Z\rangle_s)\} > P(\Lambda_{s,t})K(t - s)$. But

$$\begin{aligned} E\left\{1_{\Lambda_{s,t}}(\langle Z, Z\rangle_t - \langle Z, Z\rangle_s)\right\} &= \lim E\left\{1_{\Lambda_{s,t}}\Sigma_{t_j\in\mathcal{P}^n}E\left\{\left(Z_{t_{j+1}} - Z_{t_j}\right)^2|\mathcal{F}_{t_j}\right\}\right\} \\ &\leqslant \lim E\left\{1_{\Lambda_{s,t}}K(t - s)\right\} = P(\Lambda_{s,t})K(t - s), \end{aligned}$$

a contradiction. (Here $\mathcal{P}^n$ are partitions of $(s, t]$.) Thus $P(\Lambda_{s,t}) = 0$. Letting $\Lambda = \cup_{s,t\in\mathbb{Q}}\Lambda_{s,t}$ and using the right continuity of $\langle Z, Z\rangle.$, we conclude that almost all the paths of $\langle Z, Z\rangle.$ are Lipschitz. This completes the proof.

(3.6) THEOREM. *If Z satisfies a $K\Delta t$-condition, then Z is a special semimartingale with decomposition $Z = M + A$ where M is a locally square-integrable martingale* *and A has adapted, Lipschitz continuous paths. Moreover, both M and A are themselves $K\Delta t$ processes.*

PROOF. By Lemma (3.4) we know Z is a special semimartingale. Let $Z = M + A$ be its canonical decomposition; that is, M is a local martingale, and A is previsible with paths of bounded variation. By Lemma (3.5) Z is quasi-left-continuous, and this implies that A has continuous paths: if A jumps at a time T, it must be previsible, and $A_T, A_{T-} \in \mathcal{F}_{T-}$. Since $E\{M_T|\mathcal{F}_{T-}\} = M_{T-}$, we have $\Delta A_T = -\Delta M_T$ and $\Delta A_T = E\{\Delta A_T|\mathcal{F}_{T-}\} = E\{\Delta M_T|\mathcal{F}_{T-}\} = 0$.

Since A is continuous, it is locally bounded, and $Z_t \in L^2(dP)$ by the $K\Delta t$ hypothesis. Thus $M = Z - A$ is locally square-integrable. We therefore have

$$\begin{aligned}[M, M]_t &= \langle M^c, M^c\rangle_t + \Sigma_{s\leqslant t}\Delta M_s^2 \\ &= \langle M^c, M^c\rangle_t + \Sigma_{s\leqslant t}\Delta Z_s^2 \\ &= [Z, Z]_t\end{aligned}$$

where we have used the continuity of A. We write $\langle M, M\rangle_t$ for the dual previsible projection of $[M, M]_t$. It is well known that $M_t^2 - \langle M, M\rangle_t$ is a martingale. Hence for $s < t$,

$$\begin{aligned}E\{(M_t - M_s)^2|\mathcal{F}_s\} &= E\{M_t^2 - M_s^2|\mathcal{F}_s\} \\ &= E\{\langle M, M\rangle_t - \langle M, M\rangle_s|\mathcal{F}_s\} \\ &\leqslant E\{\langle Z, Z\rangle_t - \langle Z, Z\rangle_s|\mathcal{F}_s\} \\ &\leqslant K(t - s)\end{aligned}$$

where the Lipschitz continuity of $\langle Z, Z\rangle_t$ established in Lemma (3.5) has been used. Thus M is $K\Delta t$.

It remains to show that A is $K\Delta t$ and has Lipschitz continuous paths. First, observe that

$$\begin{aligned}E\{(A_t - A_s)^2|\mathcal{F}_s\} &\leqslant 2E\{(Z_t - Z_s)^2 + (M_t - M_s)^2|\mathcal{F}_s\} \\ &\leqslant 4K(t - s).\end{aligned}$$

Thus it remains to establish only the Lipschitz continuity of the paths. From the proof of Lemma (3.4) we have $Z_t + Kt = M_t + C_t$, where $C_t = A_t + Kt$ is increasing. Also, from the proof given above we have shown that C_t is continuous. Moreover, $C_t = Z_t + Kt - M_t$ and hence is in L^2 because $E\{M_t^2\} = E\{[M, M]_t\} = E\{[Z, Z]_t\} < \infty$ shows that M_t is in L^2. By a lemma of Doléans-Dade [10], page 90, if $\mathcal{P}^n$ are partitions of $(s, t]$ with mesh $\mathcal{P}^n \to 0$, then

$$\Sigma_{t_j\in\mathcal{P}^n}E\{C_{t_{j+1}} - C_{t_j}|\mathcal{F}_{t_j}\} = C_{t,s}^n \to C_t - C_s$$

as $n \to \infty$, with convergence in L^2. If $C_{t,s}^n - (t - s) \equiv A_{t,s}^n$ then $A_{t,s}^n \to A_t - A_s$ in L^2. But $|E\{A_{t_{j+1}} - A_{t_j}|\mathcal{F}_{t_j}\}| = |E\{Z_{t_{j+1}} - Z_{t_j}|\mathcal{F}_{t_j}\}| \leqslant K(t_{j+1} - t_j)$. Let n' be a sequence of integers such that $A_{t,s}^{n'} \to A_t - A_s$ a.s., and let $\Lambda_{t,s}$ be the exceptional set. Off $\Lambda_{t,s}$ we have

$$|A_t - A_s| = \lim_{n'\to\infty}|A^n_{t,s}|$$
$$\leqslant \lim_{n'\to\infty}\Sigma_{t_j\in\mathcal{P}^n}|E\{A_{t_{j+1}} - A_{t_j}|\mathcal{F}_{t_j}\}|$$
$$\leqslant \Sigma_{t_j\in\mathcal{P}^n}K(t_{j+1} - t_j) = K(t - s).$$

Let $\Lambda = \cup_{t,s\in\mathbf{Q}}\Lambda_{t,s}$, so $P(\Lambda) = 0$. The continuity of the paths of A implies off Λ the paths of A are Lipschitz. This completes the proof.

We now consider processes satisfying hypotheses (3.2) rather than (3.1).

(3.7) THEOREM. *If Z is a process that is $K\Delta t$ after small amendments then Z is a semimartingale. Moreover, Z is quasi-left-continuous.*

PROOF. Let Z^n be $K\Delta t$ processes and T^n stopping times increasing to ∞ such that $Z^n_{t\wedge T^n}$ and $Z_{t\wedge T^n}$ are indistinguishable. For each n $(Z^n_{t\wedge T^n})_{t\geqslant 0}$ is a semimartingale by Lemma (3.4); thus Z is a local semimartingale. But Meyer [11], page 311, has shown that a local semimartingale is a semimartingale.

Let $(T^n_k)_{k\geqslant 1}$ be a sequence of totally inaccessible stopping times exhausting the jumps of $(Z^n_{t\wedge T^n})_{t\geqslant 0}$. Then $(T^n_k)_{k\geqslant 1, n\geqslant 1}$ exhaust the jumps of Z and are totally inaccessible, so Z is quasi-left-continuous.

4. First order integrals. In this section we show that if Z is $K\Delta t$ or $K\Delta t$ after small amendments (and hence a semimartingale as shown in Section 3) and if H is an appropriate integrand, then

$$H * Z = {}^3H\cdot Z \tag{4.1}$$

where $H * Z$ denotes the star integral $(\int_0^t H_s dZ_s)_{t\geqslant 0}$, ${}^3H$ denotes the previsible projection of H, and ${}^3H\cdot Z = ({}^3H)\cdot Z$ is the dot integral.

The following definition was given by McShane [9], page 133:

(4.2) DEFINITION. A process H is a *simple process* if there exist points $0 \leqslant t_0 < t_1 < \cdots < t_k < \infty$ such that $H_t = H_i$ on $[t_{i-1}, t_i)$, with $H_i \in \mathcal{F}_{t_{i-1}}, {}_{1\leqslant i\leqslant k}$, and $H_t = 0$ otherwise.

(4.3) LEMMA. *Let $M \in \mathfrak{M}_{\text{loc}}$ and be $K\Delta t$, and let H be a bounded simple process. Then $H * M = {}^3H\cdot M$.*

PROOF. Suppose H is of the form $H_t = h1_{[s,u)}(t)$ with $h \in \mathcal{F}_s$. Let $k = E\{h|\mathcal{F}_{s-}\}$. Then

$${}^3H_t = k1_{[s]}(t) + h1_{(s,u)}(t).$$

Thus

$$\begin{aligned}{}^3H\cdot M_t = {}& k\Delta M_s 1_{\{t\geqslant s\}}\\ &+h(M_t - M_s)1_{(s,\infty)}(t)\\ &-h\Delta M_u 1_{[u,\infty)}(t).\end{aligned}$$

Since by Lemma (3.5) M is quasi-left-continuous it does not jump at fixed times. Hence

$$^3H \cdot M_t = h(M_{t\wedge u} - M_{t\wedge s}).$$

By Lemma (5.8) of [9], we have

$$H * M_t = h(M_{t\wedge u} - M_{t\wedge s}).$$

The same arguments give in the general case that

$$^3H \cdot M_t = \Sigma_{i=1}^k H_i(M_{t\wedge t_i} - M_{t\wedge t_{i-1}})$$

and Lemma (5.8) of [9] yields the equality.

(4.4) LEMMA. *Let $A \in \mathcal{Q}$ be $K\Delta t$ and let H be a bounded simple process. Then $H * A = H \cdot A = {}^3H \cdot A$.*

PROOF. Recall that for $A \in \mathcal{Q}$ $H \cdot A$ refers to the Lebesgue-Stieltjes integral of H with respect to A. Let $H_t = \Sigma_{i=1}^k h_i 1_{[t_{i-1}, t_i)}$. Then the Stieltjes integral $H \cdot A_t = \Sigma_{i=1}^k h_i(A_{t\wedge t_i} - A_{t\wedge t_{i-1}})$. It is well known that $\{H \neq {}^3H\}$ is contained in a countable union of graphs of stopping times. Since H is simple, a.s. the set $\{t : H_t \neq {}^3H_t\}$ is finite. The quasi-left-continuity of the paths of A then implies $H \cdot A = {}^3H \cdot A$. Lemma (5.8) of [9] then gives $H * A = H \cdot A = {}^3H \cdot A$.

Given a process $A \in \mathcal{Q}$, we define a (signed) measure on $R_+ \times \Omega$ by

$$\mu_A(H) = E\int_0^\infty H_s dA_s. \tag{4.5}$$

Then μ_t denotes the measure $dt \times dP$.

(4.6) THEOREM. *Let Z be $K\Delta t$ and let H be jointly measurable and locally bounded. Then $H * Z = {}^3H \cdot Z$.*

PROOF. First note that if $T^m > T^n$ then ${}^3H^{T^n}$ and ${}^3H^{T^m}$ agree on $[0, T^n]$, and so the previsible projection of H can be defined. By optional stopping, we can assume without loss of generality that H is bounded and hence $E\int_0^t H_s^2 ds < \infty$ for each t. By Theorem (3.6) we know that Z is a special semimartingale with decomposition

$$Z = M + A$$

where $M \in \mathfrak{M}_{\text{loc}}$ and $K\Delta t$, and $A \in \mathcal{Q}$, $K\Delta t$, and has Lipschitz continuous paths. Note that since the paths are Lipschitz the measure μ_A is absolutely continuous with respect to μ_t. For H a bounded simple process by Lemmas (4.3) and (4.4) we have $H * Z = {}^3H \cdot Z$.

Suppose H is bounded and jointly measurable. Doob [5], pages 440–442, has shown there exist bounded "simple processes" H^n such that H^n converges to H in $L^2(d\mu_t)$. By Theorem (5.9) of [9] we have

$$\begin{aligned} E\{(H^n * Z - H * Z)_t^2\}^{\frac{1}{2}} &= E\{((H^n - H) * Z)_t^2\}^{\frac{1}{2}} \\ &\leqslant CE\{\textstyle\int_0^t (H^n - H)_s^2 ds\}^{\frac{1}{2}} \end{aligned}$$

where C is a constant. Thus for each t, $H^n * Z_t \to H * Z_t$ in $L^2(dp)$.

Consider now ${}^3H \cdot Z$. Let $\{n'\}$ be a sequence of integers such that $H^{n'} \to H$ a.e. $(d\mu_t)$. By the absolute continuity of the measure, also $H^{n'} \to H$ a.e. $(d\mu_A)$. By the dominated convergence theorem it follows that $\int (H^{n'} - H)^2 d\mu_A \to 0$.

We have

$$
\begin{aligned}
(4.7) \qquad & E\{({}^3H^{n'} \cdot Z - {}^3H \cdot Z)_t^2\} \\
& \leq 2E\{(({}^3H^{n'} - {}^3H) \cdot M)_t^2\} + 2E\{(({}^3H^{n'} - {}^3H) \cdot A_t)^2\} \\
& \leq 2E\{({}^3H^{n'} - {}^3H)^2 \cdot \langle M, M\rangle_t\} + 2E\{((H^{n'} - H) \cdot A_t)^2\} \\
& \leq 2E\{({}^3(H^{n'} - H))^2 \cdot \langle M, M\rangle_t\} + 2E\{(Kt)((H^{n'} - H)^2 \cdot A_t)\}
\end{aligned}
$$

where the Cauchy-Schwarz inequality and the elementary inequality $(a + b)^2 \leq 2a^2 + 2b^2$ have been used. It is easy to see that the previsible projection is a contraction (cf. Yor [15]). Then (4.7) becomes

$$
\begin{aligned}
(4.8) \qquad & E\{({}^3H^{n'} \cdot Z - {}^3H \cdot Z)_t^2\} \\
& \leq 2E\{{}^3(H^{n'} - H)^2 \cdot \langle M, M\rangle_t\} + 2Kt\mu_A((H^n - H)^2) \\
& = 2E\{(H^{n'} - H)^2 \cdot \langle M, M\rangle_t\} + 2Kt\mu_A((H^n - H)^2) \\
& = 2E\{(H^{n'} - H)^2 \cdot \langle Z, Z\rangle_t\} + 2Kt\mu_A((H^n - H)^2)
\end{aligned}
$$

where $\langle Z, Z\rangle_t$ is the dual previsible projection of $[Z, Z]_t = [M, M]_t$. We showed in the proof of Lemma (3.5) that $\langle Z, Z\rangle.$ is a.s. Lipschitz continuous. Thus $\mu_{\langle Z, Z\rangle} \ll \mu_t$, and so $\mu_{\langle Z, Z\rangle}((H^{n'} - H)^2) \to 0$ as $n' \to \infty$. Thus (4.8) yields

$$
E\{({}^3H^{n'} \cdot Z - {}^3H \cdot Z)_t^2\} \leq 2\mu_{\langle Z, Z\rangle}((H^{n'} - H)^2) + 2Kt\mu_A((H^{n'} - H)^2)
$$

which tends to 0 as $n' \to \infty$. We conclude that ${}^3H^{n'} \cdot Z \to {}^3H \cdot Z$ in $L^2(dP)$.

We have shown ${}^3H^{n'} \cdot Z = H^{n'} * Z$, and both converge to the same limit. Thus ${}^3H \cdot Z_t = H * Z_t$ a.s., and hence are indistinguishable by the right continuity of the paths. This completes the proof.

(4.9) THEOREM. *Let Z be $K\Delta t$ after small amendments and H be jointly measurable and locally bounded. Then $H * Z = {}^3H \cdot Z$.*

PROOF. By Theorem (3.7) we know that Z is a semimartingale, so ${}^3H \cdot Z$ is well defined. Let T^n be stopping times increasing a.s. to ∞ and Z^n be $K\Delta t$ processes such that $Z_{t \wedge T^n} = Z^n_{t \wedge T^n}$. Meyer [11], page 307, has shown that ${}^3H \cdot Z_{t \wedge T^n}$ is indistinguishable from ${}^3H \cdot Z^n_{t \wedge T^n}$. McShane [9], page 125, has shown the analogous result for the star integral. An application of Theorem (4.6) completes the proof.

5. Second Order Integrals. Since in general martingales do not have paths of finite variation, it is necessary to define second order integrals which arise in the appropriate version of Itô's lemma and in the study of stochastic differential equations. We denote the second order integral of McShane by $H * ZX$, which is written $\int_0^t H(s)dZ(s)dX(s)$ in [9], page 124.

We let $[Z, Z]_t = \langle Z^c, Z^c\rangle_t + \Sigma_{s\leqslant t}\Delta Z_s^2$, be the quadratic variation process; $[Z, Z]$ is an increasing process. Note that if Z has continuous paths, then $[Z, Z]_t = \langle Z, Z\rangle_t = \langle Z^c, Z^c\rangle_t$ (see [11]).

We show in this section that if H is an appropriate integrand and Z, X are $K\Delta t$ after small amendments, then

(5.1) $$H * ZX = {}^3H\cdot[Z, X]_t.$$

We shall need the following classical result on square-integrable martingales (cf. Meyer [10], page 83):

(5.2) THEOREM. *Let M be a quasi-left-continuous square integrable martingale. Then*
$$M = M' + \Sigma_{n\geqslant 1}A_n = M' + M''$$
where M' is a continuous martingale, $A_n = \Delta M_{T_n}1_{\{t\geqslant T_n\}} - (\Delta M_{T_n}1_{\{t\geqslant T_n\}})^3$, and where $(T_n)_{n\geqslant 1}$ is a sequence of stopping times exhausting the jumps of M. The series $\Sigma_{n\geqslant 1}A_n$ is convergent in $\mathfrak{M}$, and M'' is orthogonal to each martingale $N \in \mathfrak{M}$ without a discontinuity in common with M''.

(5.3) LEMMA. *Let $M \in \mathfrak{M}_{\text{loc}}$ be $K\Delta t$. Let H be a bounded "simple process," as defined in* (4.2). *Then*
$$H * MM = {}^3H\cdot[M, M].$$

PROOF. For ease of notation we write $[M]_t$ for $[M, M]_t$ and we let $[M]_{t,s} = [M]_t - [M]_s$. We make use of (δ, δ^*) partitions, which are defined in [9], page 124. Fix a t and let $\mathcal{P}^n$ be a sequence of (δ, δ_n^*) partitions of $[0, t]$ with $\delta_n^* \to 0$ as $n\to\infty$. Let

$$I = {}^3H\cdot[M]_t$$
(5.4) $$J_n = \Sigma_{t_j\in\mathcal{P}^n}H_{\tau_j}[M]_{t_{j+1}, t_j}$$
$$V_n = \Sigma_{t_j\in\mathcal{P}^n}H_{\tau_j}(M_{t_{j+1}} - M_{t_j})^2.$$

We will show that $J_n - V_n \to 0$ in L^1 and that $J_n - I \to 0$ in L^1 as $n\to\infty$. We follow a technique used by Meyer in [10].

Fix a k and let $M_s^t = M_{s\wedge t}$,

(5.5) $$M^t = P + Q + R = N + R$$

where P is a continuous martingale, Q is the sum of the first n compensated jumps, and R is the sum of the rest of the compensated jumps. Using the notation of (5.4) we have

(5.6) $$E\{|J_n - V_n|\}$$
$$\leqslant E\left\{|\Sigma H_{\tau_j}([M]_{t_{j+1}, t_j} - (N_{t_{j+1}} - N_{t_j})^2|\right\}$$
$$+ E\left\{|\Sigma H_{\tau_j}\big(2(N_{t_{j+1}} - N_{t_j})(R_{t_{j+1}} - R_{t_j}) + (R_{t_{j+1}} - R_{t_j})^2\big)|\right\}.$$

Consider the second term on the right side of (5.6). Let $\Delta_j N = (N_{t_{j+1}} - N_{t_j})$. Then

$$(5.7)\qquad E\left\{|\Sigma H_{\tau_j}(2\Delta_j N\Delta_j R + (\Delta_j R)^2)|\right\}$$
$$\leqslant 2E\left\{(\Sigma H_{\tau_j}^2(\Delta_j N)^2)^{\frac{1}{2}}(\Sigma(\Delta_j R)^2)^{\frac{1}{2}}\right\} + E\left\{\Sigma|H_{\tau_j}|(\Delta_j R)^2\right\}$$
$$\leqslant 2K^2 E\left\{\Sigma(\Delta_j N)^2\right\}^{\frac{1}{2}} E\left\{\Sigma(\Delta_j R)^2\right\}^{\frac{1}{2}} + KE\left\{\Sigma(\Delta_j R)^2\right\}$$

where we have used the Schwarz inequality for sums, the Cauchy-Schwarz inequality and where K is the bound for H. Thus (5.7) yields

$$E\left\{|\Sigma H_{\tau_j}(2\Delta_j N\Delta_j R + (\Delta_j R)^2)|\right\} \leqslant 2K^2 E(M_t^2)\|R\|_2 + K\|R\|_2$$

where $\|R\|_2 = E(R_\infty^2)$, which, by Theorem (5.2), is arbitrarily small if n is chosen large enough. We now consider the first term on the right side of (5.6). Recall that $N = P + Q$ as given in (5.5). Thus

$$(5.8)\qquad E\left\{|\Sigma H_{\tau_j}([M]_{t_{j+1},t_j} - (\Delta_j N)^2|\right\}$$
$$= E\left\{|\Sigma H_{\tau_j}([P]_{t_{j+1},t_j} + [Q]_{t_{j+1},t_j} + [R]_{t_{j+1},t_j} - (\Delta_j N)^2)|\right\}$$
$$\leqslant E\left\{|\Sigma H_{\tau_j}([P]_{t_{j+1},t_j} - (\Delta_j P)^2)|\right\}$$
$$+ E\left\{|\Sigma H_{\tau_j}([Q]_{t_{j+1},t_j} - (\Delta_j Q)^2|\right\}$$
$$+ E\left\{|\Sigma H_{\tau_j}[R]_{t_{j+1},t_j}|\right\}$$
$$+ E\left\{|\Sigma H_{\tau_j}(\Delta_j P)(\Delta_j Q)|\right\}.$$

We first observe that the third term on the right in (5.8) may be estimated by

$$(5.9)\qquad E\left\{|\Sigma H_{\tau_j}[R]_{t_{j+1},t_j}|\right\}| \leqslant KE\left\{\Sigma[R]_{t_j}^{t_{j+1}}\right\}$$
$$\leqslant KE\left\{[R]_t\right\}$$
$$\leqslant K\|R\|_2^2$$

which is arbitrarily small, for n large enough. Consider next the last term on the right in (5.8). Meyer [10] has shown $\Sigma(\Delta_j P)(\Delta_j Q)$ are uniformly integrable; also $\Sigma H_{\tau_j}(\Delta_j P)(\Delta_j Q) \leqslant K(\Sigma(\Delta_j P)^2)^{\frac{1}{2}}(\Sigma(\Delta_j Q)^2)^{\frac{1}{2}}$ is uniformly integrable. Moreover,

$$|\Sigma H_{\tau_j}(\Delta_j P)(\Delta_j Q)| \leqslant K \sup_j|\Delta_j P|\,|Q|_\infty$$

which tends to 0 a.s. by the uniform continuity of the paths of P on $[0, t]$, and the fact that the paths of Q on $[0, t]$ are a.s. of finite total variation. Thus $E\{|\Sigma H_{\tau_j}(\Delta_j P)(\Delta_j Q)|\} \to 0$ as $n \to \infty$.

As for the second term on the right in (5.8) we see that $[Q]_{t_{j+1},t_j} = \Sigma_{t_j < s \leqslant t_{j+1}}(\Delta Q_s)^2 = \Sigma_{t_j < s \leqslant t_{j+1}}\Delta Q_s^2$, where we write ΔQ_s^2 for $(\Delta Q_s)^2$. The sums $\Sigma H_{\tau_j}(\Sigma_{t_j < s \leqslant t_{j+1}}\Delta Q_s^2 - (\Delta_j Q)^2)$ are uniformly integrable since H is bounded, $\Sigma_{s \leqslant t}\Delta Q_s^2 \in L^1$, and, as is well known, the sums $\Sigma(\Delta_j Q)^2$ are uniformly integrable (see, e.g., Meyer [11], page 356). Moreover,

$$|\Sigma H_{\tau_j}(\Sigma_{t_j<s\leqslant t_{j+1}}\Delta Q_s^2 - (\Delta_j Q)^2)| \leqslant 2K\Sigma|(\Sigma\Delta Q_s)(\Delta_j Q^c) + (\Delta_j Q^c)^2| \tag{5.10}$$

where $Q_t = \Sigma_{s\leqslant t}\Delta Q_s + Q_t^c$ is a pathwise decomposition. Since Q has paths of finite variation on $[0, t]$, if $|Q|_t$ denotes the total variation on $[0, t]$, (5.10) becomes

$$|\Sigma H_{\tau_j}(\Sigma\Delta Q_s^2 - (\Delta_j Q)^2)| \leqslant 4K \sup_j|Q^c_{t_{j+1}} - Q^c_{t_j}|\,|Q_t|$$

which tends to 0 as $n \to \infty$ by the uniform continuity of the paths of Q^c.

Consider finally the first term on the right of (5.8). By optional stopping, if necessary, we may assume without loss of generality that both P and $\langle P\rangle$ are bounded. Since $[P] = \langle P\rangle$ for P continuous we have that

$$E\left\{\left(\Sigma H_{\tau_j}\left(\langle P\rangle_{t_{j+1}, t_j} - (\Delta_j P)^2\right)\right)^2\right\} = E\left\{\Sigma H_{\tau_j}^2\left(\langle P\rangle_{t_{j+1}, t_j} - (\Delta_j P)^2\right)^2\right\}$$

where we have used the fact that $P^2 - \langle P\rangle$ is a martingale. Thus the first term on the right in (5.8) is dominated by

$$K^2E\left\{\Sigma\left(\langle P\rangle_{t_{j+1}, t_j} - (\Delta_j P)^2\right)^2\right\} \leqslant 2K^2E\left\{\Sigma\left(\langle P\rangle_{t_{j+1}, t_j}\right)^2 + \Sigma(\Delta_j P)^4\right\}.$$

Now

$$\begin{aligned}\Sigma\left(\langle P\rangle_{t_{j+1}, t_j}\right)^2 &\leqslant \left(\sup\langle P\rangle_{t_{j+1}, t_j}\right)\Sigma\langle P\rangle_{t_{j+1}, t_j}\\ &\leqslant \left(\sup_j\langle P\rangle_{t_{j+1}, t_j}\right)\langle P.\rangle_t\end{aligned}$$

which tends to 0 as $n \to \infty$, and which is dominated by $(\langle P\rangle_t)^2 \in L^1$. Also

$$\Sigma(\Delta_j P)^4 \leqslant \left(\sup_j(\Delta_j P)^2\right)\Sigma(\Delta_j P)^2$$

which tends to 0 by the uniform continuity of the paths of P if one chooses a subsequence such that $\Sigma(\Delta_j P)^2 \to \langle P\rangle_t$ a.s. Moreover, $\sup_j(\Delta_j P)^2$ is bounded by $4K^2$ where K is a bound for P, and $\Sigma(\Delta_j P)^2$ is uniformly integrable. Thus $E\{\Sigma(\Delta_j P)^4 \leqslant d \to 0$ as $n \to \infty$.

Thus we see that each of the four terms on the right side of (5.8) tends to 0 as $n \to \infty$. We conclude that $J_n - V_n \to 0$ in L^1 for J_n, V_n as given in (5.4). It remains to show that $J_n - I \to 0$ in L^1.

Let $L_s^n = \Sigma H_{\tau_j}1_{[t_j, t_{j+1})}(s)$ and

$$\mu(\Lambda) = E\{1_\Lambda\cdot[M]_t\}.$$

Then

$$\begin{aligned}E\{|J_n - I|\} &= E\left\{|(L^n - {}^3H)\cdot[M]_t|\right\}\\ &\leqslant E\left\{|L^n - {}^3H|\cdot[M]_t\right\}\\ &= \mu(|L^n - {}^3H|)\\ &\leqslant \|\mu\|^{\frac{1}{2}}\left(\mu(L^n - {}^3H)^2\right)^{\frac{1}{2}}.\end{aligned} \tag{5.11}$$

Now $L^n = {}^3L^n\mu$ – a.e., since $L^n = {}^3L^n$ except possibly at the fixed partition points t_j; but μ does not put mass at previsible stopping times because M is quasi-left-continuous. Thus we have

$$\text{(5.12)} \qquad \begin{aligned} \mu\big((L^n - {}^3H)^2\big) &= \mu\big(({}^3L^n - {}^3H)^2\big) \\ &\leq \mu\big({}^3(L^n - H)^2\big) \\ &= E\big\{(L^n - H)^2 \cdot \langle M\rangle_t\big\}. \end{aligned}$$

But $\lim L^n_\cdot(\omega) = H_\cdot(\omega)$ for almost all ω and at most countable many s. Since $\langle M\rangle_\cdot(\omega)$ is continuous for almost all ω, we have by the dominated convergence theorem that

$$\text{(5.13)} \qquad \lim_{n\to\infty} E\big\{(L^n - H)^2 \cdot \langle M\rangle_t\big\} = 0.$$

Combining (5.13) with (5.11) and (5.12) yields that $J_n - I \to 0$ in L^1 as $n \to \infty$.

We conclude that $H * MM_t = {}^3H \cdot [M, M]_t$ a.s. for each t, and by the right continuity of the paths the processes $(H * MM_t)_{t\geq 0}$ and $({}^3H \cdot [M, M]_t)$ are indistinguishable. The lemma is established.

(5.14) Lemma. *Let Z be $K\Delta t$ and let H be a bounded "simple process" as in* (4.2). *Then $H * ZZ = {}^3H \cdot [Z, Z]$.*

Proof. Let $Z = M + A$ be the canonical decomposition of Z, where $M \in \mathfrak{M}_{\text{loc}}$ and $K\Delta t$ and A has adapted, Lipschitz continuous paths (Theorem (3.6)). For a sequence of (δ, δ^*) partitions define

$$\text{(5.15)} \qquad \begin{aligned} A_n &= \Sigma H_{\tau_j}\big(M_{t_{j+1}} - M_{t_j}\big)^2 \\ B_n &= 2\Sigma H_{\tau_j}\big(M_{t_{j+1}} - M_{t_j}\big)\big(A_{t_{j+1}} - A_{t_j}\big) \\ C_n &= \Sigma H_{\tau_j}\big(A_{t_{j+1}} - A_{t_j}\big)^2. \end{aligned}$$

Then

$$\text{(5.16)} \qquad \Sigma H_{\tau_j}\big(Z_{t_{j+1}} - Z_{t_j}\big)^2 = A_n + B_n + C_n.$$

Letting K be a bound for H we have

$$\text{(5.17)} \qquad \begin{aligned} E\{|B_n|\} &\leq 2K^2E\Big\{\big(\Sigma(\Delta_jM)^2\big)^{\frac{1}{2}}\big(\Sigma(\Delta_jA)^2\big)^{\frac{1}{2}}\Big\} \\ &\leq 2K^2E\big\{\Sigma(\Delta_jM)^2\big\}^{\frac{1}{2}}E\big\{\Sigma(\Delta_jA)^2\big\}^{\frac{1}{2}} \\ &\leq 2K^2E\big\{M_t^2\big\}^{\frac{1}{2}}E\big\{(\sup_j|\Delta_jA|)(|A|_t)\big\}^{\frac{1}{2}} \end{aligned}$$

with $\Delta_jX = (X_{t_{j+1}} - X_{t_j})$, with $|A|_t$ the total variation of the path on $[0, t]$, and where the Schwarz inequality has been used twice. The uniform continuity of the paths of A implies that $\sup_j|\Delta_jA|$ tends to 0 as $n \to \infty$ (and so δ^* tends to 0). Moreover, we have $\sup_j|\Delta_jA| \leq Kt$ and so the dominated convergence theorem and (5.17) imply that $E\{|B_n|\} \to 0$ as $n \to \infty$.

Consider C_n in (5.16). Then

$$E\{|C_n|\} \leq KE\left\{\Sigma(A_{t_{j+1}} - A_{t_j})^2\right\}$$
$$\leq KE\left\{(\sup_j|\Delta_j A|)(|A|_t)\right\},$$

and by the arguments previously made, the right side above tends to zero. Finally, by Lemma (5.3), A_n in (5.16) tends to ${}^3H \cdot [M]_t$ in L^1; since $[M] = [Z] = [Z, Z]$, the proof is complete.

(5.18) THEOREM. *Let H be jointly measurable and locally bounded. Let Z, X be $K\Delta t$. Then*

$$H * ZX = {}^3H \cdot [Z, X].$$

PROOF. Suppose we have the result for bounded H with $E\int_0^t H_s^2 ds < \infty$. Then the result holds in general if $H * ZX_{t\wedge T^n} = H^{T^n} * ZX_{t\wedge T^n} = {}^3H^{T^n} \cdot [Z, X]_{t\wedge T^n} = {}^3H \cdot [Z, X]_{t\wedge T^n}$. But these are simple consequences of the definition of the integrals and the elementary fact that $[Z, X]_{t\wedge T} = [Z, X^T]_t$. We now assume without loss of generality that H is jointly measurable and bounded, and that $E\int_0^\infty H_s^2 ds < \infty$. Then we know ([9], page 138) that there exist bounded simple processes H^n such that $E\int_0^\infty |H_s^n - H_s| ds < \varepsilon_n$, where $\varepsilon_n \to 0$. By Theorem (6.4) of [9] we have $E\{|H^n * ZZ_t - H * ZZ_t|\} \leq CE\int_0^t |H_s^n - H_s| ds \to 0$ as $n \to \infty$.

On the other hand, we have

$$E\{{}^3H^n \cdot [Z]_t - {}^3H \cdot [Z]_t\} \leq E\{|{}^3H^n - {}^3H| \cdot [Z, Z]_t\}$$
$$\leq E\{{}^3|H^n - H| \cdot [Z, Z]_t\}$$
$$= E\{|H^n - H| \cdot \langle Z, Z\rangle_t\}.$$

We know from Lemma (3.5) that the paths of $\langle Z, Z\rangle$ are Lipschitz continuous. Thus if $\mu_{\langle Z\rangle}(\Lambda) = E\{1_\Lambda \cdot \langle Z, Z\rangle_t\}$, by choosing a sequence of integers $\{n'\}$ such that $H^{n'} \to H$ a.e. μ_t, we have $H^{n'} \to H$ a.e. $\mu_{\langle Z\rangle}$. Therefore, by the dominated convergence theorem, we find $\mu_{\langle Z\rangle}(|H^{n'} - H|) \to 0$ as $n' \to \infty$.

We have shown in Lemma (5.14) that $H^n * ZZ_t = {}^3H^n \cdot [Z, Z]_t$ and the above argument shows that $H^{n'} * ZZ_t$ and ${}^3H^{n'} \cdot [Z, Z]_t$ converge in L^1 to $H * ZZ_t$ and ${}^3H \cdot [Z, Z]_t$, respectively. Thus $H * ZZ_t = {}^3H \cdot [Z, Z]_t$ a.s. The right continuity of the paths shows that $H * ZZ$ is indistinguishable from ${}^3H \cdot [Z, Z]$. Finally, by the polarization identities $H * ZX = {}^3H \cdot [Z, X]$. This completes the proof.

(5.19) COROLLARY. *Let H be locally bounded and let Z, X be $K\Delta t$ after small amendments. Then*

$$H * ZX = {}^3H \cdot [Z, X].$$

PROOF. First note by Theorem (3.7) that Z and X are semimartingales; hence $[Z, X]$ is defined. The result follows by an application of Theorem (2.5) of [9] and Theorem 27 of [11], page 307.

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