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A CENTRAL LIMIT THEOREM FOR PIECEWISE MONOTONIC MAPPINGS OF THE UNIT INTERVAL

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It is shown that if, for a piecewise C^2 mapping of the unit interval into itself where the absolute value of the derivative is greater than 1, an invariant measure is weak-mixing, then a central limit theorem holds for a class of real Hölder functions.

0. Introduction. It has been proven by Lasota and Yorke [5] that if $\tau : [0, 1] \mapsto [0, 1]$ is a piecewise C^2 mapping (see Definition 1 below) where $\inf_{x\in[0,1]}|\tau'| > 1$, then there exists an invariant measure which is absolutely continuous with respect to Lebesgue measure and has a density of bounded variation. Also Li and Yorke [7] proved the existence of ergodic measures for such mappings. Finally Bowen [2] has proven that if an invariant measure for the mapping is weak-mixing, then the "natural" extension (see [10]) of the mapping is measure isomorphic to a Bernoulli shift. With this last result, it will be shown that a central limit theorem is true for a class of real Hölder functions. The argument used is an adaptation of Bunimovitch's paper on the central limit theorem for the billiard dynamical system [3].

1. Preliminaries and lemmas.

DEFINITION 1. A transformation $\tau : [0, 1] \mapsto R$ will be called *piecewise* C^2 if there is a partition of [0, 1], $\mathcal{P} = \{(0, a_1), (a_1, a_2), \cdots, (a_{r-1}, 1)\}$ where (a_i, a_{i+1}) is an open interval, so that, for each $i = 1, \cdots, r$, $\tau_i = \tau|(a_{i-1}, a_i)$ can be extended to the closed interval $[a_{i-1}, a_i]$ as a C^2 function.

From here on, $\tau : [0, 1] \mapsto [0, 1]$ is to be a piecewise C^2 function with $s = \inf|\tau'| > 1$ and $\mathcal{P}$ is as in Definition 1. As mentioned before, τ possesses invariant measures each of which has a density of bounded variation. Let μ be such an invariant measure and p be the associated density function of bounded variation.

DEFINITION 2. If $\mathcal{P}$ and $\mathcal{Q}$ are two partitions of [0, 1], define new partitions $\mathcal{P} \vee \mathcal{Q} = \{A \cap B : A \in \mathcal{P}, B \in \mathcal{Q}\}$, $\tau^{-n}\mathcal{P} = \{\tau^{-n}A : A \in \mathcal{P}\}$, and $\mathcal{P}_m^M = \bigvee_{n=m}^{M} \tau^{-n}\mathcal{P} = \tau^{-m}\mathcal{P} \vee \cdots \vee \tau^{-M}\mathcal{P}$. In the case where $m = 0$, let $\mathcal{P}_M = \mathcal{P}_0^M$. The sets which belong to a given partition are to be called *atoms* of the partition.

Suppose that there are $B \in \mathcal{P}_{M+m}$ and $B' \in \mathcal{P}_{M+m}$ for which $\tau^m B = \tau^m B' \in \mathcal{P}_M$. Then $\tau_{|B}^m : B \mapsto \tau^m B$ is one-to-one and so is $\tau_{|B'}^m : B' \mapsto \tau^m B'$. Define $\eta : B \mapsto B'$ by $\eta = (\tau_{|B'}^m)^{-1} \circ (\tau_{|B}^m)$ where $(\tau_{|B'}^m)^{-1}$ is the inverse function of $\tau_{|B'}^m$. Thus, for

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$x \in B$, $\eta x = y \in B'$ such that $\tau^m x = \tau^m y$. η is one-to-one, onto and C^2. (The idea for η comes from [9].)

LEMMA 1. *Given $\beta > 0$, there is an $M = M(\beta)$ so that for each $m \geqslant 0$, one can find a collection of atoms $\mathfrak{B}_{M+m} \subset \mathfrak{P}_{M+m}$ with*

(1) $\tau^m B \in \mathfrak{P}_M$ *for* $B \in \mathfrak{B}_{M+m}$;

(2) *if* $\tau^m B = \tau^m B'$ *for* $B, B' \in \mathfrak{B}_{M+m}$, *then*
$\dfrac{p(\eta x)|\eta'(x)|}{\mu(B')} \Big/ \dfrac{p(x)}{\mu(B)} \in [e^{-\beta}, e^{\beta}]$ *where* $\mu(A) = \int_A p(x)\,dx$;

(3) $\mu(\cup \mathfrak{B}_{M+m}) > 1 - \beta$.

PROOF. Parts (1) and (3) follow from Lemma 1 of Bowen's paper.

Since τ is piecewise C^2 and $s > 1$, one can find a constant d for which $|\tau'(u)/\tau'(v)| \in [e^{-d|u-v|}, e^{d|u-v|}]$ for $u, v \in [a_{i-1}, a_i]$. Then for $u, v \in B \in \mathfrak{P}_{M+m}$, $|\tau^k u - \tau^k v| \leqslant s^{-(M+m-k)}$ and

(1)
$$\left|\frac{(\tau^m)'(u)}{(\tau^m)'(v)}\right| = \prod_{k=0}^{m-1}\left|\frac{\tau'(\tau^k u)}{\tau'(\tau^k v)}\right| \in \left[e^{-d*s^{-M}}, e^{d*s^{-M}}\right] \quad \text{where}$$

$$d * s^{-M} = d\Sigma_{j\geqslant 0} s^{-M-j} = \frac{ds^{-M}}{1 - s^{-1}}.$$

(2)
$$\left|\frac{\eta'(u)}{\eta'(v)}\right| = \left|\left[(\tau^m_{|B'})^{-1} \circ (\tau^m_{|B}(u))\right]' \Big/ \left[(\tau^m_{|B'})^{-1} \circ (\tau^m_{|B}(v))\right]'\right|$$

$$= \left|\frac{(\tau^m_{|B})'(u)}{(\tau^m_{|B'})' \circ \left[(\tau^m_{|B'})^{-1} \circ (\tau^m_{|B}(u))\right]}\right| \cdot \left|\frac{(\tau^m_{|B'})' \circ \left[(\tau^m_{|B'})^{-1}(\tau^m_{|B}(f))\right]}{(\tau^m_{|B})'(v)}\right|$$

$$\in \left[e^{-2d*s^{-M}}, e^{2d*s^{-M}}\right] \qquad \text{for} \quad u, v \in B \in \mathfrak{B}_{M+m}, \qquad \text{by using} \quad (1).$$

By Lemma 2 of Bowen's paper, for M large, $p(x)$ and $p(\eta x)$ will each vary by at most a multiplicative factor in $[e^{-\beta/6}, e^{\beta/6}]$. From (2) and this last comment, $p(x) \in K_B[e^{-\beta/6}, e^{\beta/6}]$, $p(\eta x) \in K_{B'}[e^{-\beta/6}, e^{\beta/6}]$, and $|\eta'(x)| \in K_\eta[e^{-\beta/6}, e^{\beta/6}]$ where K_B, $K_{B'}$, and K_η are constants. Thus $p(\eta x)|\eta'(x)|/p(x) \in (K_{B'}K_\eta/K_B)[e^{-\beta/2}, e^{\beta/2}]$. Using a change of variable,

$$\mu(B') = \int_{B'} p(y)\,dy = \int_B p(\eta x)|\eta'(x)|\,dx$$

$$= \int_B \left[\frac{p(\eta x)|\eta'(x)|}{p(x)}\right] p(x)\,dx$$

$$\in \mu(B)\left(\frac{K_{B'}K_\eta}{K_B}\right)\left[e^{-\beta/2}, e^{\beta/2}\right],$$

$$\text{or} \quad \frac{\mu(B)}{\mu(B')} \in \frac{K_B}{K_{B'}K_\eta}\left[e^{-\beta/2}, e^{\beta/2}\right].$$

Consequently,

$$\frac{p(\eta x)|\eta'(x)|}{\mu(B')} \Big/ \frac{p(x)}{\mu(B)} \in [e^{-\beta}, e^{\beta}].$$

LEMMA 2. *Suppose f is a bounded measurable real function defined on* [0, 1]. *Given* $\beta > 0$, *suppose* $M = M(\beta)$ *and* $\mathfrak{B}_{M+m}$ *are as in Lemma* 1. *If* $B, B' \in \mathfrak{B}_{M+m}$ *with* $\tau^m B = \tau^m B'$, *then*

$$\left| \frac{1}{\mu(B)} \int_B \exp\left[i\lambda \Sigma_{j=l}^{L-1} f(\tau^j x) \right] d\mu(x) - \frac{1}{\mu(B')} \int_{B'} \exp\left[i\lambda \Sigma_{j=l}^{L-1} f(\tau^j x) \right] d\mu(x) \right|$$

$\leqslant (e^{\beta} - 1)$ *for any integers* $L > l \geqslant m$,

$\leqslant 2\|f\|_\infty |\lambda| m + (e^{\beta} - 1)$ *otherwise.*

PROOF. Let $\eta : B \to B'$ be defined as before. Denote $\exp[i\lambda\Sigma_{j=l}^{L-1} f(\tau^j x)]$ by $F_\lambda(x, l, L)$.

$$\begin{aligned}
\Delta &= \left| \frac{1}{\mu(B)} \int_B F_\lambda(x, l, L)\, d\mu(x) - \frac{1}{\mu(B')} \int_{B'} F_\lambda(x, l, L)\, d\mu(x) \right| \\
&\leqslant \left| \frac{1}{\mu(B)} \int_B F_\lambda(x, l, L)\, d\mu(x) - \frac{1}{\mu(B)} \int_B F_\lambda(\eta x, l, L)\, d\mu(x) \right| \\
&\quad + \left| \frac{1}{\mu(B)} \int_B F_\lambda(\eta x, l, L)\, d\mu(x) - \frac{1}{\mu(B')} \int_B F_\lambda(\eta x, l, L)\, d\mu(\eta x) \right| \\
&\leqslant \frac{1}{\mu(B)} \int_B |F_\lambda(x, l, L) - F_\lambda(\eta x, l, L)|\, d\mu(x) \\
&\quad + \left| \frac{1}{\mu(B)} \int_B F_\lambda(\eta x, l, L) \left[1 - \frac{d\mu(\eta x)/\mu(B')}{d\mu(x)/\mu(B)} \right] d\mu(x) \right| \\
&= \frac{1}{\mu(B)} \int_B \left| 1 - \exp\left\{ i\lambda \Sigma_l^{L-1} [f(\tau^j x) - f(\tau^j \eta x)] \right\} \right| d\mu(x) \\
&\quad + \left| \frac{1}{\mu(B)} \int_B F_\lambda(\eta x, l, L) \left[1 - \frac{p(\eta x)|\eta'(x)|}{\mu(B')} \Big/ \frac{p(x)}{\mu(B)} \right] d\mu(x) \right| \\
&\leqslant \frac{1}{\mu(B)} \int_B |\lambda| \left| \Sigma_l^{L-1} [f(\tau^j x) - f(\tau^j \eta x)] \right| d\mu(x) + (e^{\beta} - 1)
\end{aligned}$$

by Lemma 1.1 and by the fact that, for all $\alpha \in \mathbb{R}$, $|e^{i\alpha} - 1| \leqslant |\alpha|$.

$$\left| \Sigma_l^{L-1} [f(\tau^j x) - f(\tau^j \eta x)] \right| \leqslant \Sigma_l^{L-1} |f(\tau^j x) - f(\tau^j \eta x)| \leqslant 2\|f\|_\infty \max(m - l, 0)$$

since for all $j \geqslant m$, $\tau^j x = \tau^j(\eta x)$ by the definition of η. Therefore if $L > l \geqslant m$, $\Delta \leqslant (e^{\beta} - 1)$ and otherwise $\Delta \leqslant 2\|f\|_\infty |\lambda| m + (e^{\beta} - 1)$.

DEFINITION 3 ([8]). For $\mathcal{P}$ and $\mathcal{Q}$ two partitions of [0, 1], $\mathcal{P}$ and $\mathcal{Q}$ are *ε-independent*, written $\mathcal{P} \perp^{\varepsilon} \mathcal{Q}$, if

$$\Sigma_{A \in \mathcal{P}} \Sigma_{B \in \mathcal{Q}} |\mu(A \cap B) - \mu(A)\mu(B)| < \varepsilon.$$

A partition $\mathcal{P}$ is called *weak-Bernoulli* if for each $\varepsilon > 0$, there is an m such that for all $J > 0$, $K > 0$, $\mathcal{P}_J \perp^{3\varepsilon^2} \mathcal{P}_{J+m}^{J+K+m}$ ([8]).

In his paper, Bowen proves the Bernoulliness of the "natural" extension by proving that the partition $\mathcal{P}$ is weak-Bernoulli. It is this property of $\mathcal{P}$ that is needed to prove the central limit theorem.

2. Statement and proof of the theorem.

DEFINITION 4 ([3]). A measurable, essentially bounded, real function f, defined on [0, 1] with the Lebesgue σ-algebra of [0, 1] and a measure μ which is τ-invariant, i.e., invariant relative to τ, *obeys a central limit theorem* if, for some positive constant σ, for any fixed $z \in \mathbb{R}$,

$$\lim_{L \to \infty} \mu\left\{ x : \frac{1}{\sigma L^{\frac{1}{2}}} \left[\Sigma_0^{L-1} f(\tau^j x) - L\bar{f} \right] < z \right\} = \frac{1}{(2\pi)^{\frac{1}{2}}} \int_{-\infty}^{z} \exp\left(\frac{-u^2}{2} \right) du$$

where $\bar{f} = \int_0^1 f \, d\mu$ and du is Lebesgue measure.

THEOREM. *Let τ, $\mathcal{P}$, and μ be as above. Suppose μ is an ergodic measure for which τ is weak-mixing. Also suppose that f is a Hölder continuous real function defined on* [0, 1] *with exponent $\delta \in (0, 1]$ such that*

(1) *$D_L(f) \sim cL$ as $L \to \infty$ where $D_L(f) = \int_0^1 [\Sigma_0^{L-1} f(\tau^j x) - L\bar{f}]^2 \, d\mu$ and $c > 0$;*

(2) *for any $\varepsilon > 0$, there exist $N(\varepsilon)$ and $L(\varepsilon)$ for which for each $L > L(\varepsilon)$, $\frac{1}{D_L(f)} \int_{\Omega_\varepsilon} [\Sigma_0^{L-1} f(\tau^j x) - L\bar{f}]^2 \, d\mu < \varepsilon$ where $\Omega_\varepsilon = \{x : |\Sigma_0^{L-1} f(\tau^j x) - L\bar{f}| > N(\varepsilon)(D_L(f))^{\frac{1}{2}}\}$.*

Then f obeys a central limit theorem and the constant σ can be taken to be $c^{\frac{1}{2}}$.

PROOF. (Without loss of generality, one can assume $\bar{f} = 0$.) It is shown in Bowen's paper that, if μ makes τ weak-mixing, then $\mathcal{P} = \{(0, a_1), (a_1, a_2), \cdots, (a_{r-1}, 1)\}$ where (a_i, a_{i+1}) is an open interval is a weak-Bernoulli generator. If $\mathcal{Q} \leqslant \mathcal{P}_J$, i.e., the partition $\mathcal{Q}$ is no finer than $\mathcal{P}_J$, then $\Sigma_{Q \in \mathcal{Q}} \Sigma_{P \in \mathcal{P}_{J+m}^{J+K+m}} |\mu(P \cap Q) - \mu(P)\mu(Q)| < 3\varepsilon^2$. In particular, this is true for $\mathcal{P}_I^J \leqslant \mathcal{P}_J$ for $0 \leqslant I < J$.

Given $\beta > 0$, choose $M = M(\beta^2/3)$ according to Lemma 1, i.e., there is a collection of atoms $\mathfrak{B} \subset \mathcal{P}_{M+(l-M)}$ for which, if $l - M > 0$, $\tau^{l-M}B \in \mathcal{P}_M$ for $B \in \mathfrak{B}$; if $\tau^{l-M}B = \tau^{l-M}B'$ for $B, B' \in \mathfrak{B}$, then $(p(\eta x)|\eta'(x)|/\mu(B'))/(p(x)/\mu(B)) \in [\exp(-\beta^2/3), \exp(\beta^2/3)]$, and $\mu(\cup \mathfrak{B}) > 1 - \beta^2/3$. (The choice of l will be made later.) For $D \in \mathcal{P}_M$, let $A_D = \{B \in \mathfrak{B} : \tau^{l-M}B = D\}$. Let $A_1, A_2, \cdots, A_k$ be the nonempty A_D and $A_0 = [0, 1]/\cup_{j=1}^k A_j$. By the choice of A_j, $j = 1, \cdots, k$, $\mu(A_0) \leqslant \beta^2/3$. Notice that

$A_D \subset \tau^{-l+M}D$ and that each $A_j, j = 1, \cdots, k$, is contained in a union of atoms of $\mathcal{P}^l_{l-M}$, say, $\tilde{A}_j \subset \mathcal{P}^l_{l-M}$. Moreover $\tilde{A}_j \setminus (\tilde{A}_j \cap A_0) = A_j$, $j = 1, \cdots, k$. Choose m now so that $\mathcal{P}^l_{l-M} \perp^{\beta/3^{\frac{1}{2}}} \mathcal{P}^{l-M-m}_{l-2M-m}$, i.e., $\Sigma_{A \in \mathcal{P}^l_{l-M}} \Sigma_{B \in \mathcal{P}^{l-M-m}_{l-2M-m}} |\mu(A \cap B) - \mu(A)\mu(B)| < \beta^2$ for $l \geqslant 3M + m$.

$$\Sigma^k_{j=0} \Sigma_{B \in \mathcal{P}^{l-M-m}_{l-2M-m}} |\mu(A_j \cap B) - \mu(A_j)\mu(B)|$$

$$\begin{aligned}
&\leqslant \Sigma^k_{j=1} \Sigma_{B \in \mathcal{P}^{l-M-m}_{l-2M-m}} |\mu(A_j \cap B) - \mu(A_j)\mu(B)| + 2\mu(A_0) \\
&\leqslant \Sigma^k_{j=1} \Sigma_{B \in \mathcal{P}^{l-M-m}_{l-2M-m}} |\mu((\tilde{A}_j \setminus (\tilde{A}_j \cap A_0)) \cap B) \\
&\quad - \mu(\tilde{A}_j \setminus (\tilde{A}_j \cap A_0))\mu(B)| + \frac{2}{3}\beta^2 \\
&= \Sigma^k_{j=1} \Sigma_{B \in \mathcal{P}^{l-M-m}_{l-2M-m}} |\mu(\tilde{A}_j \cap B) - (\tilde{A}_j \cap A_0 \cap B) \\
&\quad - \mu(\tilde{A}_j)\mu(B) + (\tilde{A}_j \cap A_0)\mu(B)| + \tfrac{2}{3}\beta^2 \\
&\leqslant \Sigma^k_{j=1} \Sigma_{B \in \mathcal{P}^{l-M-m}_{l-2M-m}} |\mu(\tilde{A}_j \cap B) \\
&\quad - \mu(\tilde{A}_j)\mu(B)| + 2\mu(A_0) + \tfrac{2}{3}\beta^2 \\
&\leqslant \Sigma_{A \in \mathcal{P}^l_{l-M}} \Sigma_{B \in \mathcal{P}^{l-M-m}_{l-2M-m}} |\mu(A \cap B) \\
&\quad - \mu(A)\mu(B)| + \tfrac{4}{3}\beta^2 < \beta^2 + \tfrac{4}{3}\beta^2 < 3\beta^2.
\end{aligned}$$

Hence $\{A_0, A_1, \cdots, A_k\} \perp^{\beta} \mathcal{P}^{l-M-m}_{l-2M-m}$. Consequently there is a collection of atoms $\mathcal{S} \subset \mathcal{P}^{l-M-m}_{l-2M-m}$ for which $\mu(\cup \mathcal{S}) > 1 - \beta$ and for $B \in \mathcal{S}$, $\Sigma^k_{j=0} |\mu(A_j|B) - \mu(A_j)| < \beta$ ([8]).

Let $\phi_L(\lambda) = \int_0^1 \exp[i\lambda\Sigma^{L-1}_{j=0} f(\tau^j x)]\, d\mu(x)$, the characteristic function of $\Sigma^{L-1}_{j=0} f \circ \tau^j$ with respect to μ and $F_\lambda(x, l, L)$ be as in Lemma 2. To prove the theorem, it suffices to show that $\phi_L(\lambda/(D_L(f))^{\frac{1}{2}}) \to \exp(-\lambda^2/2)$ as $L \to \infty$ uniformly for λ in each finite interval. The argument is based on making an estimate of $|\phi_L(\lambda) - \phi_{L-l}(\lambda)\phi_l(\lambda)|$.

$$\begin{aligned}
|\phi_L(\lambda) - \phi_{L-l}(\lambda)\phi_l(\lambda)| &\leqslant |\phi_L(\lambda) - \Sigma_{\tilde{C} \in \tau^{-M-m}\mathcal{P}^{l-M-m}_{l-2M-m}} \mu(\tilde{C}) \textstyle\int_{\tilde{C}} F_\lambda(x, 0, l)\, d\mu(x|\tilde{C}) \\
&\times \textstyle\int_{\tilde{C}} F_\lambda(x, l, L)\, d\mu(x|\tilde{C})| + |\Sigma_{\tilde{C} \in \tau^{-M-m}\mathcal{P}^{l-M-m}_{l-2M-m}} \mu(\tilde{C}) \int_{\tilde{C}} F_\lambda(x, 0, l)\, d\mu(x|\tilde{C}) \\
&\textstyle\int_{\tilde{C}} F_\lambda(x, l, L)\, d\mu(x|\tilde{C}) - \phi_{L-l}(\lambda)\phi_l(\lambda)| = \Delta_1 + \Delta_2.
\end{aligned}$$

We will begin by estimating Δ_2. One observes that, since μ is invariant with respect to τ, if $0 \leqslant l < L$, then

$$\begin{aligned}
\phi_{L-l}(\lambda) &= \textstyle\int_0^1 \exp\left[i\lambda\Sigma^{L-l-1}_{j=0} f(\tau^j x)\right] \cdot d\mu(x) \\
&= \textstyle\int_0^1 \exp\left[i\lambda\Sigma^{L-1}_{j=l} f(\tau^j x)\right] d\mu(x) \\
&= \textstyle\int_0^1 F_\lambda(x, l, L)\, d\mu(x).
\end{aligned}$$

If $\mathcal{U} = \left\{\tilde{C} \in \tau^{-M-m}\mathcal{P}^{l-M-m}_{l-2M-m} : \mu(\cup \mathcal{B}|\tilde{C}) > 1 - \beta\right\}$, then

$$1 - \frac{\beta^2}{3} < \mu(\cup \mathcal{B}) = \Sigma_{\tilde{C} \in \mathcal{U}} \mu(\tilde{C}) \mu(\cup \mathcal{B} | \tilde{C}) + \Sigma_{\tilde{C} \notin \mathcal{U}} \mu(\tilde{C}) \mu(\cup \mathcal{B} | \tilde{C})$$
$$\leqslant \Sigma_{\tilde{C} \in \mathcal{U}} \mu(\tilde{C})$$
$$+ (1 - \beta) \Sigma_{\tilde{C} \notin \mathcal{U}} \mu(\tilde{C})$$
$$= \mu(\cup \mathcal{U}) + (1 - \beta)[1 - \mu(\cup \mathcal{U})]$$
$$=> \beta - \frac{\beta^2}{3} < \beta \mu(\cup \mathcal{U}) \quad \text{or} \quad 1 - \frac{\beta}{3} < \mu(\cup \mathcal{U}).$$

Now considering $\tau^{-M-m}\mathcal{S}$ where $\mathcal{S} \subset \mathcal{P}_{l-2M-m}^{l-M-m}$ is defined above, $\tau^{-M-m}\mathcal{S} \cap \mathcal{U}$ has μ-measure at least $1 - \frac{4}{3}\beta$. For $\tilde{C} \in \tau^{-M-m}\mathcal{S}$,

$$|\phi_{L-l}(\lambda) - \textstyle\int_{\tilde{C}} F_\lambda(x, l, L)\, d\mu(x|\tilde{C})|$$
$$\leqslant |\phi_{L-l}(\lambda) - \textstyle\int_{\tilde{C}} F_\lambda(x, l + M + m, L + M + m)\, d\mu(x|\tilde{C})|$$
$$+ |\textstyle\int_{\tilde{C}} F_\lambda(x, l + M + m, L + M + m)\, d\mu(x|\tilde{C})$$
$$- \textstyle\int_{\tilde{C}} F_\lambda(x, l, L)\, d\mu(x|\tilde{C})|$$
$$\leqslant |\textstyle\int_0^1 F_\lambda(x, l, L)\, d\mu(x) - \int_{\tilde{C}} F_\lambda(x, l + M + m, L + M + m)\, d\mu(x|\tilde{C})|$$
$$+ \textstyle\int_{\tilde{C}} |F_\lambda(x, l + M + m, L + M + m)$$
$$- F_\lambda(x, l, L)|\, d\mu(x|\tilde{C})$$
$$\leqslant |\Sigma_{j=1}^k \mu(A_j) \textstyle\int_{A_j} F_\lambda(x, l, L)\, d\mu(x|A_j)$$
$$- \textstyle\int_{\tilde{C}} F_\lambda(x, l + M + m, L + M + m)\, d\mu(x|\tilde{C})|$$
$$+ \frac{\beta^2}{3} + |\lambda| \textstyle\int_{\tilde{C}} |\Sigma_{l+M+m}^{L+M+m-1} f(\tau^j x) - \Sigma_l^{L-1} f(\tau^j x)| \cdot d\mu(x|\tilde{C})$$
$$\leqslant |\Sigma_{j=1}^k \mu(A_j) \textstyle\int_{A_j} F_\lambda(x, l, L)\, d\mu(x|A_j)$$
$$- \textstyle\int_{\tilde{C}} F_\lambda(x, l + M + m, L + M + m)\, d\mu(x|\tilde{C})|$$
$$+ \frac{\beta^2}{3} + 2\|f\|_\infty (M + m)|\lambda|.$$

For $\tilde{C} \in \tau^{-M-m}\mathcal{S} \cap \mathcal{U}$, let $\tilde{C}' = \tilde{C} \cap (\cup \mathcal{B})$.

$$\Gamma = |\textstyle\int_{\tilde{C}} F_\lambda(x, l + M + m, L + M + m)\, d\mu(x|\tilde{C})$$
$$- \textstyle\int_{\tau^{M+m}\tilde{C}} F_\lambda(x, l, L)\, d\mu(x|\tau^{M+m}\tilde{C})|$$
$$\leqslant |\textstyle\int_{\tilde{C}'} F_\lambda(x, l + M + m, L + M + m)\, d\mu(x|\tilde{C})$$
$$- \textstyle\int_{\tau^{M+m}\tilde{C}} F_\lambda(x, l, L)\, d\mu(x|\tau^{M+m}\tilde{C}) + [1 - \mu(\cup \mathcal{B} | \tilde{C})]$$
$$= |\textstyle\int_{\tilde{C}'} F_\lambda(\tau^{M+m}x, l, L)\, d\mu(x|\tilde{C})$$
$$- \textstyle\int_{\tau^{M+m}C} F_\lambda(x, l, L)\, d\mu(x|\tau^{M+m}\tilde{C})| + \beta.$$

$$\int_{\tilde{C}'} F_\lambda(\tau^{M+m}x, l, L)\, d\mu(x|\tilde{C})$$
$$= \Sigma_{B\in\mathfrak{B}_{|\tilde{C}'}} \int_B F_\lambda(\tau^{M+m}x, l, L)\, d\mu(x|\tilde{C})$$
$$= \Sigma_{B\in\mathfrak{B}_{|\tilde{C}'}} \frac{\mu(B)}{\mu(\tilde{C})} \int_{\tau^{M+m}B} F_\lambda(x, l, L)\, d\mu(\tau^{-M-m}x|B)$$
$$= \Sigma_{B\in\mathfrak{B}_{|\tilde{C}'}} \frac{\mu(B)}{\mu(\tilde{C})} \int_{\tau^{M+m}B} F_\lambda(x, l, L) \left[\frac{d\mu(\tau^{-M-m}x|B)}{d\mu(x|\tau^{M+m}B)} \right] d\mu(x|\tau^{M+m}B)$$
$$\in \Sigma_{B\in\mathfrak{B}_{|\tilde{C}'}} \frac{\mu(B)}{\mu(\tilde{C})} \int_{\tau^{M+m}B} F_\lambda(x, l, L)\, d\mu(x|\tau^{M+m}B) \left[\exp\left(\frac{-\beta^2}{3}\right), \exp\left(\frac{\beta^2}{3}\right) \right]$$

by an argument analogous to the one found in Lemma 1. Because $\tilde{C} \in \tau^{-M-m}\mathcal{P}_{l-2M-m}^{l-M-m}$ and μ is τ-invariant,

$$\Sigma_{B\in\mathfrak{B}_{|\tilde{C}'}} \frac{\mu(B)}{\mu(\tilde{C})} \int_{\tau^{M+m}B} F_\lambda(x, l, L)\, d\mu(x|\tau^{M+m}B)$$
$$= \Sigma_{B\in\mathfrak{B}_{|\tilde{C}'}} \frac{\mu(B)}{\mu(\tau^{M+m}C)} \int_{\tau^{M+m}B} F_\lambda(x, l, L)\, d\mu(x|\tau^{M+m}B)$$
$$= \Sigma_{\tilde{B}\in\mathcal{P}_{l-M-m|\tau^{M+m}\tilde{C}}} \int_{\tilde{B}} F_\lambda(x, l, L)\, d\mu(x|\tau^{M+m}\tilde{C}) \frac{\mu(\tau^{-M-m}\tilde{B} \cap \cup\mathfrak{B})}{\mu(\tilde{B})}$$
$$= \Sigma_{\tilde{B}\in\mathcal{P}_{l-M-m|\tau^{M+m}\tilde{C}}} \int_{\tilde{B}} F_\lambda(x, l, L)\, d\mu(x|\tau^{M+m}\tilde{C})\mu(\cup\mathfrak{B}\,|\tau^{-M-m}\tilde{B}).$$

Consequently,

$$\Gamma \leqslant \Sigma_{\tilde{B}\in\mathcal{P}_{l-M-m|\tau^{M+m}\tilde{C}}} \left|\int_{\tilde{B}} F_\lambda(x, l, L)\, d\mu(x|\tau^{M+m}\tilde{C})\right|$$
$$\times\left[1 - \mu(\cup\mathfrak{B}\,|\tau^{-M-m}\tilde{B})\right]\exp\left(\frac{\beta^2}{3}\right) + \beta$$
$$= \Sigma_{\tilde{B}\in\mathcal{P}_{l-M-m|\tau^{M+m}\tilde{C}}} \frac{\mu(\tilde{B})}{\mu(\tau^{M+m}\tilde{C})}\left[1 - \mu(\cup\mathfrak{B}\,|\tau^{-M-m}\tilde{B})\right]\exp\left(\frac{\beta^2}{3}\right) + \beta.$$

Let $\mathfrak{U}' = \{\tau^{-M-m}\tilde{B} : \mu(\cup\mathfrak{B}\,|\tau^{-M-m}\tilde{B}) > 1 - \beta^{\frac{1}{2}}\}$. Observe that, since

$$\tau^{-M-m}\tilde{B} \subset \tau^{-M-m}(\tau^{M+m}\tilde{C}) = \tilde{C},$$
$$\mu(\cup\mathfrak{B}\,|\tau^{-M-m}\tilde{B}) = \mu(\cup\mathfrak{B} \cap \tau^{-M-m}\tilde{B} \cap \tilde{C})/\mu(\tau^{-M-m}\tilde{B} \cap \tilde{C})$$
$$= \mu(\cup\mathfrak{B}\,|\tau^{-M-m}\tilde{B}|\tilde{C}).$$
$$1 - \beta < \mu(\cup\mathfrak{B}\,|\tilde{C})$$
$$= \Sigma_{\tau^{-M-m}\tilde{B}\in\mathfrak{U}'}\, \mu(\tau^{-M-m}\tilde{B}|\tilde{C})\mu(\cup\mathfrak{B}\,|\tau^{-M-m}\tilde{B}|\tilde{C})$$
$$+\Sigma_{\tau^{-M-m}\tilde{B}\notin\mathfrak{U}'}\, \mu(\tau^{-M-m}\tilde{B}|\tilde{C})\mu(\cup\mathfrak{B}\,|\tau^{-M-m}\tilde{B}|\tilde{C})$$
$$\leqslant \mu(\cup\mathfrak{U}'|\tilde{C}) + (1 - \beta^{\frac{1}{2}})[1 - \mu(\cup\mathfrak{U}'|\tilde{C})]$$
$$\Rightarrow \quad \text{for} \quad \beta < 1, 0 < \beta^{\frac{1}{2}} - \beta < \beta^{\frac{1}{2}}\mu(\cup\mathfrak{U}'|\tilde{C}),$$
$$\text{i.e., } \mu(\cup\mathfrak{U}'|\tilde{C}) > 1 - \beta^{\frac{1}{2}}.$$

Hence

$$\Gamma \leqslant \Big\{ \Sigma_{\tau^{-M-m}\tilde{B} \in \mathfrak{U}'} \frac{\mu(\tilde{B})}{\mu(\tau^{M+m}\tilde{C})} \big[1 - \mu(\cup \mathfrak{B} | \tau^{-M-m}\tilde{B})\big]$$

$$+ \Sigma_{\tau^{-M-m}\tilde{B} \notin \mathfrak{U}'} \frac{\mu(\tilde{B})}{\mu(\tau^{M+m}\tilde{C})} \big[1 - \mu(\cup \mathfrak{B} | \tau^{-M-m}\tilde{B})\big] \Big\} \exp\Big(\frac{\beta^2}{3}\Big) + \beta$$

$$\leqslant \Big\{ \Sigma_{\tau^{-M-m}\tilde{B} \in \mathfrak{U}'} \beta^{\frac{1}{2}} \frac{\mu(\tau^{-M-m}\tilde{B})}{\mu(\tilde{C})} + \Sigma_{\tau^{-M-m}\tilde{B} \notin \mathfrak{U}'} \frac{\mu(\tau^{-M-m}\tilde{B})}{\mu(\tilde{C})} \Big\}.$$

$\exp(\beta^2/3) + \beta < \{ \beta^{\frac{1}{2}} \mu(\cup \mathfrak{U}'|\tilde{C}) + [1 - \mu(\cup \mathfrak{U}'|\tilde{C})]\} \exp(\beta^2/3) + \beta \leqslant 2\beta^{\frac{1}{2}} \exp(\beta^2/3) + \beta$. Thus for $\tilde{C} \in \tau^{-M-m}\mathfrak{S} \cap \mathfrak{U}$,

$$(3) \qquad |\phi_{L-l}(\lambda) - \textstyle\int_{\tilde{C}} F_\lambda(x, l, L)\, d\mu(x|\tilde{C})|$$

$$\leqslant |\Sigma_{j=1}^k \mu(A_j) \textstyle\int_{A_j} F_\lambda(x, l, L)\, d\mu(x|A_j)$$

$$- \textstyle\int_{\tau^{M+m}\tilde{C}} F_\lambda(x, l, L)\, d\mu(x|\tau^{M+m}\tilde{C})|$$

$$+ \Gamma + \frac{\beta^2}{3} + 2\|f\|_\infty (M + m)|\lambda|$$

$$\leqslant |\Sigma_{j=1}^k \mu(A_j) \textstyle\int_{A_j} F_\lambda(x, l, L)\, d\mu(x|A_j)$$

$$- \textstyle\int_{\tau^{M+m}\tilde{C}} F_\lambda(x, l, L)\, d\mu(x|\tau^{M+m}\tilde{C})|$$

$$+ 2\beta^{\frac{1}{2}} \exp\Big(\frac{\beta^2}{3}\Big) + \beta + \frac{\beta^2}{3} + 2\|f\|_\infty (M + m)|\lambda|.$$

Let B_j be an arbitrarily fixed atom of $\mathfrak{B}_{|A_j}$, $j = 1, \cdots, k$. For $j = 1, \cdots, k$,

$\int_{A_j} F_\lambda(x, l, L)\, d\mu(x|A_j) - \int_{B_j} F_\lambda(x, l, L)\, d\mu(x|B_j)$

$$= \frac{1}{\mu(A_j)} \Sigma_{B \in \mathfrak{B}_{|A_j}} \mu(B) \big[\textstyle\int_B F_\lambda(x, l, L)\, d\mu(x|B) - \int_{B_j} F_\lambda(x, l, L)\, d\mu(x|B_j) \big].$$

By Lemma 2 and by seeing that $l > l - M$,

$$|\textstyle\int_B F_\lambda(x, l, L)\, d\mu(x|B) - \int_{B_j} F_\lambda(x, l, L)\, d\mu(x|B_j)| \leqslant \exp\Big(\frac{\beta^2}{3}\Big) - 1.$$

Now one obtains

$$(4) \qquad |\textstyle\int_{A_j} F_\lambda(x, l, L)\, d\mu(x|A_j) - \int_{B_j} F_\lambda(x, l, L)\, d\mu(x|B_j)| \leqslant \exp\Big(\frac{\beta^2}{3}\Big) - 1.$$

$\int_{\tau^{M+m}\tilde{C}} F_\lambda(x, l, L)\, d\mu(x|\tau^{M+m}\tilde{C})$

$$= \Sigma_{j=0}^k \textstyle\int_{\tau^{M+m}\tilde{C} \cap A_j} F_\lambda(x, l, L)\, d\mu(x|\tau^{M+m}\tilde{C})$$

$$= \Sigma_{j=0}^k \mu\big(A_j|\tau^{M+m}\tilde{C}\big) \textstyle\int_{\tau^{M+m}\tilde{C} \cap A_j} F_\lambda(x, l, L)\, d\mu\big(x|A_j|\tau^{M+m}\tilde{C}\big).$$

Notice that for $j = 1, \cdots, k$, $\tau^{M+m}\tilde{C} \cap A_j$ consists of atoms $B \in \mathfrak{B}_{|A_j}$. Thus

$\int_{\tau^{M+m}\tilde{C} \cap A_j} F_\lambda(x, l, L)\, d\mu\big(x|A_j|\tau^{M+m}\tilde{C}\big)$

$$= \Sigma' \mu\big(B \cap \tau^{M+m}\tilde{C}|A_j|\tau^{M+m}\tilde{C}\big) \cdot \textstyle\int_B F_\lambda(x, l, L)\, d\mu(x|B)$$

where "Σ'" is the summation only over those atoms of $\mathcal{B}_{|A_j}$ intersecting $\tau^{M+m}\tilde{C}$. Using Lemma 2,

$$(5) \qquad |\textstyle\int_{\tau^{M+m}\tilde{C}\cap A_j} F_\lambda(x, l, L)\, d\mu(x|A_j|\tau^{M+m}\tilde{C})$$
$$-\textstyle\int_{B_j} F_\lambda(x, l, L)\, d\mu(x|B_j)| \leqslant \Sigma'\mu(B \cap \tau^{M+m}\tilde{C}|A_j|\tau^{M+m}\tilde{C})|\textstyle\int_B F_\lambda(x, l, L)\, d\mu(x|B)$$
$$-\textstyle\int_{B_j} F_\lambda(x, l, L)\, d\mu(x|B_j)| \leqslant \exp\left(\frac{\beta^2}{3}\right) - 1.$$

From (3), (4), and (5) one obtains for $\tilde{C} \in \tau^{-M-m}\mathcal{S} \cap \mathcal{U}$,

$$|\phi_{L-l}(\lambda) - \textstyle\int_{\tilde{C}} F_\lambda(x, l, L)\, d\mu(x|\tilde{C})|$$
$$\leqslant |\Sigma_{j=1}^k \mu(A_j)\textstyle\int_{B_j} F_\lambda(x, l, L)\, d\mu(x|B_j)$$
$$-\Sigma_{j=1}^k \mu(A_j|\tau^{M+m}\tilde{C})\textstyle\int_{B_j} F_\lambda(x, l, L)\, d\mu(x|B_j)|$$
$$+\Sigma_{j=1}^k \mu(A_j)|\textstyle\int_{B_j} F_\lambda(x, l, L)\, d\mu(x|B_j)$$
$$-\textstyle\int_{A_j} F_\lambda(x, l, L)\, d\mu(x|A_j)| + \Sigma_{j=1}^k \mu(A_j|\tau^{M+m}\tilde{C})$$
$$\times\left[\Sigma'\mu(B \cap \tau^{M+m}\tilde{C}|A_j|\tau^{M+m}\tilde{C})\right.$$
$$\left.\cdot|\textstyle\int_B F_\lambda(x, l, L)\, d\mu(x|B) - \textstyle\int_{B_j} F_\lambda(x, l, L)\, d\mu(x|B_j)|\right]$$
$$+2\beta^{\frac{1}{2}}\exp\left(\frac{\beta^2}{3}\right) + \beta + \frac{\beta^2}{3} + 2\|f\|_\infty(M+m)|\lambda|$$
$$\leqslant \Sigma_{j=1}^k|\mu(A_j) - \mu(A_j|\tau^{M+m}\tilde{C})| + 2\left[\exp\left(\frac{\beta^2}{3}\right) - 1\right] + 2\beta^{\frac{1}{2}}\exp\left(\frac{\beta^2}{3}\right)$$
$$+\beta + \frac{\beta^2}{3} + 2\|f\|_\infty(M+m)|\lambda|$$
$$< 2\beta + \frac{\beta^2}{3} + 2\left[\exp\left(\frac{\beta^2}{3}\right) - 1\right] + 2\beta^{\frac{1}{2}}\exp\left(\frac{\beta^2}{3}\right) + 2\|f\|_\infty(M+m)|\lambda|.$$

Therefore,

$$\Delta_2 \leqslant |\Sigma_{\tilde{C}\in\tau^{-M-m}\mathcal{S}\cap\mathcal{U}}\mu(\tilde{C})\textstyle\int_{\tilde{C}} F_\lambda(x, 0, l)\, d\mu(x|\tilde{C})$$
$$\times\left[\textstyle\int_{\tilde{C}} F_\lambda(x, l, L)\, d\mu(x|\tilde{C}) - \phi_{L-l}(\lambda)\right]| + \tfrac{8}{3}\beta$$
$$\leqslant \Sigma_{\tilde{C}\in\tau^{-M-m}\mathcal{S}\cap\mathcal{U}}\mu(\tilde{C})|\textstyle\int_{\tilde{C}} F_\lambda(x, l, L)\, d\mu(x|\tilde{C}) - \phi_{L-l}(\lambda)| + \tfrac{8}{3}\beta$$
$$< \tfrac{14}{3}\beta + \frac{\beta^2}{3} + 2\left[\exp\left(\frac{\beta^2}{3}\right) - 1\right]$$
$$+2\beta^{\frac{1}{2}}\exp\left(\frac{\beta^2}{3}\right) + 2\|f\|_\infty(M+m)|\lambda|$$
$$< 5\beta + 2\left[\exp\left(\frac{\beta^2}{3}\right) - 1\right]$$
$$+2\beta^{\frac{1}{2}}\exp\left(\frac{\beta^2}{3}\right) + 2\|f\|_\infty(M+m)|\lambda| \qquad \text{for} \quad \beta < 1.$$

As for Δ_1,

$$\begin{aligned}
\Delta_1 &= |\phi_L(\lambda) - \Sigma_{\tilde{C}\in\tau^{-M-m}\mathcal{P}_{l-2M-m}^{l-M-m}}\mu(\tilde{C})\textstyle\int_{\tilde{C}}F_\lambda(x,0,l)\\
&\quad \times d\mu(x|\tilde{C})\textstyle\int_{\tilde{C}}F_\lambda(x,l,L)\,d\mu(x|\tilde{C})|\\
&\leqslant |\phi_L(\lambda) - \Sigma_{B\in\mathcal{P}_l}\mu(B)\textstyle\int_B F_\lambda(x,0,l)\\
&\quad \times d\mu(x|B)\textstyle\int_B F_\lambda(x,l,L)\,d\mu(x|B)|\\
&\quad + |\Sigma_{B\in\mathcal{P}_l}\mu(B)\textstyle\int_B F_\lambda(x,0,l)\,d\mu(x|B)\textstyle\int_B F_\lambda(x,l,L)\,d\mu(x|B)\\
&\quad - \Sigma_{\tilde{C}\in\tau^{-M-m}\mathcal{P}_{l-2M-m}^{l-M-m}}\mu(C)\textstyle\int_{\tilde{C}}F_\lambda(x,0,l)\,d\mu(x|\tilde{C})\textstyle\int_{\tilde{C}}F_\lambda(x,l,L)\,d\mu(x|\tilde{C})|\\
&= \Gamma_1 + \Gamma_2.
\end{aligned}$$

$$\begin{aligned}
\Gamma_1 &= |\Sigma_{B\in\mathcal{P}_l}\mu(B)[\textstyle\int_B F_\lambda(x,0,L)\,d\mu(x|B) - \textstyle\int_B F_\lambda(x,0,l)\\
&\quad \times d\mu(x|B)\textstyle\int_B F_\lambda(x,l,L)\,d\mu(x|B)]|\\
&\leqslant \Sigma_{B\in\mathcal{P}_l}\mu(B)|\textstyle\int_B F_\lambda(x,l,L)\textstyle\int_B[F_\lambda(x,0,l)\\
&\quad - F_\lambda(x_1,0,l)]\,d\mu(x_1|B)\,d\mu(x|B)\\
&\leqslant \Sigma_{B\in\mathcal{P}_l}\mu(B)\textstyle\int_B\textstyle\int_B|F_\lambda(x,0,l)\\
&\quad - F_\lambda(x_1,0,l)|\,d\mu(x_1|B)\,d\mu(x|B)\\
&\leqslant \Sigma_{B\in\mathcal{P}_l}\mu(B)\textstyle\int_B\textstyle\int_B|\lambda|\,|\Sigma_0^{l-1}[f(\tau^j x) - f(\tau^j x_1)]|\\
&\quad \times d\mu(x_1|B)\,d\mu(x|B)\\
&\leqslant \Sigma_{B\in\mathcal{P}_l}\mu(B)\textstyle\int_B\textstyle\int_B|\lambda|\Sigma_0^{l-1}|f(\tau^j x) - f(\tau^j x_1)|\\
&\quad \times d\mu(x_1|B)\,d\mu(x|B).
\end{aligned}$$

Because f is Hölder with exponent $\delta \in (0, 1]$, if $x_1, x_2 \in B \in \mathcal{P}_l$ and M_f is a fixed Hölder constant for f, then $|f(\tau^j x_1) - f(\tau^j x_2)| \leqslant M_f|\tau^j x_1 - \tau^j x_2|^\delta \leqslant M_f s^{-(l-j)\delta}$ since $|x_1 - x_2| \leqslant s^{-l}$. It follows that

$$\begin{aligned}
\Sigma_0^{l-1}|f(\tau^j x_1) - f(\tau^j x_2)| &\leqslant M_f\Sigma_0^{l-1}|\tau^j x_1 - \tau^j x_2|^\delta \leqslant M_f\Sigma_0^{l-1}s^{-(l-j)\delta}\\
&= M_f\left(\frac{s^{-\delta}}{1-s^{-\delta}}\right)[1 - s^{-\delta(l-1)}] < M_f\left(\frac{s^{-\delta}}{1-s^{-\delta}}\right)\\
&= K^* \quad \text{and} \quad \Gamma_1 \leqslant |\lambda|K^*.
\end{aligned}$$

$$\begin{aligned}
\Gamma_2 &\leqslant |\Sigma_{B\in\mathfrak{B}}\mu(B)\textstyle\int_B F_\lambda(x,0,l)\,d\mu(x|B)\textstyle\int_B F_\lambda(x,l,L)\,d\mu(x|B)\\
&\quad - \Sigma_{\tilde{C}\in\mathfrak{A}}\mu(\tilde{C})\textstyle\int_{\tilde{C}}F_\lambda(x,0,l)\,d\mu(x|\tilde{C})\textstyle\int_{\tilde{C}}F_\lambda(x,l,L)\,d\mu(x|\tilde{C})| + \tfrac{1}{3}(\beta+\beta^2)\\
&\leqslant |\Sigma_{B\in\mathfrak{B}}\textstyle\int_B F_\lambda(x,0,l)\,d\mu(x)\textstyle\int_B F_\lambda(x,l,L)\,d\mu(x|B)\\
&\quad - \Sigma_{\tilde{C}\in\mathfrak{A}}\textstyle\int_{\tilde{C}'}F_\lambda(x,0,l)\,d\mu(x)\,\times
\end{aligned}$$

$$
\begin{aligned}
&\textstyle\int_{\tilde{C}} F_\lambda(x, l, L)\, d\mu(x|\tilde{C})| + \frac{4}{3}\beta + \frac{1}{3}\beta^2 \\
&\quad \leqslant |\Sigma_{B\in\mathfrak{B}} \textstyle\int_B F_\lambda(x, 0, l)\, d\mu(x) \int_B F_\lambda(x, l, L)\, d\mu(x|B) \\
&\qquad -\Sigma_{\tilde{C}\in\mathfrak{A}} \textstyle\int_{\tilde{C}'} F_\lambda(x, 0, l)\, d\mu(x) \cdot \int_{\tilde{C}'} F_\lambda(x, l, L)\, d\mu(x|\tilde{C}) \\
&\qquad +\Sigma_{\tilde{C}\in\mathfrak{A}} \textstyle\int_{\tilde{C}'} F_\lambda(x, 0, l)\, d\mu(x) \int_{\tilde{C}\setminus\tilde{C}'} F_\lambda(x, l, L)\, d\mu(x|\tilde{C})| + \frac{4}{3}\beta + \frac{1}{3}\beta^2 \\
&\quad \leqslant |\Sigma_{B\in\mathfrak{B}} \textstyle\int_B F_\lambda(x, 0, l)\, d\mu(x) \int_B F_\lambda(x, l, L)\, d\mu(x|B) \\
&\qquad -\Sigma_{\tilde{C}\in\mathfrak{A}} \textstyle\int_{\tilde{C}'} F_\lambda(x, 0, l)\, d\mu(x) \\
&\qquad \cdot \textstyle\int_{\tilde{C}'} F_\lambda(x, l, L)\, d\mu(x|\tilde{C})| + \frac{7}{3}\beta + \frac{1}{3}\beta^2 \\
&\quad \leqslant |\Sigma_{B\in\mathfrak{B}} \textstyle\int_B F_\lambda(x, 0, l)\, d\mu(x) \int_B F_\lambda(x, l, L)\, d\mu(x|B) \\
&\qquad -\Sigma_{\tilde{C}\in\mathfrak{A}} \textstyle\int_{\tilde{C}'} F_\lambda(x, 0, l)\, d\mu(x) \cdot \int_{\tilde{C}'} F_\lambda(x, l, L)\, d\mu(x|\tilde{C}')| \\
&\qquad +\Sigma_{\tilde{C}\in\mathfrak{A}} \mu(\tilde{C}') | \textstyle\int_{\tilde{C}'} F_\lambda(x, l, L) \left[1 - \dfrac{\mu(\tilde{C}')}{\mu(\tilde{C})} \right] d\mu(x|\tilde{C}')| + \frac{7}{3}\beta + \frac{1}{3}\beta^2 \\
&\quad \leqslant |\Sigma_{B\in\mathfrak{B}} \textstyle\int_B F_\lambda(x, 0, l)\, d\mu(x) \int_B F_\lambda(x, l, L)\, d\mu(x|B) \\
&\qquad -\Sigma_{\tilde{C}\in\mathfrak{A}} \textstyle\int_{\tilde{C}'} F_\lambda(x, 0, l)\, d\mu(x) \int_{\tilde{C}'} F_\lambda(x, l, L) \cdot d\mu(x|\tilde{C}')| \\
&\qquad +\Sigma_{\tilde{C}\in\mathfrak{A}} \mu(\tilde{C}')\beta + \frac{7}{3}\beta + \frac{1}{3}\beta^2 \\
&\quad \leqslant |\Sigma_{\tilde{C}\in\mathfrak{A}} \Sigma_{B\in\mathfrak{B}_{|\tilde{C}'}} \textstyle\int_B F_\lambda(x, 0, l)\, d\mu(x) [\int_B F_\lambda(x, l, L)\, d\mu(x|B) \\
&\qquad - \textstyle\int_{\tilde{C}'} F_\lambda(x, l, L)\, d\mu(x|\tilde{C}')]| \\
&\qquad + \frac{10}{3}\beta + \frac{2}{3}\beta^2 \qquad \text{(because } \cup\mathfrak{A} \text{ need not be equal to } \cup\mathfrak{B}\text{)} \\
&\quad \leqslant \Sigma_{\tilde{C}\in\mathfrak{A}} \Sigma_{B\in\mathfrak{B}_{|\tilde{C}'}} \mu(B) | \textstyle\int_B F_\lambda(x, l, L)\, d\mu(x|B) \\
&\qquad - \textstyle\int_{\tilde{C}'} F_\lambda(x, l, L)\, d\mu(x|\tilde{C}')| + 4\beta
\end{aligned}
$$

for $\beta \leqslant 1$.

Observe that $\tilde{C}'$ consists of atoms in $\mathfrak{B}$ for which $B_1, B_2 \in \mathfrak{B}_{|\tilde{C}'}$ implies $\tau^{l-M}B_1 = \tau^{l-M}B_2$, i.e., $\tilde{C}' \subset A_j$ for some $j = 1, \cdots, k$. Allowing B_j to be an arbitrarily fixed atom of $\mathfrak{B}_{|A_j|}$, by Lemma 2 and reasoning analogous to that used to derive (4),

$$|\textstyle\int_B F_\lambda(x, l, L)\, d\mu(x|B) - \int_{B_j} F_\lambda(x, l, L)\, d\mu(x|B_j)| \leqslant \exp\left(\dfrac{\beta^2}{3}\right) - 1;$$

$$|\textstyle\int_{B_j} F_\lambda(x, l, L)\, d\mu(x|B_j) - \int_{\tilde{C}'} F_\lambda(x, l, L)\, d\mu(x|\tilde{C}')| \leqslant \exp\left(\dfrac{\beta^2}{3}\right) - 1.$$

Therefore $\Gamma_2 \leqslant 2[\exp(\beta^2/3) - 1]$ and $\Delta_1 \leqslant K^*|\lambda| + 2[\exp(\beta^2/3) - 1] + 4\beta$. One now obtains that

$$(6)\quad |\phi_L(\lambda) - \phi_{L-l}(\lambda)\phi_l(\lambda)| \leqslant K^*|\lambda| + 2\left[\exp\left(\frac{\beta^2}{3}\right) - 1\right] + 4\beta + 5\beta$$

$$+2\left[\exp\left(\frac{\beta^2}{3}\right) - 1\right] + 2\beta^{\frac{1}{2}} \exp\left(\frac{\beta^2}{3}\right) + 2\|f\|_\infty(M + m)|\lambda|$$

$$< |\lambda|\left[K^* + 2\|f\|_\infty(M + m)\right]$$

$$+4\left[\exp\left(\frac{\beta^2}{3}\right) - 1\right] + 9\beta + 2\beta^{\frac{1}{2}} \exp\left(\frac{\beta^2}{3}\right)$$

$$= |\lambda|\left[K^* + 2\|f\|_\infty(M + m)\right] + V(\beta).$$

Choosing $n \geqslant 1$ and setting $L/l = n$, applying (6) successively to $\phi_L(\lambda), \phi_{L-l}(\lambda), \cdots$, one obtains for L

$$(7)\quad |\phi_L(\lambda) - \phi_l^n(\lambda)| \leqslant n\left\{|\lambda|\left[K^* + 2\|f\|_\infty(M + m)\right] + V(\beta)\right\}.$$

Substituting $\lambda(D_L(f))^{-\frac{1}{2}}$ for λ in (7),

$$(8)\quad |\phi_L\left(\lambda(D_L(f))^{-\frac{1}{2}}\right) - \phi_l^n\left(\lambda(D_L(f))^{-\frac{1}{2}}\right)|$$

$$\leqslant n\left\{|\lambda|(D_L(f))^{-\frac{1}{2}}\left[K^* + 2\|f\|_\infty(M + m)\right] + V(\beta)\right\}.$$

For $l > 0$, define nonincreasing $\beta_l > 0$ for which $\beta_l \to 0$, $-\log \beta_l = o(\log l)$, and $M_l + m_l = o(\log l)$ since $M + m$ is determined by β which in this situation will depend upon l. With this choice of β_l, $|\lambda|(cl)^{-\frac{1}{2}}[K^* + 2f_\infty(M_l + m_l)] + V(\beta_l) \to 0$ as $l \to \infty$ where c is as in hypothesis 1. Now, for l, define a positive, nondecreasing, integer-valued function of l, $n(l)$ such that

(i) $n(l) = o(-\log \beta_l)$ $(= o(o(\log l)) = o(\log l))$;

(ii) $n(l) \to \infty$ as $l \to \infty$;

(iii) $n(l + 1)$ is either $n(l)$ or $n(l) + 1$ for each l.

Because $D_L(f) \sim cL$,

$$(9)\quad n(l)\left\{|\lambda|(D_{l\,n(l)})^{-\frac{1}{2}}\left[K^* + 2\|f\|_\infty(M_l + m_l)\right] + V(\beta_l)\right\}$$

$$\sim n(l)\left\{|\lambda|(c\,l\,n(l))^{-\frac{1}{2}}\left[K^* + 2\|f\|_\infty(M_l + m_l)\right] + V(\beta_l)\right\}$$

$$\leqslant n(l)\left\{|\lambda|(cl)^{-\frac{1}{2}}\left[K^* + 2\|f\|_\infty(M_l + m_l)\right] + V(\beta_l)\right\}.$$

By the choice of $n(l)$, M_l, m_l, and β_l, one finds

$$\frac{n(l)}{l^{\frac{1}{2}}} \to 0, \quad \frac{n(l)(M_l + m_l)}{l^{\frac{1}{2}}} = \frac{o(\log l)o(\log l)}{l^{\frac{1}{2}}} = \frac{o\left[(\log l)^2\right]}{l^{\frac{1}{2}}} \to 0,$$

and, as for $n(l)V(\beta_l)$, the term $n(l)[2(\beta_l)^{\frac{1}{2}} \exp(\beta_l^2/3)]$ will approach 0 no more slowly than $2e\, n(l)(\beta_l)^{\frac{1}{2}}$ for $\beta_l \leqslant 3^{\frac{1}{2}}$, but $n(l)(\beta_l)^{\frac{1}{2}} = o(-\log \beta_l)(\beta_l)^{\frac{1}{2}} \to 0$. Thus (9) goes to 0 as $l \to \infty$, and (8) goes to 0 uniformly for λ in each finite interval.

For $L > 0$, set $l(L) = \max\{l : l\,n(l) \leqslant L\}$ and $\Lambda(L) = l(L)n(l(L))$. One wishes to show that

$$\left|\phi_L\left(\frac{\lambda}{(D_L(f))^{\frac{1}{2}}}\right) - \phi_{\Lambda(L)}\left(\frac{\lambda}{(D_{\Lambda(L)}(f))^{\frac{1}{2}}}\right)\right| \to 0$$

uniformly for λ in each finite interval, i.e., the limiting distribution for $S_L(D_L(f))^{-\frac{1}{2}}$ and the one for $S_{\Lambda(L)}(D_{\Lambda(L)}(f))^{-\frac{1}{2}}$ are the same where $S_L = \Sigma_{j=0}^{L-1} f \circ \tau^j$. Note that

$$\frac{S_L}{(D_L(f))^{\frac{1}{2}}} = \frac{S_L - S_{\Lambda(L)}}{(D_L(f))^{\frac{1}{2}}} + \frac{(D_{\Lambda(L)}(f))^{\frac{1}{2}}}{(D_L(f))^{\frac{1}{2}}} \frac{S_{\Lambda(L)}}{(D_{\Lambda(L)}(f))^{\frac{1}{2}}}.$$

Again by hypothesis 1,

$$\frac{D_{\Lambda(L)}(f)}{D_L(f)} \sim \frac{c\Lambda(L)}{cL} = 1 - \frac{L - \Lambda(L)}{L}.$$

By choice of $n(l)$,

$$\begin{aligned}\frac{L - \Lambda(L)}{L} &\leqslant \frac{L - \Lambda(L)}{\Lambda(L)} \leqslant \frac{[l(L)+1]n(l(L)+1) - l(L)n(l(L))}{l(L)n(l(L))}\\ &\leqslant \frac{[l(L)+1][n(l(L))+1] - l(L)n(l(L))}{l(L)n(l(L))}\\ &= \frac{1}{n(l(L))} + \frac{1}{l(L)} + \frac{1}{l(L)n(l(L))} \to 0\end{aligned}$$

as $L \to \infty$. Consequently $D_{\Lambda(L)}(f)/D_L(f) \sim \Lambda(L)/L \to 1$ as $L \to \infty$. If one can show that

$$\frac{1}{D_L(f)} \int_0^1 |S_L - S_{\Lambda(L)}|^2 \, d\mu \to 0 \quad \text{as} \quad L \to \infty, \tag{10}$$

then one is done because (10) implies $(D_L(f))^{-\frac{1}{2}}|S_L - S_{\Lambda(L)}| \to 0$ in probability which implies $S_L(D_L(f))^{-\frac{1}{2}}$ and $S_{\Lambda(L)}(D_{\Lambda(L)}(f))^{-\frac{1}{2}}$ have the same limiting distribution (see, e.g., [1]). Because μ is τ-invariant,

$$\int_0^1 |S_L - S_{\Lambda(L)}|^2 \, d\mu = \int_0^1 |S_{L-\Lambda(L)}|^2 \, d\mu = D_{L-\Lambda(L)}(f),$$

and

$$\frac{1}{D_L(f)} \int_0^1 |S_L - S_{\Lambda(L)}|^2 \, d\mu = \frac{D_{L-\Lambda(L)}(f)}{D_L(f)} \sim \frac{L - \Lambda(L)}{L} \to 0$$

as shown above.

Returning to $\phi_l^{n(l)}(\lambda(D_{l\,n(l)}(f)^{-\frac{1}{2}})$, and setting $L(l) = l\,n(l)$, one notices that the characteristic function is the one for a sum of $n(l)$ independent, identically distributed random variables $\xi_1, \xi_2, \cdots, \xi_{n(l)}$ with distributions given by

$$P_{\xi_i}(z) = \mu\left\{x : (D_{L(l)}(f))^{-\frac{1}{2}}\left[\Sigma_{j=0}^{l-1} f(\tau^j x)\right] < z\right\}$$

for $i = 1, \cdots, n(l)$. In order that $\phi_l^{n(l)}(\lambda(D_{L(l)}(f))^{-\frac{1}{2}}) \to \exp(-\lambda^2/2)$ uniformly for λ in each finite interval it is necessary and sufficient that the Lindeberg condition hold (see, e.g., [4]) which in this situation has the following form: for any fixed $\gamma > 0$,

$$n(l)\int_{|\xi|>\gamma} x^2\, dP_\xi(x) \to 0 \quad \text{or} \quad n(l)\int_{\mathfrak{D}_\gamma} \frac{1}{D_{L(l)}(f)}\left[\Sigma_0^{l-1} f(\tau^j x)\right]^2 d\mu(x) \to 0$$

as $l \to \infty$ where $\mathfrak{D}_\gamma = \{x : |\Sigma_0^{l-1} f(\tau^j x)| > \gamma(D_{L(l)}(f))^{\frac{1}{2}}\}$. (Notice that $(n(l)/D_{L(l)}(f)) \sim (n(l)/cL(l)) = (cl)^{-1} \sim (D_l(f))^{-1}$ and $(D_{L(l)}(f))^{\frac{1}{2}} \sim (n(l))^{\frac{1}{2}}(D_l(f))^{\frac{1}{2}}$.) By condition 2, for any given $\varepsilon > 0$, there are constants $N(\varepsilon)$ and $L(\varepsilon)$ such that if $l > L(\varepsilon)$, then $(D_l(f))^{-1}\int_{\Omega_\varepsilon}(\Sigma_0^{l-1} f \circ \tau^j)^2\, d\mu < \varepsilon$ for $\Omega_\varepsilon = \{x : |\Sigma_0^{l-1} f(\tau^j x)| > N(\varepsilon)(D_l(f))^{\frac{1}{2}}\}$. Given $\varepsilon > 0$, for l large enough, $\mathfrak{D}_\gamma \subset \Omega_\varepsilon$ and

$$n(l)\int_{\mathfrak{D}_\gamma} \frac{1}{D_{L(l)}(f)}\left(\Sigma_0^{l-1} f \circ \tau^j\right)^2 d\mu \sim \frac{1}{D_l(f)}\int_{\mathfrak{D}_\gamma}\left(\Sigma_0^{l-1} f \circ \tau^j\right)^2 d\mu$$

$$\leqslant \frac{1}{D_l(f)}\int_{\Omega_\varepsilon}\left(\Sigma_0^{l-1} f \circ \tau^j\right)^2 d\mu < \varepsilon.$$

As $l \to \infty$, $(D_l(f))^{-1}\int_{\mathfrak{D}_\gamma}(\Sigma_0^{l-1} f \circ \tau^j)^2\, d\mu \to 0$ because $\mathfrak{D}_\gamma$ will be contained in Ω_ε for smaller and smaller ε's as $l \to 0$. Since the Lindeberg condition is satisfied, $\phi_l^{n(l)}(\lambda(D_{L(l)}(f))^{-\frac{1}{2}}) \to \exp(-\lambda^2/2)$ uniformly for λ in each finite interval. It has been shown that

$$|\phi_{L(l)}\left(\lambda(D_{L(l)}(f))^{-\frac{1}{2}}\right) - \phi_l^{n(l)}\left(\lambda(D_{L(l)}(f))^{-\frac{1}{2}}\right)| \to 0$$

uniformly for λ in each finite interval. Consequently, $\phi_{L(l)}(\lambda(D_{L(l)}(f))^{-\frac{1}{2}}) \to \exp(-\lambda^2/2)$ uniformly for λ in each finite interval. Finally, because $S_L(D_L(f))^{-\frac{1}{2}}$ and $S_{\Lambda(L)}(D_{\Lambda(L)}(f))^{-\frac{1}{2}}$ have the same limiting distribution, $\phi_L(\lambda(D_L(f))^{-\frac{1}{2}}) \to \exp(-\lambda^2/2)$ uniformly for λ in each finite interval as $L \to \infty$.

REMARKS.

1. Let $B(m) = \int_0^1 f(\tau^m x) f(x)\, d\mu(x)$, the correlation function. From the results of Leonov [6], if $\Sigma_{m \geqslant 1}|mB(m)| < \infty$, then the spectral density $r(\rho)$ exists for the process $(f \circ \tau^m)_{m \geqslant 0}$ and c in condition 1 of the theorem can be taken as $2\pi r(0)$. (By spectral density, one means $B(m) = \int_{-\pi}^{\pi} \exp(i\rho m) r(\rho)\, d\rho$.)

2. As mentioned in Bunimovitch's paper [3], in order for condition (2) to be satisfied, it is sufficient that for some constant $K > 0$,

$$\limsup_{L\to\infty} \frac{1}{L^2}\int_0^1\left(\Sigma_{j=0}^{L-1} f(\tau^j x)\right)^2 d\mu(x) \leqslant K.$$

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