The Annals of Probability
1979, Vol. 7, No. 5, 909–911

CORRECTIONS TO "CENTRAL LIMIT THEOREMS FOR EMPIRICAL MEASURES"

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In [1], page 926, line 15 is *not* $\leqslant 8(2n)^{2v}\sup(H)$. Let us correct this and some other errors and obscurities.

In the proof of (2.7), "(2.2)" should be "(2.6)."

In the last three lines of page 905, replace Pr by Wichura's probability measure, say Pr_W. Take a sequence $\{h_m\}$ dense in the set of all uniformly continuous $h \notin B_{\delta,\varepsilon/2}$ on $(\mathcal{C}, d_p)$. Then

$$M := \bigcup_m B(h_m, \varepsilon/4) \in \mathfrak{B}_b \quad \text{in} \quad D_0(\mathcal{C}, P),$$

$\Pr(v_n \notin M) < \varepsilon$ for n large enough (using $P-$EM), and $M \cap B_{\delta,\varepsilon} = \emptyset$, proving (b).

In (5.4), $\varepsilon/64$ should be $\varepsilon/96$. Two lines after (5.9), page 915, $4b_{i+1}$ should be $6b_{i+1}$; line 4 from below, 2^{i+1} should be 2^{i+2}; last line, $2^i(2$ should be $2^i(4$; page 916, first line, 2 should be 4; on pages 916–917, $n \geqslant n_0$ should be $n > n_0$; on page 917, line 2, ε should be 3ε (twice). On page 921, 4th line after (7.3), $v \in \mathbb{R}^k$. Five lines further, delete "of H_N." On page 922, (7.9), N should be n; line 9, replace "$n > k \geqslant 1$" by "$n \geqslant k+2 \geqslant 2$"; in the line after (7.10), replace "$n < v$" by "$2 \leqslant n \leqslant v$."

On page 923, to clarify the choice of δ, in the proof of Theorem 7.1 after the third display, replace "Given a δ, $0 < \delta \leqslant 1$, to be chosen later," by: "For $0 < \delta \leqslant 1$, $\mathcal{C}(\delta) \subset \mathcal{C}(1)$. Thus

$$N(\delta/2, \mathcal{C}(\delta), P) \leqslant N(\delta/2, \mathcal{C}(1), P) \leqslant N\delta^{-w}$$

for $0 < \delta < 1$. Then choose a $\delta := \delta_1(\varepsilon) > 0$ such that for $0 < \gamma \leqslant \delta_1(\varepsilon)$,

$$2N\Sigma_{j=1}^{\infty} 2^{jw}\gamma^{-w}\exp(-\varepsilon^2 2^j/(8j^4\gamma)) < \varepsilon/3,$$

which is possible since: for each $\gamma > 0$, the series converges; the jth term converges to 0 as $\gamma\downarrow 0$, monotonically for $\gamma < \varepsilon^2 2^j/(8j^4 w) > 1$ for j large enough, so we have dominated convergence."

Then on page 924, after the first display, replace the first three sentences "Now . . . $\delta = \delta_1(\varepsilon)$." by "Then $S_1 < \varepsilon/3$ by choice of δ." On page 925, line 7 up, $(P_n + P_n')/2$.

In lines 5 and 4 up, replace "For fixed . . . $P_{2n}\}$" by: "Let $\mathbb{S}_{2n}$ be the σ-algebra of events $\{\langle X_1, \ldots, X_{2n}\rangle \in V\}$ where $V \in \mathcal{Q}^{2n}$ and V is preserved by all $(2n)!$

Received April 4, 1979. Revised in proof, July 10, 1979.

permutations of coordinates. Then if a measurable random variable is a function of P_{2n}, it is measurable for $\mathbb{S}_{2n}$. For any measurable set A let $r_n(A) := 2nP_{2n}(A)$. Then $\Pr\{nP_n(A) \geqslant s,\ nP_n'(A) \leqslant t | \mathbb{S}_{2n}\}$" $\cdots$.

On page 926, replace line 4, "On . . . so" by "If $s \geqslant \varepsilon n^{1/2}$ and $t = 1$, then". Replace lines 12–21, "By (7.11), $\cdots \to 0$" by: "We have $m^{\mathfrak{B}(n,\delta)}(2n) \leqslant m^{\mathcal{C}(1)}(2n)^2 \leqslant (2n)^{2v}$ by (7.11), and $Q_{n\varepsilon} \leqslant \Sigma_{i=1}^4 P_{(i)}$, where if J is the event on the left in (7.16),

$$P_{(1)} := \Pr(\exists i(\omega)) \leqslant 2\Pr(J) = 2E\Pr(J|\mathbb{S}_{2n})$$

$$\leqslant 2E\Pr\{\exists W \in \mathfrak{B}_{n\delta}: v_n(W) \geqslant \varepsilon \text{ and } v_n'(W) \leqslant n^{-1/2} | \mathbb{S}_{2n}\}.$$

For n large, $\sup\{nP(W): W \in \mathfrak{B}_{n\delta}\} < 1$, so

$$P_{(1)} \leqslant 2E\Pr\{\exists W \in \mathfrak{B}_{n\delta}: n^{1/2}P_n(W) \geqslant \varepsilon \text{ and } nP_n'(W) \leqslant 1 | \mathbb{S}_{2n}\}.$$

Let the 4^n subsets of $\{1, 2, \cdots, 2^n\}$ be $Y(1), \cdots, Y(4^n)$. Now $(X^{2n}, \mathcal{Q}^{2n})$, being Suslin, is isomorphic as a measurable space to a subset of a countable product of lines, on which there is a lexicographical linear ordering $\preccurlyeq$ with measurable graph in X^{4n}. Then we can arrange $X_1, \cdots, X_{2n}$ in order: $X_{(1)} \preccurlyeq X_{(2)} \preccurlyeq \cdots \preccurlyeq X_{(2n)}$. The $X_{(i)}$ are all $\mathbb{S}_{2n}$-measurable random variables. Thus for each $j = 1, \cdots, 4^n$, and measurable $W \in \mathcal{Q}$, let $I(j, W) := \{\omega: X_{(i)} \in W \text{ if and only if } i \in Y(j)\}$. Let $X^{(2n)}(\omega) := \langle X_{(1)}, \cdots, X_{(2n)}\rangle$. Then $X^{(2n)}$ is measurable from $\langle X^\infty, \mathbb{S}_{2n}\rangle$ into $\langle X^{2n}, \mathcal{Q}^{2n}\rangle$. For each i and j, the set $I(i, j) :=$

$$\{\langle A, B, X^{(2n)}(\omega)\rangle : \langle A, B\rangle \in \mathfrak{D}_{n\delta i},\ \omega \in I(j, (A \setminus B) \setminus D_{ni})\}$$

is $\mathbb{S} \times \mathbb{S} \times \mathcal{Q}^{2n}$ measurable in $\mathcal{C} \times \mathcal{C} \times X^{2n}$. Let

$$I(j) := \{\omega: \exists i, A, B, \langle A, B, X^{(2n)}(\omega)\rangle \in I(i, j)\}.$$

Then using (3.3), $I(j)$ is measurable for the Pr-completion of $\mathbb{S}_{2n}$. Let $A(j) := \{X_{(i)}: i \in Y(j)\}$. For any measurable set $A \in \mathcal{Q}$ let $H(A, n)$ denote the event $(n^{1/2}P_n(A) \geqslant \varepsilon \text{ and } nP_n'(A) \leqslant 1\}$. Then

$$\begin{aligned} P_{(1)} &\leqslant 2E\big(E\big(\Sigma_{1 \leqslant j \leqslant 4^n} 1_{I(j)} 1_{H(A(j), n)} | \mathbb{S}_{2n}\big)\big) \\ &= 2E\big(\Sigma_{1 \leqslant j \leqslant 4^n} 1_{I(j)} E\big(1_{H(A(j), n)} | \mathbb{S}_{2n}\big)\big) \\ &\leqslant 2E\big(\Sigma_{1 \leqslant j \leqslant 4^n} 1_{I(j)} \sup_{A \in \mathcal{Q}} \Pr(H(A, n) | \mathbb{S}_{2n})\big) \\ &\leqslant 2(2n)^{2v} E \sup_A \Pr(H(A, n) | \mathbb{S}_{2n}). \end{aligned}$$

The last conditional probability is of the hypergeometric form H treated above, with $s \geqslant \varepsilon n^{1/2}$ and $t = 1$. For $P_{(2)}$, $P_{(3)}$ and $P_{(4)}$, we replace $i(\cdot)$ by d, e and f respectively, and make the other appropriate changes. For n large, $P_{(3)} = P_{(4)} = 0$, and

$$Q_{n\varepsilon} \leqslant 4(2n)^{2v} \exp\big(-(\varepsilon n^{1/2} - 1)^2 / (8\varepsilon n^{1/2})\big) \to 0\text{"} \ldots .$$

Many thanks to Peter Gaenssler for the corrections to Theorem 5.1, to J. Berruyer and R. Carmona for noting a gap in the proof of Theorem 1.2, and to Roy Erickson and Joel Zinn for useful discussions of the proof of Theorem 7.1.

REFERENCES

[1] DUDLEY, R. M. (1978). Central limit theorems for empirical measures. *Ann. Probability* **6** 899–929.

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