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THE RANGE OF LÉVY'S N-PARAMETER BROWNIAN MOTION IN d-SPACE[1]

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Let $B^{(N,d)}$ be Lévy's N-parameter Brownian motion in d-space. It is shown that almost surely $B^{(N,d)}$ doubles the Hausdorff dimension of every Borel set in the parameter space when $d \geqslant 2N$. The dimension of the range of B is also determined in this case.

Let $B^{(N,d)}$ denote Lévy's N-parameter Brownian motion with values in d-dimensional Euclidean space; i.e., $B^{(N,d)}(t, \omega) = (B_1(t, \omega), \cdots, B_d(t, \omega)) \in R^d$, where $t \in R^N$, and the component B_i's are independent, separable Gaussian processes with zero means and covariance

$$E(B_i(s)B_i(t)) = \tfrac{1}{2}\big[|s| + |t| - |s - t|\big].$$

Here $|\ |$ is the Euclidean norm.

Let $B = B^{(N,d)}$ for simplicity. When $N = 1$, $d = 2$, Kaufman (1969) has shown that planar Brownian motion doubles the dimension of every Borel set E in R_+^1. Hawkes and Pruitt (1974) later showed that this property is satisfied by higher dimensional Brownian motion, and Pruitt (1975) gave some applications of this result.

Our main purpose is to show that B doubles the dimension of E for every Borel set E in R^N when $d \geqslant 2N$. As a consequence, the dimension of the range of B is a.s. $2N$ if $d \geqslant 2N$, a result obtained earlier by Yoder (1975) using a different approach.

It is easily verified that for $t^1, \cdots, t^n \in R^N$ and $r > 0$:

1^0. $(B(t^2) - B(t^1), \cdots, B(t^n) - B(t^1)) \sim (B(t^2 - t^1), \cdots, B(t^n - t^1))$

2^0. $(r^{\frac{1}{2}}B(t^1), \cdots, r^{\frac{1}{2}}B(t^n)) \sim (B(rt^1), \cdots, B(rt^n))$.

Here $\sim$ denotes equality in distribution.

Occasionally, c will be used to denote constants whose values are unimportant and may be different from line to line.

2. B doubles the dimension of every Borel set.

Lemma 2.1. *Let $u \in R^N$ with $|u| = 1$. Then the conditional variance of $B_1(u)$ given every variable $B_1(t) - B_1(s)$ ($|t| \geqslant 1, |s| \geqslant 1$) is greater than zero.*

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PROOF. Assume to the contrary, i.e., this lemma is false. Then there would exist a sequence X_n of random variables of the form

$$\Sigma_{j=1}^{n} a_j[B_1(t_j) - B_1(s_j)]$$

with $|t_j| \geqslant 1$, $|s_j| \geqslant 1$ such that $E|B_1(u) - X_n|^2 \to 0$. It is elementary to check that

$$E(B_1(t)B_1(s)) = c\int_{R^N} \frac{(e^{it\cdot\lambda} - 1)(e^{-is\cdot\lambda} - 1)}{|\lambda|^{N+1}}\, d\lambda.$$

This implies that the sequence of functions

$$f_n(\lambda) = \Sigma_{j=1}^{n} a_j[(e^{it_j\cdot\lambda} - 1) - (e^{is_j\cdot\lambda} - 1)]$$

converges to $(e^{iu\cdot\lambda} - 1)$ in the space

$$L^2\left(R^N, \frac{d\lambda}{|\lambda|^{N+1}}\right).$$

Let $\varphi(\lambda)$ be any nonzero rapidly decreasing function with Fourier transform

$$\hat{\varphi}(t) = \int e^{it\cdot\lambda}\varphi(\lambda)\, d\lambda$$

supported on $\{t : |t| < \frac{1}{2}\}$. Because $\varphi(\lambda)$ is rapidly decreasing $f_n(\lambda)\varphi(\lambda) - e^{iu\cdot\lambda}\varphi(\lambda)$ converges to $-\varphi(\lambda)$ in $L^2(R^N, d\lambda)$. Taking Fourier's transforms, Plancherel's theorem implies that $\widehat{f_n\varphi}(t) - \hat{\varphi}(t + u)$ converges to $-\hat{\varphi}(t)$ in $L^2(dt)$. However, this is impossible since the supports of $\widehat{f_n\varphi}(t)$ and $\hat{\varphi}(t + u)$ are in $\{t : |t| > \frac{1}{2}\}$ and the support of $-\hat{\varphi}(t)$ is in $\{t : |t| < \frac{1}{2}\}$.

REMARK 2.1. In view of the rotational symmetry of B_1, note that the conditional variance of $B_1(u)$ in Lemma 2.1 is independent of u where $|u| = 1$.

LEMMA 2.2. *Partition the unit cube U^N of R^N into 4^{nN} cubicles with sides parallel to the coordinate axes and equal to 4^{-n}. Let ε be an arbitrary positive number and A_n be the following event: there exists a sphere D of R^d with radius 2^{-n} such that $B^{-1}(D)$ intersects at least $[n^{N+2+\varepsilon}]$ of these 4^{nN} cubicles. If $d \geqslant 2N$, then* $P(\lim\sup A_n) = 0$.

PROOF. Consider the event A_n^* defined as follows: there exists a sphere D^* of R^d with radius $c2^{-n}n^{\frac{1}{2}}$ which contains at least $[n^{N+2+\varepsilon}]$ points of R^d of the form $B(k_1 4^{-n}, \cdots, k_N 4^{-n})$ where $k_1, \cdots, k_N$ are nonnegative integers less than 4^n. By the modulus of continuity of B (see Lévy (1965), page 264), if c (a constant depending on N) is sufficiently large, then a.s.

$$\lim\sup_{n\to\infty} \sup\left\{\frac{|B(t) - B(s)|}{2^{-n}n^{\frac{1}{2}}} : s, t \in U^N \text{ and } |s - t| \leqslant N^{\frac{1}{2}}4^{-n}\right\} \leqslant c.$$

Observe that if s, t lie in the same cubicle with side 4^{-n} then $|s - t| \leqslant N^{\frac{1}{2}}4^{-n}$. Hence it is enough to show that $P(\lim\sup A_n^*) = 0$.

Pick n distinct points of R^N of the form $(k_1 4^{-n}, \cdots, k_N 4^{-n})$. Let $t^1, \cdots, t^n$ be an enumeration of these n points in such a way that $|t^i - t^{i+1}| \leqslant |t^i - t^{i+k}|$ for all $1 \leqslant i < n$, $1 \leqslant k \leqslant n - i$. By properties 1^0 and 2^0 mentioned earlier, it is easy to show that

$$P(B(t^1) \in D^*, \cdots, B(t^n) \in D^*)$$

is bounded by

$$P\Big(\bigcap_{i=1}^{n-1}\big[|B(t^{i+1}) - B(t^i)| \leqslant cn^{\frac{1}{2}}2^{1-n}\big]\Big)$$

$$\leqslant \Big[P\Big(\bigcap_{i=1}^{n-1}\big[|B_1(t^{i+1}) - B_1(t^i)| \leqslant cn^{\frac{1}{2}}2^{1-n}\big]\Big)\Big]^{2N}$$

(2.1)

$$= \left[P\left[\left[\left|B_1\left(\frac{t^2 - t^1}{|t^2 - t^1|}\right)\right| \leqslant \frac{cn^{\frac{1}{2}}2^{1-n}}{(|t^2 - t^1|)^{\frac{1}{2}}}\right] \Big/ H\right]\right]^{2N}$$

$$\times \left[P\left[\bigcap_{i=2}^{n-1}\left[\left|\frac{B_1(t^{i+1}) - B_1(t^1)}{(|t^2 - t^1|)^{\frac{1}{2}}} - \frac{B_1(t^i) - B_1(t^1)}{(|t^2 - t^1|)^{\frac{1}{2}}}\right| \leqslant \frac{cn^{\frac{1}{2}}2^{1-n}}{(|t^2 - t^1|)^{\frac{1}{2}}}\right]\right]\right]^{2N}$$

where H denotes the event

$$\bigcap_{i=2}^{n-1}\left[\left|B_1\left(\frac{t^{i+1} - t^1}{|t^2 - t^1|}\right) - B_1\left(\frac{t^i - t^1}{|t^2 - t^1|}\right)\right| \leqslant \frac{cn^{\frac{1}{2}}2^{1-n}}{(|t^2 - t^1|)^{\frac{1}{2}}}\right].$$

The conditional variance of $B^1((t^2 - t^1)/|t^2 - t^1|)$ given all variables

$$B_1\left(\frac{t^{i+1} - t^1}{|t^2 - t^1|}\right) - B_1\left(\frac{t^i - t^1}{|t^2 - t^1|}\right), \qquad 2 \leqslant i < n - 1$$

cannot be smaller than the conditional variance of $B_1((t^2 - t^1)/|t^2 - t^1|)$ given every variable $B_1(t) - B_1(s)$ with $|t| \geqslant 1$ and $|s| \geqslant 1$. By Lemma 2.1 and Remark 2.1, it follows that (2.1) is bounded by

$$\left[\frac{n^{\frac{1}{2}}2^{1-n}c}{(|t^2 - t^1|)^{\frac{1}{2}}}\right]^{2N}\Big[P\Big(\bigcap_{i=2}^{n-1}\big[|B_1(t^{i+1}) - B_1(t^i)| \leqslant cn^{\frac{1}{2}}2^{1-n}\big]\Big)\Big]^{2N}$$

$$\leqslant \Pi_{i=1}^{n-1}\left[\frac{n^{\frac{1}{2}}2^{1-n}c}{(|t^{i+1} - t^i|)^{\frac{1}{2}}}\right]^{2N}$$

$$\leqslant (cn)^{Nn}\Pi_{i=1}^{n-1}\big[(k_{i+1,1} - k_{i,1})^2 + \cdots (k_{i+1,N} - k_{i,N})^2\big]^{-N/2},$$

where $(k_{i,1}4^{-n}, \cdots, k_{i,N}4^{-n})$ are the coordinates of t^i. Thus $P(A_n^*)$ is bounded by

$$(2.2) \qquad (cn)^{Nn}\binom{[n^{N+2+\varepsilon}]}{n}^{-1} \Sigma \cdots \Sigma \Pi_{i=1}^{n-1}\left[\Sigma_{j=1}^{N}(k_{i+1,j} - k_{i,j})^2\right]^{-N/2}$$

where $\Sigma \cdots \Sigma$ is taken over values of $k_{1,1}, \cdots, k_{n,N}$ greater or equal to zero and less than 4^n. Since $t^1, \cdots t^n$ are distinct points, we only sum over values of $k_{1,1}, \cdots, k_{n,N}$ with

$$(2.3) \qquad \Sigma_{j=1}^{N}(k_{i+1,j} - k_{i,j})^2 \neq 0.$$

Under condition (2.3), it is not hard to see that

$$\Sigma \cdots \Sigma\left[\Sigma_{j=1}^{N}(k_{i+1,j} - k_{i,j})^2\right]^{-N/2} \sim c \log 4^n$$

where $\Sigma \cdots \Sigma$ is taken over $k_{i,1}, \cdots, k_{i,N}$ and $(k_{i+1,1}, \cdots, k_{i+1,N})$ is held fixed. Thus $P(A_n^*)$ is bounded by

$$(cn)^{Nn}\binom{[n^{N+2+\varepsilon}]}{n}^{-1} 4^{Nn}(\log 4^n)^n.$$

Using Stirling's formula, after some computations we obtain $\Sigma P(A_n^*) < \infty$. Therefore $P(\limsup A_n^*) = 0$.

THEOREM 2.1. *For ω outside some fixed null set, the Hausdorff dimension of $B(E) = \{B(t, \omega) : t \in E\}$ is twice the Hausdorff dimension of E for every Borel set E in R^N when $d \geqslant 2N$.*

PROOF. Almost all sample functions of B belong to the Lipschitz δ-class for every $\delta < \frac{1}{2}$ (see Lévy (1965)). Therefore the dimension of $B(E)$ is not greater than twice the dimension of E. Using Lemma 2.2, it is easy to show that the dimension of $B(E)$ is at least twice the dimension of E.

Note that Theorem 2.1 does not hold when $d < 2N$. As an example, let $u \in R^d$ and let $B^{-1}(u)$ be the u level set of B over U^N, i.e.,

$$B^{-1}(u) = \{t \in U^N : B(t) = u\}.$$

Adler (1977) has shown that for almost all u, the Hausdorff dimension of $B^{-1}(u)$ is $N - d/2$ with positive probability when $d < 2N$. However, the dimension of u is zero.

As an immediate consequence of Theorem 2.1, we obtain the following:

COROLLARY 2.1. *The Hausdorff dimension of the range of B is almost surely $2N$ when $d \geqslant 2N$.*

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