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A NEW MIXING CONDITION FOR STATIONARY GAUSSIAN PROCESSES[1]

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A new mixing condition is proposed for the study of stationary Gaussian processes on R^1. If the covariance function of the process is r, assume

$$\text{Lebesgue measure}\left\{t|0 < t < T;\, r(t) > \frac{f(t)}{\ln t}\right\} = o(T^\beta)$$

as $T \to \infty$, for some $0 < \beta < 1$ and some $f(t) = o(1)$. The stated condition is weaker than those in common use, and yet it is shown to imply the same limit theorems on the distribution of the maximum of the process. Examples are given of processes which satisfy the new condition and not the previous ones.

1. Introduction. Let $\{X(t),\, t \geqslant 0\}$ be a stationary Gaussian process with mean zero, variance one. Let $r(t) = EX(0)X(t)$ and suppose $X(t)$ has continuous sample paths. In studying the asymptotic behavior of $M_T = \max_{0 \leqslant t \leqslant T} X(t)$, one requires a "local" and a "mixing" condition on the process. A standard local condition is

$$1 - r(t) \backsim C|t|^\alpha \qquad (1)$$

as $t \to 0$ for some $0 < \alpha \leqslant 2$ and a constant C. (The constant C may also be replaced by a slowly varying function without undue effect; see Berman [3], for example.) Two types of mixing conditions are commonly used. The first type is a rate of decay condition as in

$$r(t)\ln t = o(1). \qquad (2)$$

($o(1)$ will always mean as $t \to \infty$.) The second type is an integral condition; for instance,

$$\int_0^\infty |r(t)|^p dt < \infty \qquad (3)$$

for some $p > 0$. The conditions (2) and (3) are not comparable.

Consider now the following mixing condition on $r(t)$:

$$\lambda\left\{t|0 \leqslant t \leqslant T;\, |r(t)| > \frac{f(t)}{\ln t}\right\} = o(T^\beta) \qquad (4)$$

for some $0 \leqslant \beta < 1$ and some $f(t) = o(1)$. (λ denotes Lebesgue measure.) It will be shown in Section 2 that (4) is a strictly weaker condition than (2) or (3), and some examples of covariance functions satisfying (4) will also be given there.

In Section 3, the asymptotic behavior of M_T is studied. A mixing condition is used to compare, via Berman's lemma, M_T with the maximum of a suitably chosen

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sequence of independent standard normal variables. This procedure requires that a certain sum be shown to be $o(1)$. The lemma proved in Section 3 shows that this sum is $o(1)$ under (4) and $r(t) = o(1)$. Thus, in view of this lemma, (4) together with $r(t) = o(1)$ can replace the conditions (0.5) in [3], (1.4) in [2] and (3.15) in [1]. See also (5.4), (5.5) in [4], Theorems 5.2 and 5.3 in [9] and (1.3), (1.6) in [7] for more examples. Note that Berman's condition

(5) $F^{(p)}$ is absolutely continuous for some $p > 0$,

where F is the spectral distribution, is actually too weak for the asserted Theorem 3.1 in [3] to hold. For, define

$$\begin{aligned} 1 - r(t) &= \frac{1}{2e^2}|t|, \qquad |t| \leqslant e^2 \\ &= 1 - \frac{1}{\ln|t|}, \qquad |t| > e^2. \end{aligned}$$

This covariance function satisfies (5) for $p = 1$. However, the limit distribution of the normalized maximum in this case is a convolution of the extreme value distribution $\exp(-e^{-x})$ with the normal, and not the double exponential $(\exp(-e^{-x}))$ itself (cf. [8]). Theorem 3.1 of [3] is valid under a condition such as

(6) $\int_0^\infty [\sup_{x \geqslant t}|r(x)|]^p dt < \infty$

for some $p > 0$ as has been suggested by Berman in a private communication. However, (6) is stronger than (3) and therefore (4). Theorem 7.1 of [2] is correct despite an unverified statement at the bottom of page 935. But a better theorem would result with (1.4) replaced by (4).

2. Mixing conditions. Here the weakness of condition (4) is to be demonstrated.

Suppose (2) holds, i.e., $r(t)\ln t = o(1)$. Take $f(t) = r(t)\ln t$ and observe that (4) holds with $\beta = 0$.

Suppose next that (3) holds. Note that if (4) is valid for some $f(t)$ then it is true for any function that decreases at a slower rate. Thus we will assume throughout that $f(t)$ decreases sufficiently slowly, viz, not faster than $(\ln t)^{-1}$. Now let us define $A_T = \{T^\gamma \leqslant s \leqslant T \,|\, |r(s)| > (f(s)/\ln s)\}$ for some $0 < \gamma < 1$. Then

$$\infty > \int_0^T |r(s)|^p ds \geqslant \int_{T^\gamma}^T |r(s)|^p ds > \lambda(A_T)\left\{\frac{\sup_{s \geqslant T^\gamma} f(s)}{\gamma \ln T}\right\}^p.$$

Hence $\lambda(A_T) = o\{\ln T\}^{2p}$ because of the choice of $f(t)$. Obviously $\lambda(B_T) = \lambda\{0 \leqslant s \leqslant T^\gamma \,|\, |r(s)| > (f(s)/\ln s)\} = o(T^\gamma)$. The Lebesgue measure in (4) is $\lambda(A_T) + \lambda(B_T)$ and hence (4) is satisfied.

In the following, there are some examples of covariance functions for which the local condition (1) holds together with (4), and for which neither (2) nor (3) holds.

A. The first set of examples is derived from the symmetric Bernoulli convolutions. Consider the infinite convolution

$$\gamma_a(t) = \Pi_{k=1}^{\infty}\cos(a^k t) \tag{7}$$

for $0 < a < 1$. The product is absolutely convergent since $\Sigma_{k=1}^{\infty} a^{2k} < \infty$ and thus defines a singular covariance function for all $0 < a < \frac{1}{2}$([6], page 67). If $a = m^{-1}$ for $m = 3, 4, \cdots$ then

$$\limsup_{|t|\to\infty}|\gamma_{m^{-1}}(t)| = c > 0. \tag{8}$$

(See [6], page 67, or [10], page 140; (256).) Moreover, for all $0 < a < \frac{1}{2}$

$$\int_0^T|\gamma_a(t)|^2 dt = o(T^{1-(\ln 2/\ln m)}) \tag{9}$$

as $T \to \infty$. The exponent $\ln 2/\ln m$ in (9) cannot be improved ([10], page 145). Now define

$$\begin{aligned} r_{1/m}(t) &= \left(1 - \frac{|t|}{(2e^2)^{\frac{1}{2}}}\right)\gamma_{1/m}(t);\ |t| \leqslant e^2 \\ &= \frac{1}{(\ln|t|)^{\frac{1}{2}}}\gamma_{1/m}(t);\ |t| > e^2. \end{aligned} \tag{10}$$

Observe that (2) is violated, since

$$\limsup_{|t|\to\infty}(\ln t)(r_{1/m}(t)) = \limsup_{|t|\to\infty}(\ln t)^{\frac{1}{2}}\gamma_{1/m}(t) = \infty$$

from (8). Furtheremore,

$$(\text{const.})T^{1-(\ln 2/\ln m)} \geqslant \int_0^T|r_{1/m}(t)|^2 dt \geqslant (\text{const.})\frac{T^{1-(\ln 2/\ln m)}}{\ln t} \tag{11}$$

because of (9) and the remark about the exponent $\ln 2/\ln m$. Thus $r_{1/m}(t)$ cannot satisfy (3) for any $0 < p \leqslant 2$. The condition at (6) is no help, since the integral is clearly infinite for all $p > 0$.

Now $r_{1/m}(t)$ does satisfy (4) with $\beta = 1 - (\ln 2/2\ln m)$ and $f(t) = 1/\ln t$. For if not, then $|r_{1/m}(t)| > 1/(\ln t)^2$ on a set of measure $T^{1-(\ln 2/2\ln m)}$ and $\int_0^T|r_{1/m}(t)|^2 dt \geqslant T^{1-(\ln 2/2\ln m)}/(\ln T)^4$ contradicting (11). That $r_{1/m}(t)$ satisfies (1) for $\alpha = 2$ is easily verified.

B. This example is a minor modification of the one suggested by L. A. Shepp. Define

$$\begin{aligned} g(x) &= c/(2^{n/2}) \quad \text{for} \quad 2^{2^n} \leqslant x \leqslant 2^{2^n} + \frac{2n}{(n+1)^{\frac{3}{2}}}, \quad n = 0, 1, 2, \cdots \\ &= 0 \quad \text{otherwise.} \end{aligned} \tag{12}$$

Then $\int_0^\infty g^2(x)dx = \Sigma_{n=0}^{\infty}\frac{c^2}{(n+1)^{\frac{3}{2}}} = 1$ by choice of c. Theorem 4.2.4 of Lukacs [6] gives that

$$r(t) = \int_0^\infty g(x)g(t+x)dx \tag{13}$$

is an absolutely continuous covariance function. We notice that for all n large

(14) $$r(2^{2^n} + x) \geqslant \int_2^{2^n/(n+1)^{\frac{3}{2}} - x} g(y)g(2^{2^n} + x + y)dy$$

for at least all x in $[0, 2^{n-1}/(n + 1)^{\frac{3}{2}}]$. The R.H.S. above is at least

(15) $$\frac{c}{2^{n/2}} \int_2^{2^{n-1}/(n+1)^{\frac{3}{2}}} g(y)dy \geqslant \frac{c^2}{2^{n/2}} \Sigma_{K=0}^{K_n} \frac{2^{K/2}}{(K + 1)^{\frac{3}{2}}}.$$

Where K_n is chosen so that $2^{2^{K_n-1}} \leqslant 2^{n-1}/(n + 1)^{\frac{3}{2}} < 2^{2^{K_n}}$. The R.H.S. in (15) is at least

$$\frac{c^2}{2^{n/2}} \frac{2^{K_n/2}}{(K_n + 1)^{\frac{3}{2}}} > \frac{c^2}{2^{n/2}} \cdot 2^{K_n/4} > (\text{const.}) \frac{c^2}{2^{n/2}} \{(n - 1)\ln 2 - 3/2 \ln(n + 1)\}^{\frac{1}{4}}.$$

Thus for large n

(16) $$r(2^{2^n} + 2) \geqslant \frac{(\text{const.})n^{\frac{1}{4}}}{2^{n/2}}$$

for all $0 \leqslant x \leqslant 2^{n-1}/(n + 1)^{\frac{3}{2}}$. This implies that $r(t)\ln t \nrightarrow 0$ and

$$\int_0^T |r(t)|^p dt \geqslant (\text{const.}) \Sigma_{K=0}^{n_T} \frac{(K^p)^{\frac{1}{4}}}{(2^{Kp})^{\frac{1}{2}}} \cdot \frac{2^K}{(K + 1)^{\frac{3}{2}}}$$

where n_T is the largest integer for which $2^{2^{n_T}} + (2^{n_T}/(n_T + 1)^{\frac{3}{2}}) \leqslant T$. Thus for all $0 < p \leqslant 2$, $\int_0^T |r(t)|^p dt = \infty$.

To see that $r(t)$ satisfies (4), we look at t, $0 \leqslant t \leqslant T$, for which $r(t) \neq 0$. This is possible only if t is of the form which gives

$$2^{2^j} \leqslant t + y \leqslant 2^{2^j} + \frac{2^j}{(j + 1)^{\frac{3}{2}}} \qquad \text{for some } y \text{ such that}$$

$$2^{2^K} \leqslant y \leqslant 2^{2^K} + \frac{2^K}{(K + 1)^{\frac{3}{2}}}, \ 0 \leqslant K \leqslant j.$$

Let us write $y = 2^{2^K} + \theta \quad 0 \leqslant \theta \leqslant 2^K/(K + 1)^{\frac{3}{2}}$ and $t + y = 2^{2^j} + \eta$, $0 \leqslant \eta \leqslant 2^K/(j + 1)^{\frac{3}{2}}$. Then

(17) $$t = 2^{2^j} - 2^{2^K} + \eta - \theta.$$

For each K, $0 \leqslant K \leqslant j$, the length of the t-interval for which (17) is satisfied cannot exceed 2^j. Thus the total length up to 2^{2^j} is less than $j2^j$ and hence in $0 \leqslant t \leqslant T$ it is no more than $\Sigma_{j=1}^{J_0} j2^j$ where J_0 is such that $2^{2^{J_0}} \leqslant T$.

Finally, for t in the neighborhood of zero, we have

(18) $$r(t) = c^2 \Sigma_{n=0}^{\infty} 2^{-n} \left(\frac{2^n}{(n + 1)^{\frac{3}{2}}} - |t| \right).$$

Hence $r(t)$ satisfies (1) for $\alpha = 1$.

3. Asymptotic behavior of M_T. Let $X(t)$, M_T and $r(t)$ be as before and suppose (1) holds for some $0 < \alpha \leqslant 2$. The behavior of M_T is studied by discretizing the process sufficiently closely and then comparing the maximum of the discretized version to that of the independent standard normal variables via Berman's lemma (see [1] for the statement of the lemma). The normalizing constant for M_T (given as u_T below) and the choice of how closely to discretize the process (determined by value of $n(T)$ defined below), both depend on the local conditions of the process, viz., the value of α in (1). For constant $k \geqslant 0$, define

$$n = n(T) = \left[T \frac{(\ln T)^{k/2}}{(f(T^\gamma))^{\frac{1}{4}}} \right]$$

and

$$u = u(T) = (2 \ln T)^{\frac{1}{2}} + ((k-1)/2) \ln\ln T / (2 \ln T)^{\frac{1}{2}}.$$

The function $f(t)$ is as in (4) and $0 < \gamma < 1$. The choice of γ is specified in the beginning of the proof of the lemma.

The constant $k \geqslant 0$ is used for generality. The proper value of k for continuous processes satisfying (1) is $2/\alpha$. It should be noted that this value of k is also appropriate in $n(T)$. The value of n is used in comparing $M_T = \max_{0 \leqslant t \leqslant T} X_t$ with $M_n = \max_{0 \leqslant j \leqslant n} X(jT/n)$ and hence should be large enough for the discretized version to approximate the process itself. Pickand's Lemma 4.2 in [9] shows that $(2 \ln T)^{\frac{1}{2}}(M_T - M_n) \to 0$ in probability if n is chosen slightly larger than $T(\ln T)^{1/\alpha}$. Thus the choice of $n(T)$ for $k = 2/\alpha$ is in general fine enough for closeness of the maximum to that of the discretized version. (See also page 43, Chapter 6, of Leadbetter [4] and (2.6) of Mittal [7].) Berman [3] uses $n(T)$ for $k = 3/\alpha$ in (3.7). A close inspection shows that with the help of the following lemma we could change it to the above choice. For Gaussian sequences the proper normalizing constant is $u(T)$ with $k = 0$ and this choice of k in $n(T)$ gives $n = [T]$ as it should.

LEMMA. *Let $r(T)$ be $o(1)$, satisfying* (1) *and* (4). *Then*

$$n \sum_{j=[n\varepsilon/T]}^{n} \left| r\left(\frac{jT}{n}\right) \right| \exp\left\{ - \frac{u^2}{1 + \left| r\left(\frac{jT}{n}\right) \right|} \right\} = o(1) \tag{19}$$

for every $\varepsilon > 0$.

PROOF. We first note that if (4) is valid for some $f(t) = o(1)$ then it is also valid for $g(t) = \sup_{x > t} f(x)$ where $g(t) \downarrow 0$ as $t \to \infty$. Thus without loss of generality we will assume that $f(t)$ is nonincreasing.

Fix $\varepsilon > 0$ and define $\delta = \delta(\varepsilon) = \sup_{s > \varepsilon} |r(s)|$, then $0 \leqslant \delta < 1$. Split the sum in (19) in two parts—$[n\varepsilon/T] \leqslant j \leqslant [n^\gamma]$ and $j > [n^\gamma]$ for $0 < \gamma < (1-\delta)/(1+\delta)$.

The first part is at most

$$n^{1+\gamma}\exp\left\{-\frac{u^2}{1+\delta}\right\}$$
$$\leq \exp\left\{\left(1+\gamma-\frac{2}{1+\delta}\right)\ln T + \left\{\frac{(1+\gamma)k}{2}+\frac{2}{1+\delta}\right\}\ln\ln T\right\}.$$

We notice that $u^2/(1+\delta)$ contributes the term $-((k-1)/(1+\delta))\ln\ln T$ which is at most $(1/(1+\delta))\ln\ln T$. Also $-\frac{1}{4}\ln f(T^\gamma) < (\ln\ln T^\gamma)/2 < (\ln\ln T)/(1+\delta)$. The upper bound for the coefficient of $\ln\ln T$ above is some fixed positive constant, while the coefficient of $\ln T$ is a fixed negative constant because of the definition of γ. Thus the R.H.S. above is $o(1)$. We now look at the second part of the sum in (19), i.e., when $j > n^\gamma$. The only difficulty in finding an upper bound for this sum arises in being able to find appropriate upper bounds for $r(t)$. Recall the definition of set A_T in Section 2 and set

$$A_T^C = \left\{T^\gamma \leq s \leq T \,\middle|\, |r(s)| \leq \frac{f(s)}{\ln s}\right\}.$$

Due to the continuity of $r(t)$, $A_T = \cup_{m=0}^{\infty} I_m$ where I_m is the union of all intervals in A_T of length l, $(m+1)^{-1} \leq l < m^{-1}$. Let $m_0 = m_0(T) = [(\ln T)^{8/\alpha}]$. We will show that on $\cup_{m=0}^{m_0} I_m$ we can estimate $r(t)$ above by $\delta(T^\gamma)$, since the number of j such that $jT/n \in \cup_{m=0}^{m_0} I_m$ is small. On $\cup_{m_0}^{\infty} I_m$ we can find a suitable upper bound for $r(t)$ because of its smoothness (Lipschitz condition of order $\alpha/2$) and on A_T^C the upper bound for $r(t)$ is obviously $f(t)/\ln t$. The number of j in one of the intervals in I_0 is at most $[\lambda(\text{interval}) \times ((\ln T)^{k/2}/(f(T))^{\frac{1}{4}}) + 2]$. Also $\lambda(I_0) < T^\beta$ by (4) and the number of intervals in I_0 is at most $T^\beta + 1$. The number of j in the remaining intervals (of length < 1) is at most $((\ln T)^{k/2}/(f(T))^{\frac{1}{4}}) + 1$. Again there are at most $T^\beta(\ln T)^{8/\alpha}$ intervals in $\cup_{m=0}^{m_0} I_m$. Thus the maximum number of j such that $jT/n \in \cup_{m=0}^{m_0} I_m$ is at most (const.) $T^\beta((\ln T)^{8/\alpha+k/2}/(f(T))^{\frac{1}{4}})$. Now the sum in (19) with $j > n^\gamma$ and $jT/n \in \cup_0^{m_0} I_m$ is at most

$$(20) \qquad (\text{const.})\, T^{1+\beta}\frac{(\ln T)^{8/\alpha+k}}{(f(T))^{\frac{1}{2}}}\exp\{-u^2/(1+\delta(n^\gamma))\}$$

where $\delta(n^\gamma) = \sup_{j>n^\gamma}|r(jT/n)|$. Because $r(t) = o(1)$, $\delta(n^\gamma) = o(1)$. Hence the expression in (20) tends to zero as $T \to \infty$ for all $0 < \beta < 1$.

By definition of the set A_T and the continuity of $r(t)$, we know that at the endpoints of an interval in I_m, $r(t) \leq f(t)/\ln T$. On the set I_m, $r(t)$ stays above the curve $f(t)/\ln t$, but when the lengths of the intervals are small it cannot get too far above $f(t)/\ln t$. For, by the increments inequality of Loève (12.4B),

$$|r(t+h) - r(t)|^2 \leq 2(1 - r(t)).$$

Thus $|r(t+h)-r(t)| \leqslant (\text{const.})h^{\alpha/2}$ in view of (1). Then for all s and t in any one of the intervals in $\cup_{m_0}^{\infty} I_m$,

$$|r(s)-r(t)| \leqslant (\text{const.})|s-t|^{\alpha/2} \leqslant \frac{(\text{const.})}{(\ln T)^4}$$

since the largest length of the intervals in $\cup_{m_0}^{\infty} I_m$ is $(\ln T)^{-8/\alpha}$. Thus we can easily see that

$$|r(t)| \leqslant \frac{f(t)}{\gamma(\ln T)} + \frac{(\text{const.})}{(\ln T)^4} \leqslant C \cdot \frac{f(t)}{\ln T} \tag{21}$$

for all $t \in \cup_{m_0}^{\infty} I_m$ and some constant C since we have chosen $f(t)$ to decrease sufficiently slowly. If $t \in A_T^C$ then by the definition of A_T, we will have $|r(t)| \leqslant f(t)/\ln t \leqslant f(t)/\gamma \ln T$. Hence we can use (21) as the upper bound on $r(t)$ for all $t \in (\cup_{m_0}^{\infty} I_m) \cup A_T^C$. Thus the remaining sum in (19) when $j > n^\gamma$ is at most

$$\begin{aligned} Cn^2 \frac{f(T^\gamma)}{\ln T} \exp\left\{-u^2 \Big/ \left(1 + \frac{C \cdot f(T^\gamma)}{\ln T}\right)\right\} & \\ \leqslant \frac{C(f(T^\gamma))^{\frac{1}{2}}}{\ln T} \exp\left\{2Cf(T^\gamma) + \left(k - \frac{k-1}{1 + Cf(T^\gamma)/\ln T}\right)\ln\ln T\right\} & \\ \leqslant (\text{const.})(f(T^\gamma))^{\frac{1}{2}} & \end{aligned}$$

which tend to zero as $T \to \infty$.

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