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FIRST EXIT TIME OF A RANDOM WALK FROM THE BOUNDS $f(n) \pm cg(n)$, WITH APPLICATIONS

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Let $X_1, X_2, \cdots$ be i.i.d. real-valued random variables with $EX_1 = 0$, $EX_1^2 < \infty$, and $S_n = X_1 + \cdots + X_n$, $n = 1, 2, \cdots$. For a chosen positive integer m and real $c > 0$ the exit time N_c is the least integer $n \geqslant m$ such that $f(n) - cg(n) < S_n < f(n) + cg(n)$ is violated, where the functions f and g ($0 < g\uparrow\infty$) are both defined for all real $x \geqslant m$. Under certain conditions on f and g, a function ψ (unique up to an asymptotic equivalence), satisfying $\psi(x)/x \to 0$ as $x \to \infty$, is constructed on $[m, \infty)$ such that $\psi(N_c)$ is exactly exponentially bounded. This result generalizes earlier theorems of Breiman; Chow, Robbins, and Teicher; Gundy and Siegmund; Brown; and Lai. A consequence is that N_c itself is not exponentially bounded. In a multivariate generalization the X's take their values in R^d and N_c is the first exit time of L_n from $(-l(c), l(c))$, where $L_n = n\Phi(S_n/n) - h(n)$, and certain conditions are imposed on Φ and h. Here $\psi(x) = \int_m^x h(t)t^{-1}\,dt$. The results are applied to show, both in the sequential F-test and in the Savage-Sethuraman sequential rank-order test, that for certain distributions of the X's the stopping time is not exponentially bounded.

1. Introduction. Let $X, X_1, X_2, \cdots$ be i.i.d. real-valued random variables, with $EX = 0$, $EX^2 < \infty$, and $S_n = X_1 + \cdots + X_n$, $n = 1,2, \cdots$. The random walk S_n is allowed to proceed as long as it stays between the bounds $f(n) \pm cg(n)$, where $c > 0$, and f and $g(> 0)$ are real-valued functions defined on the positive integers. In this paper certain aspects will be studied of the first exit time (or stopping time) N_c of the random walk; i.e., N_c is the least integer $n \geqslant 1$ such that

$$f(n) - cg(n) < S_n < f(n) + cg(n) \tag{1.1}$$

is violated. The interest lies in "widening" bounds, that is, g is eventually nondecreasing and $g(n) \to \infty$ as $n \to \infty$. The function f may take both positive and negative values, although the case where f is also increasing to ∞ ("tilted" bounds) is of greatest interest.

The discussion will be facilitated by employing the notion of *exact exponential boundedness*, defined below. Also, it will be convenient to adopt throughout this paper Vinogradov's symbol $\ll$ instead of Landau's big 0 notation. Thus, the order relation $f_1(x) \ll f_2(x)$ between two positive real-valued functions f_1 and f_2 means that there exist constants $c > 0$ and x_0 such that $f_1(x) \leqslant cf_2(x)$ for all $x > x_0$. The domain of the functions may be an arbitrary subset of the real line, but in this

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paper the domain will be either an interval extending to infinity on the right, or the set of all integers greater than a given integer. In the latter case x is usually written as n or k. In Section 4 comparison will be made between sequences of nonnegative random variables, say $\{X_n\}$ and $\{Y_n\}$ (the latter is usually a sure sequence). Then $X_n \ll Y_n$ means that there exist a nonrandom integer n_0 and a nonrandom number c such that $X_n \leqslant cY_n$ for all $n > n_0$.

DEFINITION 1.1. A random variable Z with values in a Euclidean space will be said to be *exactly exponentially bounded* if there exist constants ρ_1, ρ_2, with $0 < \rho_1 \leqslant \rho_2 < 1$, such that

$$\rho_1^x \ll P(\|Z\| > x) \ll \rho_2^x, \tag{1.2}$$

in which $\|Z\|$ is the norm of Z. If Z satisfies the right-hand order relation in (1.2), it is said to be *exponentially bounded.*

It is easy to establish that Z is exponentially bounded if and only if $E \exp(t\|Z\|) < \infty$ for some $t > 0$. It is also true that the left-hand order relation in (1.2) implies (but is not implied by) $E \exp(t\|Z\|) = \infty$ for some $t > 0$. Therefore, (1.2) implies that there exists $t_0 > 0$ such that $E \exp(t\|Z\|) < \infty$ or $= \infty$ according as $t <$ or $> t_0$.

The study in this paper was motivated by a problem concerning the stopping time N of invariant sequential probability ratio tests. Such tests are usually based on a sequence $X, X_1, X_2, \cdots$ of i.i.d. random vectors from which is formed (depending on the testing situation) a sequence $L_1, L_2, \cdots$ of real-valued statistics, where L_n depends only on $X_1, \cdots, X_n$; then N is the first exit of L_n from a fixed interval (l_1, l_2). The distribution of N depends on l_1, l_2, and on the true distribution P of X (where P need not belong to the model that produced the sequence $\{L_n\}$). It is desirable that N be exponentially bounded under P for every choice of l_1, l_2. If this is not the case, then P is termed *obstructive*. In all testing problems studied, examples of both kinds of distributions P have been found (see, e.g., [15], where also references to other work are listed).

It is often possible to prove exponential boundedness of N for a large family of distributions P (cf. [15]). Of the remaining distributions, which can be called "suspect," it has to be investigated whether they are obstructive. Sometimes the problem reduces to one where $L_n = n\Phi(\bar{X}_n)$, in which $\bar{X}_n = S_n/n$ and Φ satisfies some smoothness conditions. Then Theorem 2.1 in [13] may be applicable (a better version is Proposition 3.2.1 in [15]), from which obstructiveness of P can be concluded. But two examples have appeared, one the sequential F-test ([15], Section 3.3), and the other a nonparametric test for the equality of two distribution functions ([16], Section 6), in which for the suspect distributions P the problem reduces to one where $L_n = n\Phi(\bar{X}_n) - a \log n$, with $\Phi \geqslant 0$ and $a > 0$. In the simplest possible situation of this type, X is real-valued, $EX = 0$, $EX^2 < \infty$, and $L_n = n\bar{X}_n^2 - a \log n$. Also, without loss of generality, take the stopping bounds symmetric: $-l_1 = l_2 = l$, say. Then the first exit time of L_n from the bounds $\pm l$

comes no sooner than the first exit time of S_n from $c_1(n \log n)^{\frac{1}{2}} \pm c_2(n/\log n)^{\frac{1}{2}}$ with the positive constants c_1 and c_2 depending on l and a. This is of the type (1.1).

The random walk problem of (1.1) is treated in Sections 2 and 3 for various classes of functions f, g. Some of the results are generalized in Section 4 to the multivariate case where X is vector-valued, but require more assumptions. The results are applied in Section 5 to the statistical examples mentioned earlier. The basic method is an extension of that in [9] and consists of finding a function ψ having the property $\psi(x) \to \infty$ but $\psi(x)/x \to 0$ as $x \to \infty$, such that $\psi(N_c)$ is exactly exponentially bounded (for sufficiently large c). It follows then that N_c itself is not exponentially bounded. For instance, when $f(n)$ and $g(n)$ are proportional to $(n \log n)^{\frac{1}{2}}$ and $(n/\log n)^{\frac{1}{2}}$, respectively, Theorem 2.1 shows that $\psi(x)$ is asymptotically equivalent to $\frac{1}{2}(\log x)^2$. Consequently, for large enough c, $(\log N_c)^2$ is exactly exponentially bounded and therefore N_c is not exponentially bounded.

The results contained in Sections 2 and 4 are of two kinds. One is of the type that proves

$$\rho_1^x \ll P\{\psi(N_c) > x\} \tag{1.3}$$

for some $\rho_1 > 0$ and c sufficiently large. The other is of the type

$$P\{\psi(N_c) > x\} \ll \rho_2^x \tag{1.4}$$

for some $\rho_2 < 1$ and all c, i.e., $\psi(N_c)$ is exponentially bounded for every c. Lack of exponential boundedness of N_c follows from (1.3) alone, whereas (1.4) merely provides added information about the behavior of the tail probabilities of N_c. The relations (1.3) and (1.4) together imply that $\psi(N_c)$ is exactly exponentially bounded.

The functions f and g need to be defined only in the positive integers. However, it will be much more convenient to define $f(x)$ and $g(x)$ for all real $x \geqslant 1$ and express some of the conditions on the growth of these functions in terms of their derivatives. This is only done for convenience. It is, of course, not essential and differences instead of derivatives could have been used. It is also convenient sometimes not to have to define f and g on $[1, \infty)$ but only on $[m, \infty)$, where m is some positive integer. Then the stopping time N_c will be redefined as the smallest integer $n \geqslant m$ such that (1.1) is violated. This does of course not at all affect the behavior of the tail probabilities of N_c.

2. First exit time of S_n from the stopping bounds $f(n) \pm cg(n)$. In this section the tail distribution of N_c will be investigated, where N_c was defined in Section 1 as the smallest positive integer such that (1.1) is violated. Under certain conditions on f and g we shall give a simple method of constructing (in terms of f and g alone) a strictly increasing continuous function ψ with $\lim_{x\to\infty}\psi(x) = \infty$ such that $\psi(N_c)$ is exactly exponentially bounded, i.e., for some $0 < \rho_1 \leqslant \rho_2 < 1$

$$\rho_1^x \ll P\{\psi(N_c) > x\} \ll \rho_2^x. \tag{2.1}$$

Using the bijectivity of ψ, (2.1) is equivalent to

$$\rho_1^{\psi(x)} \ll P(N_c > x) \ll \rho_2^{\psi(x)}. \tag{2.2}$$

Suppose ψ^* is another strictly increasing continuous function such that $\lim_{x\to\infty}\psi^*(x) = \infty$ and $\psi^*(N_c)$ is also exactly exponentially bounded for some $c > 0$. Then it follows from (2.2) that ψ and ψ^* are asymptotically equivalent in the sense that $\psi(x) \ll \psi^*(x) \ll \psi(x)$. Hence the strictly increasing continuous function ψ satisfying (2.1) is unique up to asymptotic equivalence. In fact, by (2.2), $\psi(x)$ has to be asymptotically equivalent to $|\log P(N_c > x)|$.

In [9] the case where $f \equiv 0$ is studied under the assumption that $g(t) \ll t^{\frac{1}{2}}$ and g is eventually nondecreasing with $\limsup_{t\to\infty} g(at)/g(t) < \infty$ for every $a > 1$. Earlier, Breiman [2], Chow, Robbins, and Teicher [5], Gundy and Siegmund [6], and Brown [3] have considered the particular example $f \equiv 0$ and $g(t) = t^{\frac{1}{2}}$, where $\psi(N_c)$ in (2.1) turns out to be $\log N_c$. These earlier results are extended in [9] to other lower-class boundaries $g(t)$ as an answer to an open question raised by Breiman in [2], pages 15–16. For the sequential tests mentioned in Section 1 and treated in Section 5, however, the stopping region in terms of some suitably defined random walk not only involves the widening bounds $\pm cg(n)$, but is also tilted with tilt $f(n)$ at stage n. Theorem 2.1 below extends the results of [9] to such tilted regions when the tilt f varies sufficiently slowly with g. It also gives a simpler description of the function ψ satisfying (2.1) than [9] does. Portnoy ([10]), using different methods, studies the first exit time from the lower boundary $f(n) - cg(n)$ in the absence of the upper boundary $f(n) + cg(n)$, and for such one-sided exit times is led to lower bounds of the type (1.3) with essentially the same ψ as defined in (2.5). Other results, touching on ours, were obtained by Kesten [7] for Brownian motion rather than random walk. The function ψ defined by (2.4) also figures prominently in his Proposition 2.5 (see his Remark 2.7).

Before stating Theorem 2.1 it will be advantageous to state separately the conditions placed on f and g, both the ones needed for Theorem 2.1 and those needed for Theorem 2.2. The same assumptions on f and g will be needed again in Section 4. Limits of functions of a real argument, say x, will be understood to be taken as $x \to \infty$. The same holds for lim inf, lim sup, and order relations. If a statement is qualified by a phrase such as "for large c," or "for all large c," or "if c is large," this means that there exists c_0 such that the statement is true for all $c > c_0$.

ASSUMPTION 2.1. Let m be a positive integer and let f, g be real-valued functions on $[m, \infty)$ satisfying the following conditions: (i) g is positive, continuous, and eventually nondecreasing with $\lim g(x) = \infty$; (ii) $\limsup x^{-\frac{1}{2}}g(x) < \infty$; (iii) $\limsup g(ax)/g(x) < \infty$ for every $a > 1$; (iv) f is continuously differentiable and $\limsup|f'(x)|g(x) < \infty$.

ASSUMPTION 2.2. Let m be a positive integer and f, g real-valued functions on $[m, \infty)$ satisfying the following conditions: (i) f is continuously differentiable and for all $a > 1$, $\liminf[\min_{x\leq y\leq ax} f'(y)/f'(x)] > 0$, $\limsup[\max_{x\leq y\leq ax} f'(y)/f'(x)] < \infty$; (ii) g is positive and $\liminf f'(x)g(x) > 0$; (iii) $\lim f'(x) = 0$; (iv) $\liminf x^{\frac{1}{2}}f'(x) > 0$.

ASSUMPTION 2.3. Let f, g satisfy the conditions of Assumption 2.2, and, in addition, (i) $g(x) = o(f(x))$, (ii) $\int_m^x (f'(t))^2\,dt \ll f^2(x)/x$.

THEOREM 2.1. *Suppose* $X, X_1, X_2, \cdots$ *are* i.i.d. *real-valued random variables with* $EX = 0$, $0 < EX^2 < \infty$ *and* $S_n = X_1 + \cdots + X_n$. *Let* f, g *satisfy Assumption* 2.1 *and define, for* $c > 0$,

$$N_c = \inf\{n \geqslant m : S_n \geqslant f(n) + cg(n) \text{ or } S_n \leqslant f(n) - cg(n)\}. \tag{2.3}$$

Set

$$\psi(x) = \int_m^x (g(t))^{-2}\,dt, \qquad x \geqslant m. \tag{2.4}$$

Then $\log x \ll \psi(x)$, $\psi(x)/x \to 0$, $\psi(N_c)$ *is exponentially bounded for every* c, *and* $\psi(N_c)$ *is exactly exponentially bounded for all large* c.

The proof of Theorem 2.1 will be given in Section 3. For a typical example arising in our applications, take $m \geqslant 2$, $f(x) = b(x \log x)^{\frac{1}{2}}$, $g(x) = (x/\log x)^{\frac{1}{2}}$, where $b > 0$. Then clearly Assumption 2.1 is satisfied and $\psi(x) \sim \frac{1}{2}(\log x)^2$. Hence $(\log N_c)^2$ is exactly exponentially bounded for all large c. More generally, if h is a continuously differentiable function on $[m, \infty)$ such that $\inf h(x) > 0$, $\limsup xh'(x)/h(x) < 1$, $\liminf xh'(x)/h(x) > -\infty$, and $\limsup h(x)/h(ax) < \infty$ for all $a > 1$, then $f(x) = (xh(x))^{\frac{1}{2}}$ and $g(x) = (x/h(x))^{\frac{1}{2}}$ satisfy Assumption 2.1. An example of such a function h is $h(x) = x^{\gamma}(\log)^{\beta}$ in which $0 < \gamma < 1$ and β is any real number. In this case $\psi(x) \sim \gamma^{-1}x^{\gamma}(\log x)^{\beta}$.

The condition (iv) in Assumption 2.1 means that f cannot vary too fast compared with the growth of g. In particular, since $g(x) \to \infty$, it implies $f'(x) \to 0$. Theorem 2.1 says that the ψ function in this case is defined in terms of g alone when g satisfies (i) of Assumption 2.1. What happens if $\lim |f'(x)|\, g(x) = \infty$? In view of our applications in this situation we shall only consider the case where f is eventually monotone with $g(x) = o(f(x))$ and $f'(x) \to 0$. By replacing, if necessary, X_i with $-X_i$, it further suffices to restrict to the case where $f'(x) > 0$ eventually. Further assumptions on f and g, embodied in Assumptions 2.2 and 2.3, entail that the ψ function that makes the next theorem true is defined entirely in terms of f'.

THEOREM 2.2. *Suppose* $X, X_1, X_2, \cdots$ *are* i.i.d. *with* $EX = 0$, $0 < EX^2 < \infty$ *and* $S_n = X_1 + \cdots + X_n$. *Let* f, g *satisfy Assumption* 2.2 *and, for* $c > 0$, *define* N_c *by* (2.3). *Set*

$$\psi(x) = \int_m^x (f'(t))^2\,dt, \qquad x \geqslant m. \tag{2.5}$$

Then $\log x \ll \psi(x)$, $\psi(x)/x \to 0$, *and there exists* $\rho_1 > 0$ *such that* (1.3) *holds for all large* c. *Hence* N_c *is not exponentially bounded for large* c. *If* f, g *satisfy Assumption* 2.3 *and* X *is exponentially bounded, then there exists* $\rho_2 < 1$ *such tnat* (1.4) *holds for every* c. *Hence, under these assumptions* $\psi(N_c)$ *is exactly exponentially bounded for all large* c.

Theorem 2.2 will be proved in Section 3. Note that (iii) and (iv) of Assumption 2.2 imply that $x^{\frac{1}{2}} \ll f(x) = o(x)$. To give an example of functions f satisfying Assumption 2.3, let $\frac{1}{2} < \alpha < 1$ and let $f(x) = x^{\alpha}h(x)$, where h is a positive continuously differentiable function such that

$$\lim xh'(x)/h(x) = 0. \tag{2.6}$$

Note that (2.6) implies that $h(x) + (h(x))^{-1} = o(x^{\gamma})$ for all $\gamma > 0$. It can be verified from (2.6) that $f(x) = x^{\alpha}h(x)$ satisfies (i), (iii) and (iv) of Assumption 2.2. To show that (ii) of Assumption 2.3 is also satisfied, use integration by parts and (2.6) to obtain that

$$\begin{aligned} \textstyle\int_m^x (f'(t))^2\,dt &= (\alpha^2 + o(1))\textstyle\int_m^x t^{2\alpha-2}h^2(t)\,dt \\ &= \{\alpha^2/(2\alpha - 1) + o(1)\}x^{2\alpha-1}h^2(x). \end{aligned}$$

Hence if $0 < g(x) = o(x^{\alpha}h(x))$ and $g(x) \gg x^{1-\alpha}/h(x)$ (so that (ii) of Assumption 2.2 and (i) of Assumption 2.3 hold) and X is exponentially bounded, then $N_c^{2\alpha-1}h^2(N_c)$ is exactly exponentially bounded for all large c. Examples of functions h satisfying (2.6) include $(\log_k x)^{\beta}$, where β is any real number, k is a positive integer, and $\log_k$ means the k-times iterated logarithm. More generally, combinations of such functions, such as sums and products, also satisfy (2.6).

REMARK. In [9] where the exit time is studied for the case $f \equiv 0$, it is assumed that $X_1, X_2, \cdots$ are independent with $EX_n = 0$, $EX_n^2 = 1$, $n = 1, 2, \cdots$, and

$$n^{-1}\textstyle\sum_{i=k+1}^{k+n} EX_i^2 I_{[|X_i| > \varepsilon n^{1/2}]} \to 0 \tag{2.7}$$

uniformly in k as $n \to \infty$ for every $\varepsilon > 0$. The condition (2.7) is a uniform version of the Lindeberg condition and is obviously satisfied when $X_1, X_2, \cdots$ are i.i.d. with zero mean and finite variance. This formulation in [9] was motivated by the earlier work of Gundy and Siegmund [6] and Brown [3] who studied the connection between the central limit theorem and the finiteness of moments of N_c in the special case $f \equiv 0$ and $g(x) = x^{\frac{1}{2}}$. Since our results in this paper are mainly motivated by their statistical applications, we have stated the theorems in this section only for the i.i.d. case. However, Theorem 2.1 and the first part of Theorem 2.2 still remain true in the more general setting where $X_1, X_2, \cdots$ are independent with $EX_n = 0$, $EX_n^2 = \sigma^2$, $n = 1, 2, \cdots$ $(0 < \sigma < \infty)$ such that the uniform Lindeberg condition (2.7) is satisfied. Moreover, the second part of Theorem 2.2 still remains true in this more general setting if it is assumed that there exists $a > \sigma^2$ and $t_0 > 0$ such that $E\exp(tX_n) \leqslant 1 + \frac{1}{2}at^2$ for all $0 \leqslant t \leqslant t_0$ and $n = 1, 2, \cdots$ (this is obviously fulfilled in the i.i.d. case if X is exponentially bounded). These extensions of Theorems 2.1 and 2.2 can be proved by a straightforward modification of the arguments in Section 3, together with the application of a uniform invariance principle established in [9].

3. Proofs of Theorems 2.1 and 2.2.

LEMMA 3.1. *Let f, g satisfy Assumption* 2.1 *and define* $\psi(x)$ *by* (2.4). *Then* ψ *is strictly increasing and* $\log x \ll \psi(x) = o(x)$. *Let* $\nu = \psi^{-1}$ *be the inverse of* ψ, *so that* $\nu'(x) = g^2(\nu(x))$ *and therefore* ν *is also strictly increasing to* ∞. *Define, for any positive integer* k, $n(k) = [\nu(k)]$ $(=$ *greatest integer* $\leqslant \nu(k))$, *and define* $\Delta n_k = n(k+1) - n(k)$. *The following statements hold as* $k \to \infty$:

$$\Delta n_k \to \infty \quad \text{and} \quad n(k+1) \ll n(k), \tag{3.1}$$

$$\max_{n(k) \leqslant y \leqslant n(k+1)} |f(y) - f(n(k))| \ll (\Delta n_k)^{\frac{1}{2}}, \tag{3.2}$$

$$(\Delta n_k)^{\frac{1}{2}} \ll g(n(k)), \tag{3.3}$$

$$g(n(k+1)) \ll (\Delta n_k)^{\frac{1}{2}}. \tag{3.4}$$

PROOF. To prove (3.1) note that

$$\nu(x+1) - \nu(x) = \nu'(x^*) = g^2(\nu(x^*)), \tag{3.5}$$

where $x < x^* < x + 1$. Since $x \to \infty$, so does x^*, and then so do $\nu(x^*)$ and $g(\nu(x^*))$, using Assumption 2.1 (i). Since Δn_k differs from $\nu(k+1) - \nu(k)$ by at most 1, the first statement of (3.1) follows. By Assumption 2.1 (ii) there exists $A > 0$ such that $g^2(t) \leqslant A^{-1}t$ for t sufficiently large. Therefore, for x sufficiently large,

$$\begin{aligned} 1 = \psi(\nu(x+1)) - \psi(\nu(x)) &= \textstyle\int_{\nu(x)}^{\nu(x+1)} (g(t))^{-2}\, dt \\ &\geqslant A \textstyle\int_{\nu(x)}^{\nu(x+1)} t^{-1}\, dt = A \log(\nu(x+1)/\nu(x)), \end{aligned} \tag{3.6}$$

and so $\nu(x+1) \ll \nu(x)$. This proves the second part of (3.1) since $0 \leqslant \nu(k) - n(k) < 1$.

The relations (3.3) and (3.4) follow easily from (3.5) and Assumption 2.1 (iii). To prove (3.2), first note that $f'(x) \to 0$ by Assumption 2.1 (iv). Then for k sufficiently large and $n(k) \leqslant y \leqslant n(k+1)$,

$$\begin{aligned} |f(y) - f(n(k))| &\leqslant \textstyle\int_{n(k)}^{n(k+1)} |f'(t)|\, dt \\ &= \textstyle\int_{\nu(k)}^{\nu(k+1)} |f'(t)|\, dt + o(1) \\ &\ll \textstyle\int_{\nu(k)}^{\nu(k+1)} (g(t))^{-1}\, dt + o(1) \qquad \text{by Assumption 2.1 (iv)} \\ &= \textstyle\int_k^{k+1} g(\nu(u))^{-1} \nu'(u)\, du + o(1) \\ &= \textstyle\int_k^{k+1} g(\nu(u))\, du + o(1) \\ &\ll g(n(k)) \text{ by (3.1)} \qquad \text{and Assumption 2.1 (iii).} \end{aligned} \tag{3.7}$$

Then use (3.4) to obtain (3.2). □

PROOF OF THEOREM 2.1. Set $\Delta n_k = n(k+1) - n(k)$, $\Delta f_k = f(n(k+1)) - f(n(k))$, and $\Delta S_k = S_{n(k+1)} - S_{n(k)}$, $k = 1, 2, \cdots$. In view of (3.2) and (3.4) there exists a positive constant B such that

$$|\Delta f_k| + 2cg(n(k+1)) \leqslant B(\Delta n_k)^{\frac{1}{2}} \tag{3.8}$$

for all $k \geqslant k_1$, say. By the central limit theorem, using the first part of (3.1), $P\{|\Delta S_k| < B(\Delta n_k)^{\frac{1}{2}}\} \to p_1 < 1$. Therefore, by (3.8) there exists $k_2 \geqslant k_1$ and $p < 1$ such that $k \geqslant k_2$ implies

$$P\{|\Delta S_k| < |\Delta f_k| + 2cg(n(k+1))\} < p. \tag{3.9}$$

Without loss of generality it is assumed that $g(x)$ is nondecreasing for $x \geqslant n(k_2)$, using Assumption 2.1 (i). Then for $k \geqslant k_2$,

$$\begin{aligned} P\{\psi(N_c) > k\} &= P\{N_c > \nu(k)\} = P\{N_c > n(k)\} \\ &\leqslant \Pi_{i=k_2}^{k-1} P\{|\Delta S_i| < |\Delta f_i| + 2cg(n(i+1))\} \\ &< p^{k-k_2} \quad \text{by (3.9).} \end{aligned} \tag{3.10}$$

Hence $\psi(N_c)$ is exponentially bounded for every c.

It remains to show (1.3) for large c. It is sufficient to demonstrate the existence of $\rho_1 > 0$ such that for all large c

$$\rho_1^k \ll P\{\psi(N_c) > k\}. \tag{3.11}$$

By (3.2) and (3.3) one can choose $\gamma > 0$ and $k_3 \geqslant k_2$ such that for $k \geqslant k_3$

$$\max_{n(k) < y \leqslant n(k+1)} |f(y) - f(n(k))| \leqslant \gamma g(n(k)). \tag{3.12}$$

Define the following events:

$$A_{k,c} = \left[N_c > n(k), |S_{n(k)} - f(n(k))| \leqslant \frac{c}{4} g(n(k))\right], \tag{3.13}$$

$$\begin{aligned} B_{k,c} &= \left[\max_{n(k) < j \leqslant n(k+1)} |S_j - S_{n(k)}| \leqslant \frac{c}{4} g(n(k))\right] \\ &\cap \left[-\frac{c}{4} g(n(k)) \leqslant \Delta S_k - \Delta f_k \leqslant 0\right], \end{aligned} \tag{3.14}$$

$$\begin{aligned} D_{k,c} &= \left[\max_{n(k) < j \leqslant n(k+1)} |S_j - S_{n(k)}| \leqslant \frac{c}{4} g(n(k))\right] \\ &\cap \left[0 \leqslant \Delta S_k - \Delta f_k \leqslant \frac{c}{4} g(n(k))\right]. \end{aligned} \tag{3.15}$$

Then for $c \geqslant 4\gamma$ and $k \geqslant k_3$, by (3.12),

$$\max_{n(k) < y \leqslant n(k+1)} |f(y) - f(n(k))| \leqslant \frac{c}{4} g(n(k)). \tag{3.16}$$

Using (3.16) it can easily be checked that $A_{k+1,c}$ contains the union of the events $A_{k,c} \cap [S_{n(k)} \geqslant f(n(k))] \cap B_{k,c}$, and $A_{k,c} \cap [S_{n(k)} < f(n(k))] \cap D_{k,c}$. Since $B_{k,c}$ and $D_{k,c}$ are independent of $A_{k,c}$, it follows that

$$P(A_{k+1,c} | A_{k,c}) \geqslant \min(PB_{k,c}, PD_{k,c}). \tag{3.17}$$

By (3.2) and (3.3) there exist constants $\alpha, \beta > 0$, and $k_4 \geqslant k_3$ such that $|\Delta f_k| \leqslant$

$\alpha(\Delta n_k)^{\frac{1}{2}}$ and $g(n(k)) \geqslant \beta(\Delta n_k)^{\frac{1}{2}}$ if $k \geqslant k_4$. Hence for $c > 8\alpha/\beta$ and $k \geqslant k_4$,

$$PB_{k,c} \geqslant P\Big\{\max_{j \leqslant \Delta n_k}|S_j| \leqslant \frac{c}{4}\beta(\Delta n_k)^{\frac{1}{2}} \quad \text{and} \quad -\Big(\frac{c}{4}\beta - \alpha\Big)(\Delta n_k)^{\frac{1}{2}} \leqslant S_{\Delta n_k} \leqslant -\alpha(\Delta n_k)^{\frac{1}{2}}\Big\} = b_{k,c}, \quad \text{say;} \tag{3.18}$$

$$PD_{k,c} \geqslant P\Big\{\max_{j \leqslant \Delta n_k}|S_j| \leqslant \frac{c}{4}\beta(\Delta n_k)^{\frac{1}{2}} \quad \text{and} \quad \alpha(\Delta n_k)^{\frac{1}{2}} \leqslant S_{\Delta n_k} \leqslant \Big(\frac{c}{4}\beta - \alpha\Big)(\Delta n_k)^{\frac{1}{2}}\Big\} = d_{k,c}, \quad \text{say.} \tag{3.19}$$

By Donsker's functional central limit theorem ([1], page 72), assuming without loss of generality that $EX^2 = 1$, as $k \to \infty$,

$$b_{k,c} \to P\Big\{\max_{0 \leqslant t \leqslant 1}|W(t)| \leqslant \frac{c}{4}\beta, -\Big(\frac{c}{4}\beta - \alpha\Big) \leqslant W(1) \leqslant -\alpha\Big\}, \tag{3.20}$$

$$d_{k,c} \to P\Big\{\max_{0 \leqslant t \leqslant 1}|W(t)| \leqslant \frac{c}{4}\beta, \alpha \leqslant W(1) \leqslant \frac{c}{4}\beta - \alpha\Big\}, \tag{3.21}$$

where $W(t)$ is the standard Wiener process. Therefore, choosing $c_1 > \max(4\gamma, 8\alpha/\beta)$, by (3.17)–(3.21) there exists $k_5 \geqslant k_4$ and $\rho_1 > 0$ such that for $k \geqslant k_5$ and $c = c_1$

$$P(A_{k+1,c}|A_{k,c}) > \rho_1. \tag{3.22}$$

Since obviously the events $B_{k,c}$ and $D_{k,c}$ are nondecreasing in c, so is the right-hand side of (3.17). Consequently, (3.22) is valid for $k \geqslant k_5$ and $c \geqslant c_1$. Moreover, $A_{k,c}$ defined in (3.13) is obviously also nondecreasing in c, and for any fixed k, $PA_{k,c} \to 1$ as $c \to \infty$. Thus, there exists $c_2 \geqslant c_1$ and $p > 0$ such that $PA_{k_5,c} > p$ if $c > c_2$. Then for such c and $k > k_5$, $P(\psi(N_c) > k) = P(N_c > \nu(k)) = P(N_c > n(k)) \geqslant PA_{k,c} \geqslant PA_{k_5,c}\Pi_{i=k_5}^{k-1}P(A_{i+1,c}|A_{i,c}) \geqslant p\rho_1^{k-k_5}$, using (3.22), so that (3.11) has been shown to hold. ☐

Lemma 3.2. *Let f, g satisfy Assumption* 2.2. *Define ψ by* (2.5) *and let $\nu = \psi^{-1}$ so that $\nu'(x) = (f'(\nu(x)))^{-2}$. Define $n(k)$ and Δn_k as in Lemma* 3.1. *Then* (3.1), (3.2), *and* (3.3) *still hold.*

Proof. The proof of (3.1) proceeds as in Lemma 3.1, using Assumption 2.2 (iii) and (iv). Since $\nu'(x) = (f'(\nu(x)))^{-2}$ and (3.1) holds, Assumption 2.2 (i) implies that there exist $\delta_2 > \delta_1 > 0$ such that

$$\delta_1\nu'(k) \leqslant \nu'(y) \leqslant \delta_2\nu'(k), \qquad k \leqslant y \leqslant k+1, \tag{3.23}$$

for sufficiently large k. Write $\nu(k+1) - \nu(k) = \nu'(x^*)$ with $k < x^* < k+1$, and note that Δn_k differs from $\nu(k+1) - \nu(k)$ by at most 1. Then using (3.23) one obtains

$$\nu'(k) \ll \Delta n_k \ll \nu'(k). \tag{3.24}$$

In order to prove (3.2), first observe that by Assumption 2.2 (iv) f is eventually increasing. Therefore, it suffices to show (3.2) with its left-hand side replaced by

$f(n(k+1)) - f(n(k))$. Compute

(3.25)

$$f(n(k+1)) - f(n(k)) = \int_{\nu(k)}^{\nu(k+1)} f'(t)\,dt + o(1)$$
$$= \int_k^{k+1} f'(\nu(u))\nu'(u)\,du + o(1)$$
$$= \int_k^{k+1} (\nu'(u))^{\frac{1}{2}}\,du + o(1) \qquad \text{since} \quad \nu'(u) = (f'(\nu(u)))^{-2}$$
$$\leqslant \delta_2^{\frac{1}{2}}(\nu'(k))^{\frac{1}{2}} + o(1) \qquad \text{by} \quad (3.23).$$

Then use (3.24) to obtain (3.2). Lastly, to prove (3.3) use Assumption 2.2 (i) to obtain $g(n(k))f'(n(k)) \ll g(n(k))f'(\nu(k)) = g(n(k))(\nu'(k))^{-\frac{1}{2}}$. From this and Assumption 2.2 (ii) it follows that $\liminf g(n(k))(\nu'(k))^{-\frac{1}{2}} > 0$. Then use the right-hand inequality in (3.24) to obtain (3.3). □

LEMMA 3.3. *Let f, g be real-valued functions on $[m, \infty)$ such that $g > 0$, $\lim x^{-1}f(x) = 0$, $\lim x^{-\frac{1}{2}}f(x) = \infty$, and $\lim g(x)/f(x) = 0$. Let $\psi : [m, \infty) \to [0, \infty)$ be continuous and eventually increasing, satisfying the condition*

$$\psi(x+1) \ll x^{-1}f^2(x). \tag{3.26}$$

Let $X, X_1, X_2, \cdots$ be i.i.d. *real-valued random variables with $EX = 0$ and X exponentially bounded, and let N_c be as defined in* (2.3). *Then $\psi(N_c)$ is exponentially bounded for every $c > 0$.*

PROOF. Fix any $c > 0$. Without loss of generality it may be assumed that ψ is strictly increasing on $[m, \infty)$ and $\lim \psi(x) = \infty$. Also, $f(x) \to \infty$ so that f is eventually positive. For convenience it will be assumed in the proof that f is positive everywhere. Let $\nu = \psi^{-1}$ and set $n(k) = [\nu(k)]$. Then ν is strictly increasing and $\lim \nu(x) = \infty$. Take $a > EX^2$, then there exists $t_0 > 0$ such that for $0 \leqslant t \leqslant t_0$

$$E\exp(tX) \leqslant \exp\left(\tfrac{1}{2}at^2\right). \tag{3.27}$$

Since $f(x) = o(x)$ one can choose k_0 such that for $k \geqslant k_0$

$$f(n(k)) < 2at_0 n(k). \tag{3.28}$$

Furthermore, since $g(x) = o(f(x))$, it may be assumed that k_0 is so large that for $k \geqslant k_0$

$$cg(n(k)) < \tfrac{1}{2}f(n(k)). \tag{3.29}$$

Compute, for $0 \leqslant t \leqslant t_0$,

$$P\{\psi(N_c) > k\} = P\{N_c > \nu(k)\} = P\{N_c > n(k)\}$$
$$\leqslant P\{S_{n(k)} > f(n(k)) - cg(n(k))\}$$
$$\leqslant P\left\{S_{n(k)} > \tfrac{1}{2}f(n(k))\right\} \qquad \text{by} \quad (3.29) \tag{3.30}$$
$$\leqslant \exp\left[-\tfrac{1}{2}tf(n(k))\right]E\exp(tS_{n(k)})$$
$$\leqslant \exp\left[-\tfrac{1}{2}tf(n(k)) + \tfrac{1}{2}at^2 n(k)\right] \qquad \text{by} \quad (3.27).$$

In the inequality (3.30) set $t = f(n(k))/(2an(k))$, which is $< t_0$ by (3.28), to obtain that for $k \geqslant k_0$

$$P\{\psi(N_c) > k\} \leqslant \exp\left[-f^2(n(k))/(8an(k))\right]. \tag{3.31}$$

By (3.26) there exists $\rho > 0$ such that $f^2(n(k))/n(k) > \rho\psi(\nu(k)) = \rho k$ for k sufficiently large. Substitution of this into (3.31) leads immediately to the exponential boundedness of $\psi(N_c)$. □

PROOF OF THEOREM 2.2. The part of the theorem dealing with (1.3) follows from Lemma 3.2 and the fact that in our previous proof of (1.3) in Theorem 2.1 only (3.1), (3.2), and (3.3) were used (but not (3.4)). The part dealing with (1.4) follows from Lemma 3.3 after showing that the conditions of that lemma are satisfied. Now it follows from Assumption 2.2 (iii) that $f(x) = o(x)$, and from Assumption 2.2 (iv) that $\psi(x)$ given by (2.5) is $\gg \log x$. By Assumption 2.3 (ii),

$$\psi(x) \ll x^{-1}f^2(x). \tag{3.32}$$

Since $\lim \psi(x) = \infty$, it follows from (3.32) that $\lim x^{-\frac{1}{2}}f(x) = \infty$. Moreover, in view of Assumption 2.2 (iii), $\psi(x + 1) - \psi(x) \to 0$, and therefore (3.32) implies that (3.26) holds. □

4. Stopping time of a higher dimensional random walk. In this section the random walk $S_n = \Sigma_1^n X_i$ is based on a sequence $X_1, X_2 \cdots$ of i.i.d. random variables taking values in R^d, with $d \geqslant 1$. Let X be a random variable with the same distribution as that of the X_i. It will be assumed throughout that X has a finite covariance matrix Σ. In some propositions X will be required to be exponentially bounded. As in Section 1 it will be convenient to put $S_n/n = \overline{X}_n$. The stopping time of the random walk will be governed by a sequence of statistics $L_1, L_2, \cdots$, with L_n real-valued and depending only on $X_1, \cdots, X_n$, and for chosen $l > 0$ and integer $m \geqslant 1$

$$N = \text{smallest integer } n \geqslant m \quad \text{such that} \quad |L_n| < l \quad \text{is violated.} \tag{4.1}$$

Sometimes l will be a function of a positive variable $c : l = l(c)$, in which case N will be denoted N_c as in Sections 2 and 3. In connection with the statistical applications in Section 5, the statistic L_n is assumed to have the form

$$L_n = n\Phi(\overline{X}_n) - h(n), \qquad n = m, m + 1, \cdots, \tag{4.2}$$

in which Φ and h are real-valued functions defined on R^d and $[m, \infty)$, respectively. In the theorems that follow, various assumptions will be made on Φ. On h the following assumption will be made:

ASSUMPTION 4.1. There exist an integer $m \geqslant 1$ and $0 < \eta < 1$ such that $h > 0$ on $[m, \infty)$, $h(x) \ll x^\eta$, h is continuously differentiable with $h' > 0$, $xh'(x) \leqslant h(x)$ for all large x, and $\lim\sup h(ax)/h(x) < \infty$ for all $a > 1$.

REMARK. The last two conditions in Assumption 4.1 are implied by: h' is eventually decreasing.

Functions f and g satisfying Assumption 2.1 or 2.2 will be employed again. They are now defined in terms of h and an integer $p \geqslant 1$ as follows:

$$f(x) = x^{p/(p+1)}(h(x))^{1/(p+1)}, \tag{4.3}$$

$$g(x) = x^{p/(p+1)}(h(x))^{-p/(p+1)}, \tag{4.4}$$

for $x \geqslant m$. Then it can be checked easily that f, g satisfy Assumption 2.1 if $p = 1$ and Assumption 2.2 if $p > 1$. Consequently, the function ψ will be defined by (2.4) if $p = 1$ and by (2.5) if $p > 1$, i.e.,

$$\psi(x) = \int_m^x (g(t))^{-2}\, dt \qquad \text{if} \quad p = 1, \tag{4.5}$$

$$\psi(x) = \int_m^x (f'(t))^2\, dt \qquad \text{if} \quad p > 1. \tag{4.6}$$

It follows immediately from (4.4) with $p = 1$ that (4.5) can be written

$$\psi(x) = \int_m^x t^{-1}h(t)\, dt \qquad \text{if} \quad p = 1. \tag{4.7}$$

We first consider conclusions of the type (1.3) in the following theorem. Conclusions of the type (1.4) will be given later in Theorems 4.2 and 4.3.

THEOREM 4.1. *For any given $l > 0$, let the stopping time N be defined by* (4.1) *with L_n of the form* (4.2). *It is assumed that $EX = \xi \in R^d$, $E(X - \xi)(X - \xi)' = \Sigma$ nonsingular, and Φ is of the form*

$$\Phi(x) = Q(x - \xi) + b(x)\|x - \xi\|^{p+1+\varepsilon} \tag{4.8}$$

for some $\varepsilon > 0$ and integer $p \geqslant 1$, in which Q is a homogeneous polynomial of degree $p + 1$, not everywhere $\leqslant 0$, and b is bounded on compacta. Furthermore, it is assumed that h satisfies Assumption 4.1, *with $\eta = \varepsilon/(p + 1 + \varepsilon)$. Define ψ by* (4.7) *if $p = 1$ and by* (4.6) *if $p > 1$. Then there exist constants $l > 0$ and $\rho > 0$ such that*

$$P\{\psi(N) > k\} \gg \rho^k. \tag{4.9}$$

Consequently, N is not exponentially bounded.

REMARK. The condition on Φ is satisfied with $\varepsilon = 1$ if Φ possesses all continuous partial derivatives of order $p + 2$, all derivatives of order $\leqslant p$ vanish at $x = \xi$, and Φ is bounded on compacta.

The proof of Theorem 4.1 relies on a generalization of its counterpart in Theorem 2.1 and it will be convenient to deal with this part of the proof in the following lemma.

LEMMA 4.1. *Let $X, X_1, X_2, \cdots$ be* i.i.d. *random variables with values in R^d $(d \geqslant 1)$, $EX = 0$, $EXX' = \Sigma$ nonsingular. Put $S_n = \Sigma_1^n X_i = (S_{1n}, \cdots, S_{dn})'$. Let $L_1, L_2, \cdots$ be a sequence of statistics with L_n depending only on $X_1, \cdots, X_n$. Suppose there exist functions f, g satisfying Assumption* 2.1 *or* 2.2, *and a function l on* *the positive half line, such that for every* $c > 0$ *the following two inequalities*

$$f(n) - cg(n) < S_{1n} < f(n) + cg(n) \tag{4.10}$$

$$-cg(n) < S_{in} < cg(n), \qquad i = 2, \cdots, d \tag{4.11}$$

together imply $|L_n| < l(c)$ *(in case* $d = 1$ (4.11) *is omitted). Define* N_c *by* (4.1), *with* $l = l(c)$, *and* ψ *by* (4.5) *or* (4.6) *depending on whether Assumption* 2.1 *or* 2.2 *is satisfied. Then there exist constants* $\rho > 0$ *and* $c_0 > 0$ *such that for all* $c > c_0$

$$P\{\psi(N_c) > k\} \gg \rho^k. \tag{4.12}$$

PROOF. The proof will be given for $d > 1$. The necessary modifications if $d = 1$ are obvious. The proof is in essence the same as the corresponding parts of the proofs of Theorems 2.1 and 2.2, but slightly more complicated. Only the changes will be indicated here. The quantities Δn_k and Δf_k were defined in Section 3 and will also be used here. Further, define $\Delta S_{ik} = S_{i,n(k+1)} - S_{i,n(k)}$, and

$$A_{k,c} = \left[N > n(k);\ |S_{1,n(k)} - f(n(k))| \leqslant \frac{c}{4}g(n(k)); |S_{i,n(k)}| \leqslant \frac{c}{2}g(n(k)), i = 2, \cdots, d\right]. \tag{4.13}$$

The event defined in (4.13) takes the place of $A_{k,c}$ introduced in (3.13). In (3.14) and (3.15) were introduced the two events $B_{k,c}$ and $D_{k,c}$, corresponding to the two possibilities $\Delta S_k - \Delta f_k \leqslant 0$ and $\geqslant 0$. In the present situation, however, there are 2^d events corresponding to the 2^d combinations of the signs of $\Delta S_{1k} - \Delta f_k, \Delta S_{2k}, \cdots, \Delta S_{dk}$. These events will be labeled by a vector $\lambda = (\lambda_1, \cdots, \lambda_d)$, with $\lambda_i = \pm 1$. Thus, define

$$\begin{aligned} B_{k,c}^{\lambda} &= \left[|S_{ij} - S_{i,n(k)}| \leqslant \frac{c}{4}g(n(k)), n(k) \leqslant j \leqslant n(k+1), i = 1, \cdots, d\right] \\ &\quad \cap \left[0 \leqslant \lambda_1(\Delta S_{1k} - \Delta f_k) \leqslant \frac{c}{4}g(n(k)), \lambda_i \Delta S_{ik} \geqslant 0, i = 2, \cdots, d\right]. \end{aligned} \tag{4.14}$$

Note that $B_{k,c}^{\lambda}$ is nondecreasing in c. Choosing $c > 4\gamma$ as in Section 3, where γ is defined in (3.12), it can be verified that the event $A_{k,c} \cap [\lambda_1(S_{1,n(k)} - f(n(k))) \leqslant 0; \lambda_i S_{i,n(k)} \leqslant 0, i = 2, \cdots, d] \cap B_{k,c}^{\lambda}$ implies the event $A_{k+1,c}$ for all large k. It follows that for all large k

$$P(A_{k+1,c}|A_{k,c}) \geqslant \min_{\lambda} PB_{k,c}^{\lambda}. \tag{4.15}$$

Furthermore, introducing α, β as in Section 3 and taking $c > 8\alpha/\beta$, the inequalities analogous to (3.18) and (3.19) are now

$$\begin{aligned} PB_{k,c}^{\lambda} \geqslant P\Big\{&\max_{j \leqslant \Delta n_k}|S_{ij}| \leqslant \frac{c}{4}\beta(\Delta n_k)^{\frac{1}{2}}, i = 1, \cdots, d; \\ &0 \leqslant \lambda_1\left(S_{1,\Delta n_k} - \alpha(\Delta n_k)^{\frac{1}{2}}\right) \leqslant \left(\frac{c}{4}\beta - 2\alpha\right)(\Delta n_k)^{\frac{1}{2}}; \lambda_i S_{i\Delta n_k} \\ &\geqslant 0, i = 2, \cdots, d\Big\} = b_{k,c}^{\lambda}, \qquad \text{say,} \end{aligned} \tag{4.16}$$

provided k is sufficiently large. Finally, as $k \to \infty$,

$$b_{k,c}^{\lambda} \to P\Big\{\max_{0<t<1}|W_i(t)| < \frac{c}{4}\beta, i = 1, \cdots, d; \quad 0 < \lambda_1(W_1(1) - \alpha) < \frac{c}{4}\beta - 2\alpha; \lambda_i W_i(1) \geq 0, i = 2, \cdots, d\Big\} > 0, \tag{4.17}$$

in which $(W_1(\cdot), \cdots, W_d(\cdot))'$ is d-dimensional Wiener process with covariance matrix Σ. The rest of the proof is identical to the relevent parts of the proofs of Theorems 2.1 and 2.2. □

Proof of Theorem 4.1. In the proof it will be assumed that $d > 1$. If $d = 1$, a few trivial modifications are necessary. After making an affine transformation in R^d, it may be assumed without loss of generality that $\xi = 0$ and

$$\Phi(x) = x_1^{p+1} + Q_1(x) + b(x)\|x\|^{p+1+\epsilon}, \tag{4.18}$$

in which Q_1 is a homogeneous polynomial of degree $p + 1$ which is of degree p in x_1. The functions f, g are defined by (4.3) and (4.4). The conclusion of the theorem will follow from Lemma 4.1 if it can be shown that there exists $c > 0$ such that for $n \geq m$ the inequalities (4.10) and (4.11) together imply $|L_n| < l(c)$. Take (4.10), raise all members to the power $p + 1$ and divide by n^p. The result is

$$|n\overline{X}_{1n}^{p+1} - h(n)| - c(p+1) \ll 1/h(n). \tag{4.19}$$

(The reader is reminded that the constant implied by the order relation $\ll$ is understood to be nonrandom.) From (4.19) follows

$$|\overline{X}_{1n}| \ll (h(n)/n)^{1/(p+1)}. \tag{4.20}$$

Inequalities (4.11) and (4.4) together show that

$$|\overline{X}_{in}| \ll n^{-1/(p+1)}(h(n))^{-p/(p+1)}, \qquad i = 2, \cdots, d. \tag{4.21}$$

Combining (4.20) and (4.21) it is seen that

$$\|\overline{X}_n\| \ll (h(n)/n)^{1/(p+1)} \tag{4.22}$$

so that (4.10) and (4.11) imply that $\|\overline{X}_n\|$ remains bounded. By assumption on b in (4.8) the same is then true for $b(\overline{X}_n)$: $|b(\overline{X}_n)| < B$ for all n, for some $B > 0$. In (4.18) replace x by $\overline{X}_n$ and use (4.20)–(4.22). It is found that

$$|n\Phi(\overline{X}_n) - n\overline{X}_{1n}^{p+1}| \ll 1. \tag{4.23}$$

Combining (4.2), (4.19), and (4.23) it follows that $|L_n| \ll 1$. Therefore, there exists $l = l(c)$ such that for all $n \geq m$, (4.10) and (4.11) together imply $|L_n| < l$. □

The following lemma will be used in the proof of the next theorem.

Lemma 4.2. *Let Z_n, Z be random variables with values in R^d and let $Z_n \to Z$ in law, where the distribution of Z is equivalent to d-dimensional Lebesgue measure. Let A be a set with nonempty interior. Then*

$$\liminf_{n\to\infty} \inf_{U\in O(d)} P(UZ_n \in A) > 0, \tag{4.24}$$

where $O(d)$ is the group of $d \times d$ orthogonal matrices.

PROOF. There exists an open ball A_1 and $\varepsilon > 0$ such that the ε-neighborhood of A_1 is contained in A. Let B be an upper bound for the norms of the vectors in A_1. If $U_1, U_2 \in O(d)$ and $\|U_1 - U_2\| < \varepsilon/B = \delta$, say, then $U_1x \in A_1$ implies $U_2x \in A$. Since $O(d)$ is compact, there exist $U_1, \cdots, U_k \in O(d)$ such that for every $U \in O(d)$ there is some U_i such that $\|U - U_i\| < \delta$. For this U and U_i, if $U_iZ_n \in A_1$, then $UZ_n \in A$. Therefore, for every $U \in O(d)$ and every n, $P(UZ_n \in A) \geqslant \min_{1 \leqslant i \leqslant k} P(U_iZ_n \in A_1)$. Since the distribution of U_iZ is equivalent to Lebesgue measure, $P(U_iZ \in A_1) = 2p_i$, say, with $p_i > 0$. Since $U_iZ_n \to U_iZ$ in law, there exists n_i such that $P(U_iZ_n \in A_1) > p_i$ if $n > n_i$. It follows that $P(UZ_n \in A) \geqslant \min_{1 \leqslant i \leqslant k} p_i$ for all $U \in O(d)$ and $n > \max_{1 \leqslant i \leqslant k} n_i$. □

THEOREM 4.2. *Let $EX = \xi \in R^d$, $E(X - \xi)(X - \xi)' = \Sigma$ nonsingular. Let $l > 0$ be fixed and let N be defined by* (4.1) *and* (4.2) *with h satisfying Assumption* 4.1 *for some $0 < \eta < 1$. About Φ the following is assumed. Either Case* (a): *$\Phi(x) = (x - \xi)'A(x - \xi)$ with A a symmetric $d \times d$ matrix; or Case* (b): *$\Phi \geqslant 0$, $\Phi(\xi) = 0$, there exists a neighborhood V of ξ such that on V the function Φ has continuous second partial derivatives with positive definite matrix $A(x)$, and Φ is bounded away from* 0 *outside V. Define ψ by* (4.7). *Then $\psi(N)$ is exponentially bounded.*

PROOF. Define f, g by (4.3) and (4.4) with $p = 1$, so that they satisfy Assumption 2.1. Hence the results of Lemma 3.1 apply. In particular, it follows from (3.3), (3.4), and (4.4) that

$$1 \ll \Delta n_k h(n(k))/n(k) \ll 1. \tag{4.25}$$

Here $n(k)$ and Δn_k were defined in Lemma 3.1. In the following only n of the form $n(k)$ will be considered. To prove the theorem it suffices to show that for some $p_1 > 0$ and all large k,

$$P\{N > n(k+1) | \mathcal{F}_{n(k)}\} < 1 - p_1 \quad \text{on the event} \quad [N > n(k)], \tag{4.26}$$

where $\mathcal{F}_j$ denotes the σ-field generated by $X_1, \cdots, X_j$.

For notational convenience the dependence on k will usually be suppressed in the following. Thus, n means $n(k)$, Δn means Δn_k. Also, S, $\bar{X}$, ΔS, and $\Delta \bar{X}$ mean $S_{n(k)}$, $\bar{X}_{n(k)}$, $S_{n(k+1)} - S_{n(k)}$, and $\bar{X}_{n(k+1)} - \bar{X}_{n(k)}$, respectively. Limits are taken as $k \to \infty$, which implies $n \to \infty$. If stopping has not occurred yet at stage n, then

$$n(h(n) - l) < n^2\Phi(\bar{X}) < n(h(n) + l). \tag{4.27}$$

Setting $\Delta n^2\Phi(\bar{X}) = (n + \Delta n)^2\Phi(\bar{X} + \Delta\bar{X}) - n^2\Phi(\bar{X})$, let B_n denote the event

$$-\Delta n^2\Phi(\bar{X}) > 2nl. \tag{4.28}$$

We shall only consider the case $\lim h(n) = \infty$. A straightforward modification (replacing $2nl$ in (4.28) by rnl with r sufficiently large) can be used to deal with the case $\lim h(n) < \infty$. For all large n, say $n \geqslant n_0$, $h(n) > l$. Then (4.27) and (4.28) together imply $(n + \Delta n)\Phi(\bar{X} + \Delta\bar{X}) < h(n) - l < h(n + \Delta n) - l$ so that stopping will occur by stage $n + \Delta n$. Therefore, for $n \geqslant n_0$ we have $N \leqslant n + \Delta n$ on the event $B_n \cap [N > n]$. Hence to prove (4.26) it suffices to show that there exists

$p_1 > 0$ such that for all large n, say $n \geqslant n_1$ ($\geqslant n_0$),

$$(4.29) \qquad P(B_n | \mathfrak{F}_n) > p_1 \quad \text{on the event} \quad [N > n].$$

It will be shown that if n is sufficiently large and (4.27) holds, then (4.28) can be implied by an event of the type

$$(4.30) \qquad E_n(u) = \left[u'\Delta S > c_1(\Delta n)^{\frac{1}{2}},\ \|\Delta S\| < 2c_1(\Delta n)^{\frac{1}{2}} \right]$$

with suitably chosen constant $c_1 > 0$ and random vector $u \in R^d$ such that $\|u\| = 1$ and u is $\mathfrak{F}_n$-measurable. Since $\Delta S/(\Delta n)^{\frac{1}{2}} \to N(0, \Sigma)$ in law, Lemma 4.2 can be applied with the result that for every $c_1 > 0$, there exists $p_1 > 0$ such that for all large n, $PE_n(u) > p_1$ for every fixed $u \in R^d$ with $\|u\| = 1$. It will follow then that (4.29) holds for all n.

By making a translation in R^d and projection on a linear subspace, if necessary, it may be assumed that $\xi = 0$ and A nonsingular. In Case (a) (i.e., $\Phi(x) = x'Ax$), if A is negative definite, then from (4.2) it is obvious that stopping occurs by a predetermined n so that the theorem is trivially true. It may be assumed then that A is positive definite or indefinite. By making a suitable linear transformation it can be assumed that Φ takes either of the following two forms in Case (a):

$$(4.31) \qquad \Phi(x) = \Sigma_1^d x_i^2 = \|x\|^2: \qquad \text{Case (a1)},$$

$$(4.32) \qquad \Phi(x) = \Sigma_1 x_i^2 - \Sigma_2 x_i^2: \qquad \text{Case (a2)},$$

where in (4.32) Σ_1 denotes summation over i from 1 to d_1, say, and Σ_2 over $d_1 + 1$ to d.

The left-hand inequality in (4.27) implies the following order relation:

$$(4.33) \qquad \|S\|^2 \gg nh(n).$$

In Case (a) this follows by (4.31) and (4.32) from $\|S\|^2 \geqslant n^2\Phi(\bar{X})$ (equality in Case (a1)). In Case (b) there exist $r > 0$ and $c_2 > 0$ such that $\Phi(x) < c_2\|x\|^2$ if $\|x\| < r$; then $\|\bar{X}\| > r$ or $c_2\|\bar{X}\|^2 > \Phi(\bar{X})$, i.e., $\|S\| > rn$ or $c_2\|S\|^2 > n^2\Phi(\bar{X})$.

In Case (a1) the left-hand side of (4.28) is $\|S\|^2 - \|S + \Delta S\|^2 = -2S'\Delta S - \|\Delta S\|^2$. Take in (4.30) $u = -S/\|S\|$, then $E_n(u)$ implies $-2S'\Delta S - \|\Delta S\|^2 > 2c_1\|S\|(\Delta n)^{\frac{1}{2}} - 4c_1^2\Delta n$. The first term on the right-hand side of the above inequality sign can be made $> 3nl$ by choosing c_1 sufficiently large in view of (4.25) and (4.33). With this choice of c_1 the second term is $< nl$ for all large n, by (4.25). In Case (a2) choose $u_i = -S_i/\|S\|$ for $i = 1, \cdots, d_1$, and $u_i = S_i/\|S\|$ for $i = d_1 + 1, \cdots, d$, then the same inequality for the left-hand side of (4.28) is obtained as in Case (a1).

Thus it has been shown that in Case (a), if n is large and (4.27) holds, then for suitably chosen u and c_1 in (4.30), the event $E_n(u)$ implies (4.28). It remains to be shown that the same is also true for Case (b). Without loss of generality it may be assumed in Case (b) that the positive definite matrix $A(x)$ equals the identity matrix at $x = 0$. Since $\Phi(0) = 0$ and $\Phi \geqslant 0$, grad $\Phi(0) = 0$ so that $\Phi(x) = \|x\|^2 + o(\|x\|^2)$ (as $x \to 0$). Put $H(x) = \|x\|^2$. The assumptions on Φ imply that given

$\varepsilon > 0$, there exists $\delta > 0$ such that if $\|x\| < \delta$, then $x \in V$ and

(4.34) $$|\text{grad }\Phi(x) - \text{grad }H(x)| < \varepsilon\|x\|,$$

(4.35) $$|\Phi(x) - H(x)| < \varepsilon\|x\|^2.$$

From (4.34) it follows that if $\|x\|$ and $\|x + \Delta x\|$ are both $< \delta$, then

(4.36) $$|\Delta\Phi - \Delta H| < \varepsilon\|x\| \, \|\Delta x\|,$$

in which $\Delta\Phi = \Phi(x + \Delta x) - \Phi(x)$, and similarly ΔH. Finally, on V

(4.37) $$\Phi(x) \geqslant c_2\|x\|^2$$

for some $c_2 > 0$.

Note that replacing Φ by H leads to Case (a1). In order to achieve (4.28), choose u and c_1 in (4.30) so that $-\Delta n^2 H(\bar{X}) > 3nl$ on the event $E_n(u)$, assuming that n is sufficiently large and (4.27) holds. It has been shown in Case (a1) how this can be done. With this choice of u and c_1 it remains to be shown that if n is large and (4.27) holds, then

(4.38) $$|\Delta n^2(\Phi(\bar{X}) - H(\bar{X}))| < nl \qquad \text{on the event} \quad E_n(u).$$

The assumptions on Φ imply that there exists $\varepsilon_1 > 0$ such that $\Phi(x) < \varepsilon_1$ implies $\|x\| < \delta/2$. Therefore, if n is large and (4.27) holds, then $\|\bar{X}\| < \delta/2$. This implies that on $E_n(u)$, where $\|\Delta S\| < 2c_1(\Delta n)^{\frac{1}{2}}$, $\|\bar{X} + \Delta\bar{X}\| < \delta$ for large n. It follows that (4.34)–(4.36) are valid, with x replaced by $\bar{X}$, for large n on the event $E_n(u)$, provided (4.27) holds. A computation, using (4.35) and (4.36), reveals that the left-hand side of (4.38) can be bounded by $c_3\varepsilon(\|S\| \, \|\Delta S\| + n^{-1}\|S\|^2\Delta n)$, in which c_3 is some positive constant. By (4.27) and (4.37), $\|S\|^2 \ll nh(n)$. Therefore, on $E_n(u)$, $\|S\| \, \|\Delta S\| + n^{-1}\|S\|^2\Delta n \ll (nh(n)\Delta n)^{\frac{1}{2}} + h(n)\Delta n \ll n$ by (4.25), and it follows that (4.38) will hold by taking ε small enough. □

Theorems 4.1 and 4.2 can be combined to show that if Φ satisfies the union of the conditions stated in both theorems, then for some $l_0 > 0$, $\psi(N)$ is exactly exponentially bounded whenever $l > l_0$, where ψ is defined by (4.7).

The condition in Theorem 4.2, Case (b), that Φ be bounded away from 0 outside V can be dispensed with if X is exponentially bounded, for then $\bar{X}_n$ will lie in the neighborhood V of ξ with a probability that converges to 1 exponentially fast.

THEOREM 4.3. *If in Theorem 4.2 it is also assumed that X is exponentially bounded, then the conclusion of that theorem holds if the assumption on Φ in Case* (b) *is replaced by: Φ has continuous second partial derivatives in a neighborhood of ξ with a matrix $A(x)$ that is positive definite at $x = \xi$; $\Phi(\xi) = \text{grad }\Phi(\xi) = 0$.*

PROOF. Since $A(\cdot)$ is continuous there is a neighborhood V of ξ such that A is continuous and positive definite on V. By Chernoff's theorem ([4], Theorem 1) the event $[\bar{X}_n \notin V]$ is exponentially bounded, i.e., $P(\bar{X}_n \notin V) \ll \rho_1^n$ for some $\rho_1 < 1$. Introduce the event $C_k = [\bar{X}_{n(k)} \in V]$, C_k^c its complement, where $n(k)$ is as defined in Lemma 3.1. Then

$$PC_k^c \ll \rho_1^{n(k)} \ll \rho_1^k. \tag{4.39}$$

Set $A_k = [N > n(k)]$. Now use Lemma 1 in [14], with n in that lemma replaced by k. The condition $PA_k C_k^c \ll \rho_1^k$ in that lemma is satisfied, as evidenced by (4.39). In the proof of Theorem 4.2 it was shown (see (4.26)) that for large k, $P(A_{k+1}|A_k C_k) < 1 - p_1$ for some $p_1 > 0$ so that the condition $P(A_{k+1}C_{k+1}|A_k C_k) < 1 - p_1$ of Lemma 1 in [14] is also satisfied. The conclusion $PA_k \ll \rho^k$, for some $\rho < 1$, follows. ▯

COROLLARY 4.1. *Let N be defined by* (4.1) *with L_n of the form* (4.2). *Let X be exponentially bounded and $EX = \xi$. Suppose that Φ is bounded on compacta, has continuous partial derivatives of the third order in a neighborhood of ξ, the matrix of second order partial derivatives is positive definite at ξ, and* $\Phi(\xi) = \text{grad}\,\Phi(\xi) = 0$. *Lastly, suppose that h satisfies Assumption* 4.1 *with $\eta = \frac{1}{3}$ and define ψ by* (4.7). *Then for every $l > 0$, $\psi(N)$ is exponentially bounded, and there exists l_0 such that for $l > l_0$, $\psi(N)$ is exactly exponentially bounded.*

PROOF. Follows from Theorems 4.1 and 4.3. ▯

5. Applications.

5.1. *Sequential* F-*test*. Independent observations $Z_1, Z_2, \cdots$ are made on a random vector $Z = (z_1, \cdots, z_k)'$. The (canonical) model specifies the z_i to be independently normal with common variance σ^2, while $Ez_i = \mu_i$ is known to be 0 for $i = s+1, \cdots, k$, where $1 \leqslant s < k$. Let $\gamma = \Sigma_1^q \mu_i^2/\sigma^2$, in which $1 \leqslant q \leqslant s$. The problem is to test sequentially $\gamma = \gamma_1$ against $\gamma = \gamma_2$, where it is assumed that $\gamma_1 < \gamma_2$. Let $Z_j = (z_{1j}, \cdots, z_{kj})'$, $j = 1, 2, \cdots$, and put $\bar{z}_{in} = (1/n)\Sigma_{j=1}^n z_{ij}$, $i = 1, \cdots k$. For notational convenience the summations $\Sigma_{i=1}^q, \Sigma_{i=q+1}^s, \Sigma_{i=s+1}^k$ will be abbreviated by $\Sigma_1, \Sigma_2, \Sigma_3$, respectively. Of these, the middle sum disappears if $q = s$. Define

$$Y_n = n\Sigma_1 \bar{z}_{in}^2 / \Sigma_{j=1}^n \left[\Sigma_1 z_{ij}^2 + \Sigma_2 (z_{ij} - \bar{z}_{in})^2 + \Sigma_3 z_{ij}^2\right]. \tag{5.1}$$

Lai ([8], Section 5) showed that within a uniformly bounded constant the log probability ratio L_n at the nth stage is

$$L_n = n(Y_n - \beta) - a \log n, \tag{5.2}$$

in which β and a are constants depending on γ_1 and γ_2; $0 < \beta < 1$; $a = 0$ if $\gamma_1 > 0$ and $a = c(q-1)$ for some $c > 0$ if $\gamma_1 = 0$. Let N be defined by (4.1) and let P be the true distribution of Z (not necessarily normal). Wijsman ([15], Section 3.3) showed that N is exponentially bounded under P unless

$$P\left\{\Sigma_1(z_i - \beta^{-1}\mu_i)^2 + \Sigma_2(z_i - \mu_i)^2 + \Sigma_3 z_i^2 = (\beta^{-2} - \beta^{-1})\Sigma_1 \mu_i^2\right\} = 1. \tag{5.3}$$

Furthermore, if a in (5.2) equals 0, then every P satisfying (5.3) was shown in [15] to be obstructive. It will be shown now, using Theorem 4.1, that the same conclusion can be drawn if $a \neq 0$. Note that this happens if and only if $\gamma_1 = 0$ and $q > 1$, and then $a > 0$.

In order to apply Theorem 4.1, introduce $z_0 = \Sigma_1^k z_i^2$ and take at first $X = (z_0, z_1, \cdots, z_s)'$. Define

$$\Phi(x) = \Sigma_1 x_i^2 / \left(x_0 - \Sigma_2 x_i^2\right) - \beta, \tag{5.4}$$

and define $h(t) = a \log t$. Then h satisfies Assumption 4.1 for any $m \geqslant 2$ and every $\eta > 0$, and it follows from (5.1), (5.2), and (5.4) that L_n has the required form (4.2). Equation (5.3) can be written

$$P\{z_0 - 2\beta^{-1}\Sigma_1 \mu_i z_i - 2\Sigma_2 \mu_i z_i + \beta^{-1}\Sigma_1 \mu_i^2 + \Sigma_2 \mu_i^2 = 0\} = 1 \tag{5.5}$$

so that with probability one X lies in the s-dimensional hyperplane

$$x_0 - 2\beta^{-1}\Sigma_1 \mu_i x_i - 2\Sigma_2 \mu_i x_i + \beta^{-1}\Sigma_1 \mu_i^2 + \Sigma_2 \mu_i^2 = 0. \tag{5.6}$$

In this hyperplane the denominator in (5.4) equals

$$x_0 - \Sigma_2 x_i^2 = \beta^{-1}\Sigma_1 x_i^2 - \beta^{-1}\Sigma_1 (x_i - \mu_i)^2 - \Sigma_2 (x_i - \mu_i)^2. \tag{5.7}$$

Now redefine $X = (z_1, \cdots, z_s)'$, $x = (x_1, \cdots, x_s)'$, and (using (5.4) and (5.7)):

$$\Phi(x) = \beta \Sigma_1 x_i^2 / \left[\Sigma_1 x_i^2 - \Sigma_1 (x_i - \mu_i)^2 - \beta \Sigma_2 (x_i - \mu_i)^2\right]. \tag{5.8}$$

Putting $\xi = EX = (\mu_1, \cdots, \mu_s)'$, it is immediate that $\Phi(\xi) = 0$. Expanding Φ about ξ, one finds easily

$$\Phi(x) = \beta\left(\Sigma_1 \mu_i^2\right)^{-1}\left[\Sigma_1 (x_i - \mu_i)^2 + \beta \Sigma_2 (x_i - \mu_i)^2\right] + 0(\|x - \xi\|^3). \tag{5.9}$$

From (5.8) it is obvious that Φ possesses derivatives of all orders and that Φ is bounded on compacta. From (5.9) it follows that Φ has the form (4.8) with $p = 1$, $\varepsilon = 1$, so that Theorem 4.1 applies with $\psi(x) \sim \frac{1}{2}(\log x)^2$. The conclusion (4.9) then implies that for l sufficiently large N is not exponentially bounded. Theorem 4.2 also applies since by (5.9) $\Phi(\xi) = \text{grad } \Phi(\xi) = 0$ and the matrix of second order partial derivatives is positive definite at ξ, while from (5.8) it is seen that outside any neighborhood of ξ, Φ is bounded away from 0. Hence for sufficiently large l, $(\log N)^2$ is exactly exponentially bounded. The same conclusion also follows from Corollary 4.1 since by (5.5) the support of Z, and therefore of X, is bounded.

5.2. *Sequential test for the equality of two distribution functions.* Let U, V be real-valued random variables with distribution functions F and G, respectively. Independent observations $(U_1, V_1), (U_2, V_2), \cdots$ on (U, V) are taken. Let P be the true joint distribution of U and V. Savage and Sethuraman [11] and Sethuraman [12] investigated the stopping time N of a sequential rank-order test for testing $F = G$ against the Lehmann alternative $G = F^A$, $A \neq 1$. Strongest results were obtained in [12] where it was shown that N is exponentially bounded under every P, except for P belonging to a certain family. These exceptional P's were further investigated in [16] and it was shown that all exceptional P's under which F and G

are continuous are obstructive. One discrete P was also investigated but L_n turned out to be of the form (4.2) and it is only with the results of Section 4 that obstructiveness of P can be concluded. Specifically, the expression for L_n given by (6.1) in [16] and rewritten in the notation of Section 4 is

$$L_n = n\Phi(\overline{X}_n) + \tfrac{1}{2}\log n, \tag{5.10}$$

in which $X_1, X_2, \cdots$ are i.i.d. Bernoulli variables with expectation $\xi(= p$ in [16]). The function Φ depends on A (which is assumed to be > 1) and is given by (5.11)

$$\Phi(x) = -x\log(\tfrac{1}{2}x) - x\log(x(1+\tfrac{1}{2}A)) - (1-x)\log(\tfrac{1}{2} + (A+\tfrac{1}{2})x) - (1-x)\log(1+\tfrac{1}{2}A(1+x)) + \log 4A - 2.$$

There is exactly one value of A, say A_0, and one value of ξ, say ξ_0, such that $\Phi(\xi_0) = \Phi'(\xi_0) = 0$ and $\Phi''(\xi_0) < 0$ (these values, accurate to 10 places, are $A_0 = 1.320015126$, $\xi_0 = .1401865276$). Then $-L_n$ is of the form (4.2) with $h(n) = \frac{1}{2}\log n$. Since a Bernoulli variable is bounded, Corollary 4.1 applies to show that for all sufficiently large l, $(\log N)^2$ is exactly exponentially bounded. Therefore, N itself is not exponentially bounded so that for $A = A_0$ the P under which $\xi = \xi_0$ is obstructive.

REFERENCES

[1] Billingsley, Patrick (1968). *Convergence of Probability Measures*. Wiley, New York.

[2] Breiman, Leo (1967). First exit times from a square root boundary. *Proc. Fifth Berkeley Symp. Math. Statist. Prob.* **II** 2 9–16.

[3] Brown, Bruce M. (1969). Moments of a stopping rule related to the central limit theorem. *Ann. Math. Statist.* **40** 1236–1249.

[4] Chernoff, H. (1952). A measure of asymptotic efficiency for tests of a hypothesis based on the sum of observations. *Ann. Math. Statist.* **23** 493–507.

[5] Chow, Y. S., Robbins, Herbert and Teicher, Henry (1965). Moments of randomly stopped sums. *Ann. Math. Statist.* **36** 789–799.

[6] Gundy, Richard F. and Siegmund, David (1967). On a stopping rule and the central limit theorem. *Ann. Math. Statist.* **38** 1915–1917.

[7] Kesten, Harry (1978). Branching Brownian motion with absorption. *Stochastic Processes and their Applications* **7** 9–47.

[8] Lai, Tze Leung (1975). Termination, moments and exponential boundedness of the stopping rule for certain invariant sequential probability ratio tests. *Ann. Statist.* **3** 581–598.

[9] Lai, Tze Leung (1977). First exit times from moving boundaries for sums of independent random variables. *Ann. Probability* **5** 210–221.

[10] Portnoy, Stephen (1978). Probability bounds for first exits through moving boundaries. *Ann. Probability* **6** 106–117.

[11] Savage, I. Richard and Sethuraman, J. (1966). Stopping time of a rank-order sequential probability ratio test based on Lehmann alternatives. *Ann. Math. Statist.* **37** 1154–1160.

[12] Sethuraman, J. (1970). Stopping time of a rank-order sequential probability ratio test based on Lehmann alternatives II. *Ann. Math. Statist.* **41** 1322–1333.

[13] Wijsman, R. A. (1972). A theorem on obstructive distributions. *Ann. Math. Statist.* **43** 1709–1715.

[14] WIJSMAN, R. A. (1975). Exponentially bounded stopping time of the sequential t-test. *Ann. Statist.* **3** 1006–1010.

[15] WIJSMAN, R. A. (1977). A general theorem with applications on exponentially bounded stopping time, without moment conditions. *Ann. Statist.* **5** 292–315.

[16] WIJSMAN, R. A. (1977). Obstructive distributions in a sequential rank-order test based on Lehmann alternatives. *Statistical Decision Theory and Related Topics II*, 451–469 (Shanti S. Gupta and David S. Moore, eds.). Academic Press, New York.

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