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PERFECT MIXTURES OF PERFECT MEASURES

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It is shown that all the possible cases can arise in the mixture problem with respect to perfectness of probability measures. A characterization of perfectness is obtained through properties of a countably generated sub-σ-algebra given which there is a regular conditional probability. Perfectness of a perfect mixture of perfect measures is characterized.

0. Introduction and summary. Let $(X, \mathcal{Q})$ and $Y, \mathcal{B})$ be two Borel spaces and let $\mu(x, B)$ be a transition probability on $X \times \mathcal{B}$; that is, $\mu(x, B)$ is a function defined on $X \times \mathcal{B}$ taking values in [0, 1] such that for every x in X, $\mu(x, .)$ is a probability on $\mathcal{B}$ and for every $B \in \mathcal{B}$, $\mu(. , B)$ is $\mathcal{Q}$-measurable. Let λ be a probability on $\mathcal{Q}$. The set function μ defined on $\mathcal{B}$ by

$$\mu(B) = \int \mu(x, B)\, d\lambda \qquad B \in \mathcal{B}$$

is a probability on $\mathcal{B}$. μ is called the λ-mixture of the $\mu(x, .)$'s. λ is called a mixing measure, the $\mu(x, .)$'s are called mixand measures and μ is called a mixture measure. The properties of a mixture measure depend on those of the mixing and mixand measures. We complete here the study of the role of perfectness of probability measures in the mixture problem which was started in Rodine (1966) and pursued in Ramachandran (1974).

We call a mixture measure perfect mixture or nonperfect mixture according as the mixing measure is perfect or nonperfect. In [3] two conjectures of Rodine (1966) were settled. In Section 1 we illustrate by examples that all the possible cases can arise in the mixture problem with regard to perfectness of measures. In Section 2, we give a characterization of perfectness in separable Borel spaces through properties of a countably generated sub-σ-algebra given which there is a regular conditional probability. In Section 3, using the results of Section 2, we characterize perfectness of a perfect mixture of perfect measures.

All measures considered in this paper are probabilities. $\mathcal{B}_{[0, 1]}$ denotes the Borel σ-algebra of [0, 1]. A Borel space $(X, \mathcal{Q})$ is called separable if $\mathcal{Q}$ is separable, that is, $\mathcal{Q}$ is countably generated and contains all singletons. A probability space is called separable space if the underlying Borel space is separable. For other terminology used in this paper refer to Neveu (1965).

Let $(X, \mathcal{Q}, P)$ be a probability space. P is called perfect if for every $\mathcal{Q}$-measurable real valued function f on X and every subset A of the real line for which

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$f^{-1}A \in \mathcal{A}$ there is a linear Borel set B contained in A such that $P(f^{-1}A) = P(f^{-1}B)$. It is known (see Sazonov (1962)) that:

(P1) a measure P on $(X, \mathcal{A})$ is perfect if and only if for every $\mathcal{A}$-measurable real-valued function f on X there is a linear Borel set $B(f)$ contained in $f(X)$ such that $P(f^{-1}B(f)) = 1$;

(P2) a measure is perfect if and only if its restriction to every countably generated sub-σ-algebra is perfect;

(P3) the restriction to any sub-σ-algebra of a perfect measure is perfect;

(P4) a measure on a product space is perfect if and only if every marginal is perfect.

It follows from (P1) that every discrete measure is perfect and that every 0–1 valued measure is perfect.

1. The examples. Rodine (1966) showed that a mixture of perfect measures, in general, is not perfect and conjectured that perfect mixtures of perfect measures are perfect. In [3] three examples were constructed to show that his conjecture is false and to answer other questions raised by him. Table 1 lists all the cases that can arise in the mixture problem and examples illustrating each case.

TABLE 1

No.	Mixing measure	Mixand measure	Mixture measure	Example
1	perfect	perfect	perfect	1.2
2	perfect	nonperfect	perfect	1.4
3	perfect	perfect	nonperfect	1.3
4	perfect	nonperfect	nonperfect	1.6
5	nonperfect	perfect	perfect	1.5
6	nonperfect	nonperfect	perfect	1.7
7	nonperfect	perfect	nonperfect	1.1
8	nonperfect	nonperfect	nonperfect	1.8

EXAMPLE 1.1 (*Rodine*). Let $(X, \mathcal{A}, \lambda)$ be a nonperfect probability space. Let $Y = X$, $\mathcal{B} = \mathcal{A}$ and let $\mu(x, B)$ on $X \times \mathcal{B}$ be defined by $\mu(x, B) = 1_B(x)$. Then each $\mu(x, .)$, being a 0–1 valued measure, is perfect. The mixture measure $\mu = \lambda$ is nonperfect.

EXAMPLE 1.2. Let $(X, \mathcal{A}, \lambda)$ be a perfect probability space. Let $Y = X$, $\mathcal{B} = \mathcal{A}$ and let $\mu(x, B)$ on $X \times \mathcal{B}$ be defined by $\mu(x, B) = 1_B(x)$. Then the mixture $\mu = \lambda$ and is perfect.

EXAMPLE 1.3. Example 1 in [3] where $X = [0, 1]$, $\mathcal{A} = \mathcal{B}_{[0,1]}$, λ is the Lebesgue measure on $\mathcal{B}_{[0,1]}$ and $(Y, \mathcal{B})$ is a separable space.

EXAMPLE 1.4. Example 2 in [3].

EXAMPLE 1.5. Example 3 in [3].

EXAMPLE 1.6. Let $X = \{0\}$, $\mathcal{A} = \{X, \varnothing\}$ and let λ on $(X, \mathcal{A})$ be the only probability given by $\lambda(X) = 1$, $\lambda(\varnothing) = 0$. Let $(Y, \mathcal{B}, \mu)$ be a nonperfect probability space. Define $\mu(x, B) = \mu(B)$ for all x in X. Then μ is a nonperfect measure which is a perfect mixture of nonperfect measures.

For the construction of our next two examples we need the following result:

For each $i = 1, 2$ let $(X_i, \mathcal{A}_i, \lambda_i)$ be two probability spaces, $(Y_i, \mathcal{B}_i)$ be two Borel spaces and $\mu_i(x_i, B_i)$ on $X_i \times \mathcal{B}_i$ be two transition probabilities. Let μ_i be the λ_i-mixture of $\mu_i(x_i, .)$'s. If we define $X = X_1 \times X_2$, $\mathcal{A} = \mathcal{A}_1 \otimes \mathcal{A}_2$, $\lambda = \lambda_1 \times \lambda_2$, $Y = Y_1 \times Y_2$, $\mathcal{B} = \mathcal{B}_1 \otimes \mathcal{B}_2$ and $\mu((x_1, x_2), .) = \mu_1(x_1, .) \times \mu_2(x_2, .)$ then the following lemma can be easily established.

LEMMA 1. *$\mu((x_1, x_2), B)$ is a transition probability on $X \times \mathcal{B}$ such that $\mu = \mu_1 \times \mu_2$ where μ is the λ-mixture of the $\mu((x_1, x_2), .)$'s.*

We shall call μ the product mixture of μ_1 and μ_2.

EXAMPLE 1.7. Let μ_1 and μ_2 be obtained as in Examples 1.4 and 1.5 respectively. Then by Lemma 1 and (P4) it follows that the product mixture is a perfect measure which is a nonperfect mixture of nonperfect measures.

EXAMPLE 1.8. Let μ_1 and μ_2 be obtained as in Examples 1.1 and 1.4 respectively. Then, by Lemma 1 and (P4), the product mixture is a nonperfect measure which is a nonperfect mixture of nonperfect measures.

2. A characterization of perfectness in separable spaces. Let $(Z, \mathcal{C}, P)$ be a separable probability space and let $\mathcal{A}$ be a countably generated sub-σ-algebra of $\mathcal{C}$. We assume throughout this section that there exists a regular conditional probability $\mu(z, C)$ on $Z \times \mathcal{C}$ given $\mathcal{A}$ which is proper a.s. $[P|_{\mathcal{A}}]$; that is, there exists $N \in \mathcal{A}$ with $P(N) = 0$ such that $\mu(z, .)$ is proper for all $x \notin N$. For relevant definitions and results used in this section refer to [4].

LEMMA 2. *Suppose P on $(Z, \mathcal{C})$ is perfect. If $(Z', \mathcal{C}', P')$ is a separable probability space and if $h : Z \to Z'$ is a $\mathcal{C}$-measurable map such that*
(i) $Ph^{-1} = P'$, and
(ii) h is 1–1 on $Z_0 \in \mathcal{C}$ with $P(Z_0) = 1$, then P' is perfect and $h(Z) \in \overline{\mathcal{C}'}^{P'}$ where $\overline{\mathcal{C}'}^{P'}$ is the completion of $\mathcal{C}'$ with respect to P'.

PROOF. Suppose f is a $\mathcal{C}'$-measurable real-valued function. Then $f \circ h_{Z_0}$ is a $\mathcal{C} \cap Z_0$-measurable real-valued function where $h_{Z_0} = h|_{Z_0}$. Since $(Z_0, \mathcal{C} \cap Z_0, P|_{Z_0})$ is perfect there is a linear Borel set $B \subset f \circ h_{Z_0}(Z_0)$ such that $Ph_{Z_0}^{-1}(f^{-1}B) = 1$. Hence $Ph^{-1}(f^{-1}B) = P'(f^{-1}B) = 1$ and $B \subset f(Z')$. Thus, by (P1), P' is perfect. Now take f to be 1–1. Then further $f^{-1}B \subset h(Z)$ and $P'(f^{-1}B) = 1$ implies that $h(Z) \in \overline{\mathcal{C}'}^{P'}$.

Suppose now that there is no partial selector for $\mathcal{A}$ of positive measure. Then there is an independent complement $\mathcal{A}^*$ of $\mathcal{A}$ which is countably generated (see Theorem 3 of [4]). Let α and α^* denote the quotient maps on Z with respect to atoms of $\mathcal{Q}$ and atoms of $\mathcal{Q}^*$ respectively. Let $(Z_\alpha, \mathcal{Q}_\alpha, P_\alpha)$ and $(Z_{\alpha^*}, \mathcal{Q}_{\alpha^*}, P_{\alpha^*})$ be the quotient spaces of $(Z, \mathcal{Q}, P|_{\mathcal{Q}})$ and $(Z, \mathcal{Q}^*, P|_{\mathcal{Q}^*})$ induced by α and α^* respectively. Consider the separable space $(Z', \mathcal{C}', P') = (Z_\alpha \times Z_{\alpha^*}, \mathcal{Q}_\alpha \otimes \mathcal{Q}_{\alpha^*}, P_\alpha \times P_{\alpha^*})$. Define the map $h_{\mathcal{Q}} : Z \to Z'$ by

$$h_{\mathcal{Q}}(z) = (\alpha(z), \alpha^*(z))$$

where $\alpha(z)$ and $\alpha^*(z)$ are respectively the $\mathcal{Q}$-atom and $\mathcal{Q}^*$-atom containing z. Since $\mathcal{Q}$ and $\mathcal{Q}^*$ are complementary let $Z_0 \in \mathcal{C}$ with $P(Z_0) = 1$ be such that $(\mathcal{Q} \vee \mathcal{Q}^*) \cap Z_0 = \mathcal{C} \cap Z_0$.

THEOREM 1. *Suppose there is no partial selector for* $\mathcal{Q}$ *of positive measure. Then* $h_{\mathcal{Q}}$ *is measurable and* $Ph_{\mathcal{Q}}^{-1} = P'$. *Further if* $Z_0' = h_{\mathcal{Q}}(Z_0)$ *then* $(Z_0, \mathcal{C} \cap Z_0, P|_{Z_0})$ *and* $(Z_0', \mathcal{C}' \cap Z_0', P'|_{Z_0})$ *are isomorphic.*

PROOF. Let $A_\alpha \in \mathcal{Q}_\alpha$ and $A_{\alpha^*} \in \mathcal{Q}_{\alpha^*}$. Then

$$h_{\mathcal{Q}}^{-1}(A_\alpha \times A_{\alpha^*}) = \alpha^{-1}A_\alpha \cap \alpha^{*-1}A_{\alpha^*} \quad \text{and}$$

$$\begin{aligned} P'(A_\alpha \times A_{\alpha^*}) &= P_\alpha(A_\alpha)P_{\alpha^*}(A_{\alpha^*}) \\ &= P(\alpha^{-1}A_\alpha)P(\alpha^{*-1}A_{\alpha^*}) \\ &= P(\alpha^{-1}A_\alpha \cap \alpha^{*-1}A_{\alpha^*}) \quad \text{(since } \mathcal{Q} \text{ and } \mathcal{Q}^* \text{ are independent)} \\ &= Ph_{\mathcal{Q}}^{-1}(A_\alpha \times A_{\alpha^*}). \end{aligned}$$

Thus $h_{\mathcal{Q}}$ is measurable and $P' = Ph_{\mathcal{Q}}^{-1}$.

Clearly $h_{\mathcal{Q}}$ is 1–1 on Z_0. In order to prove that $(Z_0, \mathcal{C} \cap Z_0, P|_{Z_0})$ and $(Z_0', \mathcal{C}' \cap Z_0', P'|_{Z_0'})$ are isomorphic it remains to show that $h_{\mathcal{Q}}(\mathcal{C} \cap Z_0) \subset \mathcal{C}' \cap Z_0'$. But

$$h_{\mathcal{Q}}(A \cap Z_0) = (\alpha A \times Z_{\alpha^*}) \cap Z_0' \qquad \text{if} \quad A \in \mathcal{Q}$$

and

$$h_{\mathcal{Q}}(A^* \cap Z_0) = (Z_\alpha \times \alpha^* A^*) \cap Z_0' \qquad \text{if} \quad A^* \in \mathcal{Q}^*$$

and so $h_{\mathcal{Q}}(\{\mathcal{Q} \cap Z_0, \mathcal{Q}^* \cap Z_0\}) \subset \mathcal{C}' \cap Z_0'$. It follows that

$$h_{\mathcal{Q}}(\mathcal{C} \cap Z_0) \subset \mathcal{C}' \cap Z_0'.$$

DEFINITION 1. Let $\mathcal{Q}$ be such that there is no partial selector for $\mathcal{Q}$ of positive measure. Then $\mathcal{Q}$ is said to resolve $(Z, \mathcal{C}, P)$ as a product space if $h_{\mathcal{Q}}(Z) \in \overline{\mathcal{C}'}^{P'}$.

REMARK 1. Using P1 (see Sazonov (1962), for instance) we note that $(Z_\alpha, \mathcal{Q}_\alpha, P_\alpha)$ $((Z_{\alpha^*}, \mathcal{Q}_{\alpha^*}, P_{\alpha^*}))$ is a perfect probability space if and only if $(Z, \mathcal{Q}, P|_{\mathcal{Q}})$ $((Z, \mathcal{Q}^*, P|_{\mathcal{Q}^*}))$ is a perfect probability space.

THEOREM 2. *Suppose* $\mathcal{Q}$ *is such that there is no partial selector for* $\mathcal{Q}$ *of positive measure. Then* $(Z, \mathcal{C}, P)$ *is a perfect probability space if and only if*

(i) $(Z, \mathcal{Q}, P|_{\mathcal{Q}})$ *is a perfect probability space*

(ii) $\mu(z, .)$ *is perfect for almost all* z, *and*

(iii) $\mathcal{Q}$ *resolves* $(Z, \mathcal{C}, P)$ *as a product space.*

PROOF. *Necessity.* (i) P is perfect $\Rightarrow P|_{\mathcal{A}}$ is perfect (by (P2)). (ii) follows because almost all $\mu(z, .)$ can be chosen to be compact approximable and hence to be perfect (see proof of Theorem 5 in [1]). (iii) follows by Lemma 2.

Sufficiency. Let $\mathcal{A}^*$, α, α^*, $h_{\mathcal{A}}$ and Z_0 be defined as before. By Lemma 1 of [4] there exists $N_1 \in \mathcal{A}$ with $P(N_1) = 0$ such that $z \notin N_1$ implies $\mu(z, A^*) = P(A^*)$ for all $A^* \in \mathcal{A}^*$. Since

$$\int \mu(z, Z_0)\, dP = P(Z_0) = 1$$

there exists $N_2 \in \mathcal{A}$ with $P(N_2) = 0$ such that for $z \notin N_2$, $\mu(z, Z_0) = 1$. By condition (ii) there exists $N_3 \in \mathcal{A}$ with $P(N_3) = 0$ such that for $z \notin N_3$, $\mu(z, .)$ is perfect. Let $N_0 = N_1 \cup N_2 \cup N_3 \cup N$ and let $z \notin N_0$. Consider $(\alpha(z), \mathcal{C} \cap \alpha(z), \mu(z, .))$ where $\alpha(z) \in \mathcal{A}$ is the $\mathcal{A}$-atom containing z. The map $\alpha^* : \alpha(z) \to Z_{\alpha^*}$ is measurable, 1–1 on $\alpha(z) \cap Z_0$ and $\mu(z, \alpha^{*-1}A_{\alpha^*} \cap \alpha(z)) = P(\alpha^{*-1}A_{\alpha^*}) = P_{\alpha^*}(A_{\alpha^*})$ for all $A_{\alpha^*} \in \mathcal{A}_{\alpha^*}$. Hence by Lemma 2, P_{α^*} is perfect.

By Remark 1, (i) implies that P_α is perfect. Hence by (P4), $(Z', \mathcal{C}', P')$ is a perfect probability space being the product of two perfect probability spaces. So by condition (iii) $(Z, \mathcal{C}, P)$ can be imbedded as a measurable subspace of measure one in the probability space $(Z', \overline{\mathcal{C}'^{P'}}, \overline{P'})$, which is perfect since $(Z, \mathcal{C}, P)$ is perfect (see Sazonov (1962)). Hence $(Z, \mathcal{C}, P)$ is a perfect probability space.

We shall give an example in the next section (see Example 3.1 of this paper) to show that conditions (i) and (ii) of Theorem 2 are not sufficient to ensure that $(Z, \mathcal{C}, P)$ is a perfect probability space.

Let $Z = M_0 \cup M_1 \cup M_2 \cup \cdots$ be a maximal decomposition of Z according to Theorem 1 of [4]. From the definition of a maximal decomposition given there it is easy to see that if $Z = M_0 \cup M_1 \cup M_2 \cup \cdots = M_0' \cup M_1' \cup M_2' \cup \cdots$ are two maximal decompositions of Z then $P(M_0 \Delta M_0') = 0$ because $P(M_0 \cap M_n') = P(M_0' \cap M_n) = 0$ for every $n \geq 1$. So let us denote by M_0 the essentially unique subset of Z which does not contain any partial selector for $\mathcal{A}$ of positive measure.

DEFINITION 2. We say that $\mathcal{A}$ is a product type sub-σ-algebra if, whenever $P(M_0) > 0$, $\mathcal{A} \cap M_0$ resolves the subspace $(M_0, \mathcal{C} \cap M_0, P_{M_0} = P(\,.\,)/P(M_0))$ as a product space.

THEOREM 3. *$(Z, \mathcal{C}, P)$ is a perfect probability space if and only if*

(i) *$(Z, \mathcal{A}, P|_{\mathcal{A}})$ is a perfect probability space*

(ii) *$\mu(z, .)$ is a perfect measure for almost all z and*

(iii) *$\mathcal{A}$ is a product type sub-σ-algebra.*

PROOF. The necessity follows from Theorem 2 and Definition 2. To prove sufficiency let $Z = \cup_{n=0}^{\infty} M_n$ be a maximal decomposition of Z.

For $n \geq 1$, let $A_n = \{z : \mu(z, M_n) > 0\}$. $A_n \in \mathcal{A}$ and

$$P(A_n \cap M_n) = \int_{A_n} \mu(z, M_n)\, dP = P(M_n).$$

Since M_n is a partial selector for $\mathcal{A}$, $\alpha|_{A_n \cap M_n}$ is a 1–1 measurable map from

$(A_n \cap M_n, \mathcal{C} \cap A_n \cap M_n)$ to $(\alpha A_n, \mathcal{Q}_\alpha \cap \alpha A_n)$. Further if $C \in \mathcal{C} \cap A_n \cap M_n$ then $\alpha(C) = \alpha\{z : \mu(z, C) > 0\} \in \mathcal{Q}_\alpha \cap \alpha A_n$. Hence $\alpha|_{A_n \cap M_n}$ is bimeasurable. Further the measures $P_{A_n \cap M_n}\alpha^{-1}$ and $P_\alpha|_{\alpha A_n}$ are mutually absolutely continuous. Now from (i) it follows that $(Z_\alpha, \mathcal{Q}_\alpha, P_\alpha)$ is perfect by Remark 1. Hence $(\alpha A_n, \mathcal{Q}_\alpha \cap \alpha A_n, P_\alpha|_{\alpha A_n} = P_\alpha(\,.\,)/P_\alpha(\alpha A_n))$ is perfect. So it follows that $(A_n \cap M_n, \mathcal{C} \cap A_n \cap M_n, P_{A_n \cap M_n})$ and hence $(M_n, \mathcal{C} \cap M_n, P_{M_n})$ are perfect probability spaces.

If $P(M_0) > 0$, let $A = \{z : \mu(z, M_0) > 0\} - N$. Then $A \in \mathcal{Q}$. Let $A_0 = A \cap M_0$. Then $P(A_0) = \int_A \mu(z, M_0)\, dP = P(M_0)$. (i) implies that $(A, \mathcal{Q} \cap A, P_A)$ is perfect. Observe that

$$z \in A \Rightarrow \mu(z, \alpha(z) \cap M_0) > 0$$

$$\Rightarrow \alpha(z) \cap M_0 \neq \varnothing$$

and hence $\alpha(z) \to \alpha(z) \cap M_0$ defines a σ-isomorphism between the σ-algebras $\mathcal{Q} \cap A$ and $\mathcal{Q} \cap A_0$. Hence one can show using (P1) that $(A_0, \mathcal{Q} \cap A_0, P_{A_0})$ is perfect. So $(M_0, \mathcal{Q} \cap M_0, P_{M_0})$ is perfect. By Corollary 1 of [4] and (ii), in the separable space $(M_0, \mathcal{C} \cap M_0, P_{M_0})$ there exists a regular conditional probability $\mu_0(z, C \cap M_0)$ given $\mathcal{Q} \cap M_0$ which is proper a.s. $[P|_{\mathcal{Q} \cap M_0}]$ such that almost all measures $\mu_0(z, .)$ are perfect measures. Since M_0 does not contain any partial selector for $\mathcal{Q}$ of positive measure, there is no partial selector for $\mathcal{Q} \cap M_0$ of positive measure. Hence by (iii) and Theorem 2, $(M_0, \mathcal{C} \cap M_0, P_{M_0})$ is a perfect probability space.

Finally if f is a real valued $\mathcal{C}$-measurable function on Z then $f_n = f|_{M_n}$ is $\mathcal{C} \cap M_n$-measurable for all $n = 0, 1, 2, \cdots$. For each $n \geq 0$, there is a linear Borel set $B_n \subset f_n(M_n)$ such that $P_{M_n}(f_n^{-1}(B_n)) = 1$ or $P(f_n^{-1}B_n) = P(M_n)$. Let $B_f = \cup_{n \geq 0} B_n$. Then B_f is a linear Borel set contained in $f(Z)$ such that $P(f^{-1}B_f) = \Sigma_{n \geq 0} P(M_n) = 1$. Hence by (P1), $(Z, \mathcal{C}, P)$ is a perfect probability space.

COROLLARY 1. *If $(Z, \mathcal{Q}, P|_{\mathcal{Q}})$ is perfect and if $\mu(z, .)$ is discrete for almost all z then $(Z, \mathcal{C}, P)$ is perfect.*

PROOF. We shall show that if $\mu(z, .)$ is discrete for almost all z, then $P(M_0) = 0$ where M_0 is the essentially unique set in $\mathcal{C}$ which does not contain any partial selector for $\mathcal{Q}$ of positive measure. By Theorem 3, $P(M_0) = 0$ implies that $(Z, \mathcal{C}, P)$ is perfect.

Suppose $P(M_0) > 0$. Then $A = \{z : \mu(z, M_0) > 0\}$ is $\mathcal{Q}$-measurable and $P(A) > 0$. By Proposition 1 and Corollary 1 of [4] there is a regular conditional probability $\mu_0(z, C \cap M_0)$ given $\mathcal{Q} \cap M_0$ which is proper a.s. $[P|_{\mathcal{Q} \cap M_0}]$ and which is such that for almost all z, $\mu_0(z, C \cap M_0) = \mu(z, C \cap M_0)/\mu(z, M_0)$. Now there is no partial selector for $\mathcal{Q} \cap M_0$ of positive measure. Hence by Proposition 4 of [4] $\mu_0(z, .)$ must be continuous for all $z \in A_0 \cap M_0$, $A_0 \in \mathcal{Q}$ with $P_{M_0}(A_0 \cap M_0) = 1$. Let $A_1 = A_0 \cap A$. $A_1 \in \mathcal{Q}$ and since $P(A \cap M_0) = P(A_0 \cap M_0) = P(M_0)$ we have

$P(A_1) \geqslant P(A_1 \cap M_0) = P(M_0) > 0$. Now for all $z \in A_1$, $\mu(z, .)|_{M_0}$ is continuous since $\mu_0(z, .)$ is continuous. Hence $\mu(z, .)$ is not discrete for $z \in A_1$, with $P(A_1) > 0$ which is a contradiction. Hence $P(M_0) = 0$.

3. Perfect mixtures of perfect measures. In this section, given a mixture problem where the σ-algebras considered are countably generated, we construct an associated separable space and a countably generated sub-σ-algebra of this space given which there is a regular conditional probability which is everywhere proper. We shall then use results of the preceding section to characterize the perfectness of perfect mixtures of perfect measures.

Let μ be the λ-mixture of $\mu(x, .)$'s where $(X, \mathcal{A}, \lambda)$ is a probability space with $\mathcal{A}$ countably generated, $(Y, \mathcal{B})$ is a Borel space with $\mathcal{B}$ countably generated and $\mu(x, B)$ is a transition probability given $\mathcal{A}$ on $X \times \mathcal{B}$. We denote by $(X_1, \mathcal{A}_1, \lambda_1)$ the quotient space of $(X, \mathcal{A}, \lambda)$ induced by atoms of $\mathcal{A}$ and by $(Y_1, \mathcal{B}_1, \mu_1)$ the quotient space of Y induced by atoms of $\mathcal{B}$. For $x \in X$ let x_1 denote the $\mathcal{A}$-atom containing x and for $B \in \mathcal{B}$ let B_1 denote the corresponding set in $\mathcal{B}_1$. Define $\mu_1(x_1, B_1)$ on $X_1 \times \mathcal{B}_1$ by $\mu_1(x_1, B_1) = \mu(x, B)$. Then $\mu_1(x_1, B_1)$ is a transition probability on $X_1 \times \mathcal{B}_1$. Let $Z_\mu = X_1 \times Y_1$, $\mathcal{C} = \mathcal{A}_1 \otimes \mathcal{B}_1$ and $P_\mu = \lambda\mu$ defined by

$$\lambda\mu(C) = \int \mu_1(x_1, C_{x_1})\, d\lambda_1, \qquad C \in \mathcal{C}.$$

Then $(Z_\mu, \mathcal{C}_\mu, P_\mu)$ is a separable space. If we define $\mu((x_1, y_1), C) = \mu_1(x_1, C_{x_1})$, then $\mu((x_1, y_1), C)$ is a regular conditional probability on $Z_\mu \times \mathcal{C}_\mu$ given $\mathcal{A}_1 \times Y_1$ which is everywhere proper. We call $(Z_\mu, \mathcal{C}_\mu, P_\mu)$ the separable space associated with μ. We have the following

THEOREM 4. *Suppose λ is perfect. Then μ is perfect if and only if the associated separable space $(Z_\mu, \mathcal{C}_\mu, P_\mu)$ is perfect.*

PROOF. If μ is perfect then μ_1 is perfect by Remark 1 and hence the marginal of P_μ on Y_1 is perfect. Since λ is perfect λ_1 is perfect. Thus $(Z_\mu, \mathcal{C}_\mu, P_\mu)$ is a product space in which both the marginals are perfect. Hence by (P4), $(Z_\mu, \mathcal{C}_\mu, P_\mu)$ is perfect.

If $(Z_\mu, \mathcal{C}_\mu, P_\mu)$ is perfect, by (P4) the marginal of P_μ on Y_1 and hence μ_1 is perfect. Again by Remark 1, μ is perfect.

We are now in a position to show that conditions (i) and (ii) of Theorem 2 are not sufficient to ensure that $(Z, \mathcal{C}, P)$ is a perfect probability space. It is easy to see that $\mu(x, .)$ is perfect if and only if $\mu_1(x_1, .)$ and hence $\mu((x_1, y_1), .)$ is perfect.

EXAMPLE 3.1. Consider the setup of Example 1.3. Construct the associated separable space $(Z_\mu, \mathcal{C}_\mu, P_\mu)$. λ is perfect implies $(Z_\mu, \mathcal{A}_1 \times Y_1, P_\mu|_{\mathcal{A}_1 \times Y_1})$ is perfect. Each $\mu(x, .)$ is perfect implies that each $\mu((x_1, y_1), .)$ is perfect. Thus conditions (i) and (ii) of Theorem 2 hold. But μ is nonperfect and so by Theorem 4, $(Z_\mu, \mathcal{C}_\mu, P_\mu)$ is not perfect.

Let us now turn our attention to the general mixture problem. We are given a probability space $(X, \mathcal{A}', \lambda')$, a measurable space $(Y, \mathcal{B}')$, a transition probability $\mu(x, B')$ on $X \times \mathcal{B}'$. Let μ' be the λ'-mixture of $\mu(x, .)$'s. Suppose $\mathcal{B}$ is a countably generated sub-σ-algebra of $\mathcal{B}'$ and $\mu = \mu'|_{\mathcal{B}}$. We shall denote by $\mathcal{A}$ the smallest σ-algebra of $\mathcal{A}'$ with respect to which $\mu(. , B)$, $B \in \mathcal{B}$ are all measurable. Then $\mathcal{A}$ is countably generated. We denote $\lambda'|_{\mathcal{A}}$ by λ. So μ is in fact a λ-mixture of $\mu(x, .)$'s and the σ-algebras considered are now countably generated. We can now associate the separable space $(Z_\mu, \mathcal{C}_\mu, P_\mu)$ and the sub-σ-algebra $\mathcal{A}_\mu = \mathcal{A}_1 \times Y_1$ with μ. Thus in the general mixture problem whenever μ denotes the restriction of the mixture measure to a countably generated sub-σ-algebra, we associate a separable space $(Z_\mu, \mathcal{C}_\mu, P_\mu)$ and a countably generated sub-σ-algebra $\mathcal{A}_\mu$ of this space with μ. If in addition the mixing measure and almost all mixand measures are perfect then $(Z_\mu, \mathcal{A}_\mu, P_\mu|_{\mathcal{A}_\mu})$ is perfect and there is a regular conditional probability given $\mathcal{A}_\mu$ on $Z \times \mathcal{C}_\mu$ which is almost everywhere proper and almost all of which are perfect measures. Hence we have the following characterization of perfect mixture of perfect measures.

THEOREM 5. *Let the measure μ' on $(Y, \mathcal{B}')$ be a perfect mixture of perfect measures. In order that μ' be perfect it is necessary and sufficient that for every countably generated sub-σ-algebra $\mathcal{B}$ of $\mathcal{B}'$, the sub-σ-algebra $\mathcal{A}_\mu$ of the space $(Z_\mu, \mathcal{C}_\mu, P_\mu)$ be of product type, where $\mu = \mu'|_{\mathcal{B}}$.*

PROOF. If μ' is perfect then by (P3) μ is perfect. By Theorem 4, $(Z_\mu, \mathcal{C}_\mu, P_\mu)$ is perfect. By Theorem 3, $\mathcal{A}_\mu$ is of product type.

Suppose the condition holds. By Theorems 3 and 4 it follows that the restriction of μ' to every countably generated sub-σ-algebra is perfect. By (P2), μ' is perfect.

An alternative proof to Theorem 3 of [3] is now given.

COROLLARY 2 (Theorem 3 of [3]). *Perfect mixtures of discrete measures are perfect.*

PROOF. Let the setup be as in Theorem 5 with the additional assumption that almost all mixand measures are discrete. Let $\mathcal{B}$ be a countably generated sub-σ-algebra of $\mathcal{B}'$ and let $\mu = \mu'|_{\mathcal{B}}$. Consider the sub-$\sigma$-algebra $\mathcal{A}_\mu$ of the associated separable space $(Z_\mu, \mathcal{C}_\mu, P_\mu)$. Since the mixing measure is perfect $P_\mu|_{\mathcal{A}_\mu}$ is perfect. It can be checked, since almost all mixand measures are discrete, that in the regular conditional probability given $\mathcal{A}_\mu$ almost all measures are discrete. Hence by Corollary 1, $(Z_\mu, \mathcal{C}_\mu, P_\mu)$ is perfect. By Theorem 4, μ is perfect. By (P2), μ' is perfect.

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