# MULTIPLE COVERAGE OF THE LINE

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The probability of covering [0, 1) at least $m$ times by $N$ randomly placed subintervals of length $p$ is derived for all $m, N, p$.

**1. Introduction.** Whitworth [23] derives the probability that $N$ randomly placed subarcs each of length $p$ will cover the unit circle. Votaw [19] derives the probability that $N$ random subintervals will cover the unit interval. More recently, Flatto [6], Cooke [3], Kaplan [11] and others derive bounds and limit theorems on the probability of covering the unit circle at least $m$ times.

Feller ([5] pages 75, 187), Gilbert [8], Eckler [4], Wendell [2] and others detail applications of covering the line, circle, and sphere at least once, to traffic counters, periodgram analysis, destruction of targets, and covering of viruses by antibodies. Applications of multiple coverage of the line include the complete destruction of a hardened linear target; and the full utilization of at least $m$ servers in an infinite server system subject to random demands of constant duration.

The present paper derives the probability $\zeta(m, N; p)$ of covering the unit interval at least $m$ times by $N$ randomly placed subintervals each of length $p$. The approach is to partition the unit interval in particular ways that enable application of the scanning-conditioning methodology used to derive cluster probabilities in [14], [9], [21] and [10]. The adaptation of this methodology is suggested by the following direct relationship between the cluster and coverage probabilities on the circle.

Given $N$ points distributed at random over the unit circle, let $P_c(n, N; q)$ denote the cluster probability that there exists a subarc of the unit circle of length $q$ that contains at least $n$ points. (The $n$ points are said to "cluster" within the arc of length $q$.) Let $C(m, N; p)$ denote the coverage probability that the unit circle is covered at least $m$ times by $N$ randomly placed arcs each of length $p$. The cluster and circular coverage probabilities are related by the identity

$$(1) \qquad C(m, N; p) = 1 - P_c(N - m + 1, N; 1 - p).$$

To see this, view the $N$ random points as the centers of $N$ random subarcs of length $p = 1 - q$. Let $N_p(t)$ denote the number of random points that fall in $[t, t + p)$. If $N_p(t) \geqslant m$, then the coordinate $(t + 0.5p)$ will be covered at least $m$ times. If for all $t$ on the circle, $N_p(t) \geqslant m$, then the circle is covered at least $m$ times. Alternatively, if for some $t$, $N_p(t) < m$, then in the arc of length $1 - p$ that is the complement of $[t, t + p)$, there are at least $N - m + 1$ points. This yields identity (1).

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The interval $[t, t + p)$ as $t$ goes from zero to one is called the "scanning" interval. Ajne [1] derives $P_c(n, N; 0.5)$. Rothman [16] derives $P_c(n, N; 1/L)$ for integer $L \geqslant 3$, $n \geqslant [N/2] + 1$. Wallenstein [20] derives $P_c(n, N; 1/L)$ for integer $L \geqslant 3$, $n > 2N/L$, and gives tables for $N = 3(1)60$, $n = [N/3] + 1(1)[N/2]$, $L = 5(1)10$, where $[x]$ denotes the largest integer in $x$. These results imply $m$-coverage of the circle probabilities for large subarc sizes ($p = 1 - 1/L$). This complements the asymptotic results of Flatto [6] and others.

For any given $m$, $N$, $p$, $\zeta(m, N; p) \leqslant C(m, N; p)$. To see this, split the circle at an arbitrary origin and observe that covering the resulting line implies, but is not necessarily implied by, the covering of the circle. For small covering arcs ($p$ sufficiently small), $m$ and $N$ moderate, $C(m, N; p)$ will be small, and thus be a useful upper bound for $\zeta(m, N; p)$. For $p$ small, $N$ large, Flatto's [6] asymptotic result for $C(m, N; p)$ can serve as an approximation for $\zeta(m, N; p)$; see Kaplan's [11] comment following his equation (8). For large covering arcs, the tabled results for $P_c(n, N; 1/L)$ can be used to provide an upper bound for the probability of $m$ coverage of the line. However, for $p$ large, $\zeta(m, N; p)$ and $C(m, N; p)$ can be quite different. For example, $\zeta(8, 20; 2/3) = 0.04$, while $C(8, 20; 2/3) = 0.92$. The next section derives $\zeta(m, N; p)$ for all $m$, $N$, and $p$.

**2. $m$-Covering the line.** Let $X_1, X_2, \cdots, X_N$ be independently and identically distributed according to the uniform distribution on $[0, 1)$. To each $X_i$ associate an interval $[a_i, b_i)$, where $a_i = \max(0, X_i - \frac{1}{2}p)$, $b_i = \min(1, X_i + \frac{1}{2}p)$ for $p$ a constant, $0 < p < 1$.

For each $y$ in $[0, 1)$ define $C_i(y) = 1$ if $a_i \leqslant y < b_i$, zero otherwise, for $i = 1, 2, \cdots, N$. Let $M_N(y) = \sum_{i=1}^{N} C_i(y)$. $M_N(y)$ denotes the number of times that the coordinate $y$ is covered by the $N$ subintervals of length $p$. Let $D_{N,p}$ denote $\inf_{0 \leqslant y \leqslant 1} M_N(y)$. We denote the $m$-coverage linear probability $\Pr(D_{N,p} \geqslant m)$ by $\zeta(m, N; p)$.

Recall that $N_p(t)$ denotes the number of $X$'s that fall in $[t, t + p)$. Letting $t$ go from 0 to $1 - p$ "scans" the unit interval with a subinterval of length $p$. Huntington [10] shows that the same argument can be used to find the probability that the scanning interval sometimes contains at least $r$ points and the probability that it always contains at least $m$ points. On the circle, the event that the scanning interval always contains at least $m$ points is equivalent to $m$-coverage. However, on the line, the event that the scanning interval always contains $m$ points only guarantees $m$-coverage over $[\frac{1}{2}p, 1 - \frac{1}{2}p)$. For $m$-coverage over $[0, 1)$ we require in addition at least $m$ points in each of $[0, \frac{1}{2}p)$ and $[1 - \frac{1}{2}p, 1)$.

LEMMA 1. $\inf_{0 \leqslant y \leqslant 1} M_N(y) \geqslant m$ iff $N_{p/2}(0) \geqslant m$, $N_{p/2}(1 - \frac{1}{2}p) \geqslant m$, and $\inf_{0 < t \leqslant 1-p} N_p(t) \geqslant m$.

PROOF. For $0 < t < 1 - p$, $N_p(t) \geqslant m$ iff $M_N(t + \frac{1}{2}p) \geqslant m$. Lemma 1 follows.

To derive the $m$-coverage probability $\zeta(m, N; p)$, it is convenient to treat three separate cases. In what follows, $L$ denotes the largest integer in $1/p$.

CASE 1. $p = 2/(2L + 1)$.

Since $L = [1/p]$, then $p/2 = 1 - Lp$. The interval $[0, 1)$ is subdivided into $2L + 1$ subintervals, $[(i - 1)p/2, ip/2)$, $i = 1, 2, \cdots, 2L + 1$. Let $n_i$ denote the number of points in the $i$th subinterval.

THEOREM 1a. *For $p = 2/(2L + 1)$, $L = [1/p]$, $0 < p < 1$, $L \geqslant 2$, $m \geqslant 2$, $N \geqslant (L + 1)m$,*

(2)
$$\zeta(m, N; p) = N! \, (p/2)^N \Sigma_{Q1} \det |1/c_{ij}!| \det |1/d_{ij}!|,$$

*where*

(3)
$$c_{ij} = \Sigma_{k=2L-2j+2}^{2L-2i+2} n_k - (j - i)(m - 1), \quad for \quad 1 \leqslant i \leqslant j \leqslant L,$$
$$= -\Sigma_{k=2L-2i+3}^{2L-2j+1} n_k + (i - j)(m - 1), \quad for \quad 1 \leqslant j < i \leqslant L;$$

*and*

$$d_{ij} = \Sigma_{k=2L-2j+3}^{2L-2i+3} n_k - (j - i)(m - 1), \quad for \quad 1 \leqslant i \leqslant j \leqslant L + 1,$$
$$= -\Sigma_{k=2L-2i+4}^{2L-2j+2} n_k + (i - j)(m - 1), \quad for \quad 1 \leqslant j < i \leqslant L + 1.$$

*$Q1$ is the set of all partitions of $N$ into $2L + 1$ integers $n_i$ satisfying*

(4)     $n_1 \geqslant m$; $n_i + n_{i+1} \geqslant m$, *for* $i = 2, 3, \cdots, 2L - 1$; $n_{2L+1} \geqslant m$.

*In determinants $1/x! = 0$ if $x < 0$, or $x > N$.*

PROOF. Abbreviate $N_s((i - 1)p/2)$ to $Y_i(s)$, where $i = 1, 2, \cdots, 2L + 1$, and $0 \leqslant s \leqslant p/2$. Let $B_i$ denote the event

(5)
$$\inf_{0 \leqslant s < p/2} \{ n_i + n_{i+1} - Y_i(s) + Y_{i+2}(s) \} \geqslant m.$$

Condition on the $n_i$'s and observe that conditionally $\{ B_i | i = 1, 3, \cdots, 2L - 1 \}$ is independent of $\{ B_i | i = 2, 4, \cdots, 2L - 2 \}$. Find

(6)
$$\Pr(D_{N,p} \geqslant m | \{n_i\}) = \Pr( \cap_{i=1}^{L-1} B_{2i} | \{n_i\}) \Pr( \cap_{i=1}^{L} B_{2i-1} | \{n_i\}).$$

Apply Barton and Mallow's [2] corollary to a theorem of Karlin and McGregor [12], letting $Y_{2(L-i+1)}(t)$, $n_{2(L-i+1)}$, and $\Sigma_{k=2}^{2L-2i+1} n_k - (L - i)(m - 1)$ correspond to their $A_i(h)$, $a_i$, and $\alpha_i$ respectively; to find

(7)
$$\Pr( \cap_{i=1}^{L-1} B_{2i} | \{n_i\}) = \det |1/c_{ij}!| \Pi_{i=1}^{L} n_{2i}!.$$

Similarly,

(8)
$$\Pr( \cap_{i=1}^{L} B_{2i-1} | \{n_i\}) = \det |1/d_{ij}!| \Pi_{i=1}^{L+1} n_{2i-1}!.$$

Substitute the right-hand sides of equations (7) and (8) into the right-hand side of equation (6), and average over the multinomial distribution of $\{n_i\}$ to find equation (2).

For $p = 2/(2L + 1)$, $L = 1$, there are no $B_{2i}$. We have the simpler result:

THEOREM 1b. *For $m \geqslant 2$, $N > 2m$,*

$$(9) \quad \zeta(m, N; \tfrac{2}{3}) = 1 - F_b(2m - 1, N; \tfrac{2}{3}) - 2\Sigma_{r=2m}^{N} b(r, N; \tfrac{2}{3}) F_b(m - 1, r; \tfrac{1}{2})$$

$$- b(m - 1, N; \tfrac{2}{3})\{(N - 2m + 1)F_b(N - 2m, m - 1; \tfrac{1}{2})$$

$$- \tfrac{1}{2}(m - 1)F_b(N - 2m - 1, m - 2; \tfrac{1}{2})\}$$

*where*

$$b(n, N; p) = \binom{N}{n}p^n(1 - p)^{N-n}, \quad F_b(n, N; p) = \Sigma_{i=0}^{n} b(i, N; p), \quad for \quad n \leqslant N$$

$$= 1, \quad otherwise.$$

PROOF. Divide the unit interval into three parts: $[0, \tfrac{1}{3})$, $[\tfrac{1}{3}, \tfrac{2}{3})$, $[\tfrac{2}{3}, 1)$. Let $n_1, n_2, n_3$ denote the number of points in each part. Following the approach of the proof of Theorem 1a,

$$(10) \quad \zeta(m, N; 2/3) = \Sigma_{n_1=m}^{N-m} \Sigma_{n_3=m}^{N-n_1} \Pr(B_1|n_1, n_2, n_3)N!3^{-N}/n_1!n_2!n_3!$$

where

$$(11) \quad \Pr(B_1) = \det \begin{vmatrix} 1/n_3! & 1/(N - m + 1)! \\ 1/(m - 1 - n_2)! & 1/n_1! \end{vmatrix} n_1!n_3!$$

$$= 1 - n_1!n_3!/(m - 1 - n_2)!(N - m + 1)!.$$

Substitute the right-hand side of equation (11) into equation (10), simplify and make the change of summation to

$$\Sigma_{n_2=0}^{N-2m} \Sigma_{n_3=m}^{N-n_2-m}$$

where $n_2 = N - n_1 - n_3$ to find equation (9).

CASE 2. $p < 2/(2L + 1)$, where $L = [1/p]$.

Let $b = 1 - Lp$, where $L$ is the largest integer in $1/p$. For Case 2, $b > p/2$. Subdivide $[0, 1)$ into $4L + 3$ parts:

$$[(i - 1)p, b + (2i - 3)p/2); \quad [b + (2i - 3)p/2, (2i - 1)p/2);$$

$$[(2i - 1)p/2, b + (i - 1)p);$$

$$[b + (i - 1)p, ip); \text{ for } i = 1, 2, \cdots, L; \quad [Lp, b + (2L - 1)p/2);$$

$$[b + (2L - 1)p/2, (2L + 1)p/2); \quad [(2L + 1)p/2, 1).$$

Let $n_j$ denote the number of points in the $j$th interval. The intervals alternate in length: $b - \tfrac{1}{2}p$ for the odd numbered intervals (counting from left), $p - b$ for the even numbered intervals.

THEOREM 2. *For $p < 2/(2L + 1)$, $L = [1/p] \geqslant 1$, $m \geqslant 2$, $N \geqslant (L + 1)m$,*

$$(12) \quad \zeta(m, N; p) = \Sigma_{Q2}N!(b - p/2)^M(p - b)^{N-M}\prod_{k=1}^{4} \det |1/e_{ij}^{(k)}!|$$

*where*

$$M = \Sigma_{j=0}^{2L+1} n_{2j+1},$$

*and for $k = 1, 2, 3, 4$*

$$e_{ij}^{(k)} = \Sigma_{t=4(L+1-j)+4-k}^{4(L+1-i)+4-k} n_t - (j - i)(m - 1), \quad \text{for} \quad 1 \leqslant i \leqslant j \leqslant L_k$$

$$= -\Sigma_{t=4(L+1-i)+5-k}^{4(L+1-j)+3-k} n_t + (i - j)(m - 1), \quad \text{for} \quad 1 \leqslant j < i \leqslant L_k,$$

*where $L_1 = L_2 = L_3 = L + 1$, $L_4 = L$. $Q2$ is the set of all partitions of $N$ into $4L + 3$ integers $n_i$ satisfying:*

(13)     $n_1 + n_2 \geqslant m;\ n_i + n_{i+1} + n_{i+2} + n_{i+3} \geqslant m,$

$$\text{for } i = 2, 3, \cdots, 4L - 1;\ n_{4L+2} + n_{4L+3} \geqslant m.$$

PROOF. Let $I(j)$ denote the left endpoint of the $j$th interval. Abbreviate $N_{I(j)}(s)$ to $Y_j(s)$ where $0 \leqslant s < p - b$ for $j$ even, $0 \leqslant s < b - \frac{1}{2}p$ for $j$ odd. Define the events $B_j$ for $j = 1, 2, \cdots, 4L - 1$, as follows:

(14)     $B_j \equiv \{\inf_s \{n_j + n_{j+1} + n_{j+2} + n_{j+3} - Y_j(s) + Y_{j+4}(s)\} \geqslant m\}.$

Lemma 1 implies that

(15)     $\zeta(m, N; p) = \Sigma_{Q2} \Pr(\cap_{j=1}^{4L-1} B_j | \{n_i\}) \Pr(\{n_i\}),$

where

(16)     $\Pr(\{n_i\}) = N! (b - \frac{1}{2}p)^M (p - b)^{N-M} / \Pi_{i=1}^{4L+3} n_i!.$

The events $B_{4j-r}$ are conditionally (given the $\{n_i\}$) independent of the events $B_{4j-s}$, for $r \neq s;\ r, s = 0, 1, 2, 3$. This implies that

(17)     $\Pr(\cap_{j=1}^{4L-1} B_j | \{n_i\}) = \Pi_{k=1}^{4} \Pr(\cap_{j=1}^{L} B_{4j-k} | \{n_i\}),$

where for compact notation, define $B_0 \equiv B_4$.

The remainder of the proof evaluates $\Pr(\cap_{j=1}^{L} B_{4j-k} | \{n_i\})$ by applying Barton and Mallow's [2] corollary. For example, to evaluate $\Pr(\cap_{j=1}^{L} B_{4j-1} | \{n_i\})$, let $Y_{4(L+1-j)+3}(s)$, $n_{4(L+1-j)+3}$, and $\Sigma_{t=3}^{4(L+1-j)+2} n_t - (L + 1 - j)(m - 1)$ correspond respectively to their $A_j(h)$, $a_j$, and $\alpha_j$.

CASE 3. $p > 2/(2L + 1)$, where $L = [1/p]$.

We first consider separately the subcase $p = 1/L$. Subdivide the unit interval into $2L$ parts: $[(i - 1)/2L, i/2L)$, for $i = 1, 2, \cdots, 2L$. Let $n_i$ denote the number of points in the $i$th part.

THEOREM 3a.   *For $p = 1/L$, $L$ an integer, $L \geqslant 2$, $m \geqslant 2$, $N \geqslant (L + 1)m$,*

(18)     $\zeta(m, N; 1/L) = N! (2L)^{-N} \Sigma_{Q3} \det |1/c_{ij}!| \det |1/d_{ij}^*!|,$

*where $c_{ij}$ is given by equation (3) with the $n_i$'s being the occupancy numbers for the new partition. $d_{ij}^*$ is given by the formula for $d_{ij}$ of equation (3), with $L - 1$ replacing*

$L$:

(19) $\quad d_{ij}^{*} = \sum_{k=2L-2j+1}^{2L-2i+1} n_k - (j-i)(m-1), \quad for\ 1 \leqslant i \leqslant j \leqslant L;$

$\qquad = -\sum_{k=2L-2i+2}^{2L-2j} n_k + (i-j)(m-1), \quad for\ 1 \leqslant j < i \leqslant L.$

$Q3$ *is the set of all partitions of* $N$ *into* $2L$ *integers satisfying* $n_1 \geqslant m$; $n_i + n_{i+1} \geqslant m$, *for* $i = 1, 2, \cdots, 2L - 2$; $n_{2L} \geqslant m$.

PROOF. The proof is similar to that of Theorem 1a, except that here $Y_i(s)$ denotes $N_{(i-1)/2L}(s)$ for $i = 1, 2, \cdots, 2L$ and $0 \leqslant s < 1/2L$.

For $p \neq 1/L$, subdivide the unit interval into $4L + 1$ intervals. Let $b$ denote $1 - Lp$. Here $b < p/2$. The subdivision is as follows: $[(i - 1)p, b + (i - 1)p)$; $[b + (i - 1)p, (2i - 1)p/2)$; $[(2i - 1)p/2, b + (2i - 1)p/2)$; $[b + (2i - 1)p/2, ip)$; for $i = 1, 2, \cdots, L$; $[Lp, 1)$. Let $n_i$ denote the number of points in the $i$th interval.

THEOREM 3b. *For* $p > 2/(2L + 1)$, $L = [1/p], p \neq 1/L, L \geqslant 1, m \geqslant 2, N \geqslant (L + 1)m$,

(20) $\qquad \zeta(m, N; p) = \sum_{Q4} N! b^M \left(\tfrac{1}{2}p - b\right)^{N-M} \Pi_{k=1}^{4} \det |1/d_{ij}^{(k)}!|,$

*where* $M = \sum_{j=0}^{2L} n_{2j+1}$, *and* $d_{ij}^{(3)}$ *and* $d_{ij}^{(4)}$ *are respectively given by the same formula as* $e_{ij}^{(3)}$ *and* $e_{ij}^{(4)}$ *given by equation* (12). *For* $k = 1, 2$

(21) $\quad d_{ij}^{(k)} = \sum_{t=4(L-j)+4-k}^{4(L-i)+4-k} n_t - (j-i)(m-1), \quad for\ 1 \leqslant i \leqslant j \leqslant L$

$\qquad = -\sum_{t=4(L-i)+5-k}^{4(L-j)+3-k} n_t + (i-j)(m-1), \quad for\ 1 \leqslant j < i \leqslant L.$

$Q4$ *is the set of all partitions of* $N$ *into* $4L + 1$ *integers* $n_i$ *satisfying* $n_1 + n_2 \geqslant m$; $n_i + n_{i+1} + n_{i+2} + n_{i+3} \geqslant m$, *for* $i = 2, 3, \cdots, 4L - 3$; $n_{4L} + n_{4L+1} \geqslant m$.

PROOF. The proof follows closely that of Theorem 2 with the same correspondences between $A_j(h)$, $a_j$, $\alpha_j$, and $Y_i(s)$, $n_i$ for $k = 3, 4$ cases. For $k = 1, 2$ correspondence is similar except that we substitute $L$ for $L + 1$. For example, for $k = 2, j = 1, \cdots, L, A_j(h) = Y_{4(L-j)+2}(s)$, $a_j = n_{4(L-j)+2}$, and $\alpha_j = \sum_{t=2}^{4(L-j)+1} n_t - (L - j)(m - 1)$.

The above cases together give the probability distribution of the number of subintervals, each of length $p$, required to cover the unit line $m$ times. In general, the formula are in terms of sums of products of determinants, where the sums are over partitions of $N$ into $2L$, $2L + 1$, $4L + 1$ or $4L + 3$ parts (depending on the case), where $L$ is the largest integer in $1/p$. The cases where $p = 1/L$ or $2/(2L + 1)$ involve the fewest parts.

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