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DISCRETE ANALOGUES OF SELF-DECOMPOSABILITY AND STABILITY

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Analogues are proposed for the concepts of self-decomposability and stability for distributions on the nonnegative integers. It turns out that these "discrete self-decomposable" and "discrete stable" distributions have properties that are quite similar to those of their continuous counterparts.

1. Introduction and preliminaries. A probability distribution on $\mathbb{R}$ is said to be self-decomposable (or, of class L) if its characteristic function (ch.f.) satisfies (cf. [5], page 161)

$$\varphi(t) = \varphi(\alpha t)\varphi_\alpha(t) \qquad t \in \mathbb{R};\ \alpha \in (0, 1), \tag{1.1}$$

with φ_α a ch.f. For the corresponding random variables (rv's) this means that (in distribution)

$$X = \alpha X' + X_\alpha \qquad \alpha \in (0, 1), \tag{1.2}$$

where X' and X_α are independent and X' is distributed as X. Clearly, apart from $X \equiv 0$, no lattice rv can satisfy (1.2); in fact, all nondegenerate self-decomposable (self-dec) distributions are known to be absolutely continuous (see, e.g., [3]).

In this note we propose analogues of self-decomposability and stability for distributions on $\mathbb{N}_0 := \{0, 1, 2, \cdots\}$. It turns out that the discrete self-dec distributions and the discrete stable distributions share the basic properties with their continuous counterparts. The discrete self-dec distributions, for instance, are unimodal, and the discrete stable distributions are very similar to their continuous analogues on $(0, \infty)$.

We shall need the following two lemmas for probability generating functions (p.g.f.'s), and on infinite divisibility (inf div). For a proof of the second lemma we refer to [1] and [6]. The generating function of sequences $(a_n)_0^\infty$, $(b_n)_0^\infty$, etc. will be denoted by A, B, etc.

LEMMA 1.1. *If P is a* p.g.f., *then*

$$\lim_{x\uparrow 1}(1 - x)P'(x) = 0.$$

PROOF. For $x \in [0, 1)$ we have $1 - P(x) = (1 - x)P'(\xi)$ with $\xi \in (x, 1)$. As P' is nondecreasing, we have $(1 - x)P'(x) \leqslant (1 - x)P'(\xi) = 1 - P(x) \to 0$ as $x \uparrow 1$.

LEMMA 1.2. *A* p.g.f. *P with $0 < p_0 < 1$ is* inf div iff *P has the form*

$$P(z) = \exp\{\lambda(G(z) - 1)\}, \tag{1.3}$$

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where $\lambda > 0$ *and* G *is a* (*unique*) p.g.f. *with* $G(0) = 0$. *Equivalently*, P *is* inf div iff

$$P(z) = \exp\{-\textstyle\int_z^1 R(u)\,du\}, \tag{1.4}$$

where $R(u) = \Sigma_0^\infty r_n u^n$, *with* $r_n \geqslant 0$ *and, necessarily*, $\Sigma_0^\infty r_n(n+1)^{-1} < \infty$, i.e., iff *the* p_n *satisfy*

$$(n+1)p_{n+1} = \Sigma_{k=0}^n p_k r_{n-k} \qquad n \in \mathbb{N}_0, \tag{1.5}$$

with $r_n \geqslant 0$.

2. Discrete self-decomposability. We start with analogues of (1.1) and (1.2) that operate within the set of distributions on $\mathbb{N}_0$. For definiteness we shall assume $0 < p_0 < 1$.

DEFINITION 2.1. A distribution on $\mathbb{N}_0$ with p.g.f. P is called discrete self-decomposable if

$$P(z) = P(1 - \alpha + \alpha z)P_\alpha(z) \qquad |z| \leqslant 1; \qquad \alpha \in (0, 1), \tag{2.1}$$

with P_α a p.g.f.

Equation (2.1) can be written in terms of rv's as follows:

$$X = \alpha \circ X' + X_\alpha, \tag{2.2}$$

where $\alpha \circ X'$ and X_α are independent, and X' is distributed as X. Here $\alpha \circ X$ is defined (in distribution) by its p.g.f. $P(1 - \alpha + \alpha z)$, or by

$$\alpha \circ X = \Sigma_1^X N_j, \tag{2.3}$$

where $P(N_j = 1) = 1 - P(N_j = 0) = \alpha$, all rv's being independent. It then follows that $\alpha \circ X \in \mathbb{N}_0$, with $1 \circ X = X$, $0 \circ X = 0$, and $E\alpha \circ X = \alpha EX$, as in scalar multiplication; an empty sum is zero.

We first establish the canonical form of the discrete self-decomposable p.g.f.'s.

THEOREM 2.2. *A* p.g.f. P *is discrete self-dec* iff *it has the form*

$$P(z) = \exp\left\{-\lambda \int_z^1 \frac{1 - G(u)}{1 - u}\,du\right\}, \tag{2.4}$$

where $\lambda > 0$ *and* G *is a* (*unique*) p.g.f. *with* $G(0) = 0$. *Equivalently*, P *is discrete self-dec* iff *it is* inf div *and has a canonical measure* r_n (cf. *Lemma* 1.2) *that is nonincreasing*.

PROOF. Let P be self-dec, i.e., satisfy (2.1). Then for $r > 0$ and $r(1 - \alpha_n)^{-1} \in \mathbb{N}$,

$$Q_{r,n}(z) := \{P_{\alpha_n}(z)\}^{r/(1-\alpha_n)} \tag{2.5}$$

is a p.g.f. As $P(1 - \alpha + \alpha z) = P(z) + (1 - \alpha)(1 - z)P'(z) + o\,(1 - \alpha)$ as $\alpha \uparrow 1$, by (2.1) and (2.5), with α_n such that $\alpha_n \uparrow 1$ as $n \to \infty$,

$$Q_r(z) := \lim_{n\to\infty} Q_{r,n}(z) = \exp\{-r(1 - z)P'(z)/P(z)\}. \tag{2.6}$$

As (cf. Lemma 1.1) $Q_r(z) \to 1$ as $z \uparrow 1$, by the continuity theorem for p.g.f.'s (cf. [1], page 280), Q_r is a p.g.f. for every $r > 0$. It follows that $Q := Q_1$ is infinitely divisible, and therefore by (2.6), and (1.3) applied to Q, that

$$R(z) := \frac{P'(z)}{P(z)} = -\frac{\log Q(z)}{1-z} = \lambda\frac{1-G(z)}{1-z}, \tag{2.7}$$

equivalent to (2.4). Comparing (2.4) and (1.4) we see that P is inf div, with

$$r_n = \lambda(1 - \Sigma_{j=1}^{n} g_j) = \lambda\Sigma_{j=n+1}^{\infty} g_j, \tag{2.8}$$

which is nonincreasing. Conversely, let P satisfy (2.4); this is easily seen to be the case if P is inf div with nonincreasing r_n, i.e., satisfying (2.8). Then P satisfies (2.1) with

$$P_\alpha(z) = \exp\{-\textstyle\int_z^{1-\alpha(1-z)} R(u)\,du\},$$

i.e., with $R_\alpha(z) := P_\alpha'(z)/P_\alpha(z) = R(z) - \alpha R(1 - \alpha(1-z))$, with coefficients

$$r_n - \alpha\Sigma_{k=n}^{\infty}\binom{k}{n}\alpha^n(1-\alpha)^{k-n} r_k \geqslant r_n\left\{1 - \alpha^{n+1}\Sigma_{j=0}^{\infty}\binom{n+j}{j}(1-\alpha)^j\right\} = 0,$$

where we have used the fact that r_n is nonincreasing. It follows that P_α is a (infinitely divisible) p.g.f.

The unimodality of discrete self-decomposable distributions is a corollary to the following theorem.

THEOREM 2.3. *Let* $(p_n)_0^\infty$ *and* $(r_n)_0^\infty$ *be sequences of real numbers with* $p_n \geqslant 0$, $p_0 > 0$, *and* r_n *nonincreasing. Furthermore let* p_n *and* r_n *be related by*

$$(n+1)p_{n+1} = \Sigma_{k=0}^{n} p_k r_{n-k} \qquad n \in \mathbb{N}_0. \tag{2.9}$$

Then $(p_n)_0^\infty$ *is* unimodal, i.e., $p_n - p_{n-1}$ *changes sign at most once* ($p_{-1} = 0$); p_n *is nonincreasing* iff $r_0 \leqslant 1$.

PROOF. The proof is very similar to that in [7] for self-decomposable densities on $(0, \infty)$. Putting $d_n = p_n - p_{n-1}$ and $\lambda_n = r_n - r_{n+1}$, from (2.9) we obtain by subtraction

$$(n+1)d_{n+1} = (r_0 - 1)p_n - \Sigma_{j=0}^{n-1}\lambda_j p_{n-j-1} \qquad n \in \mathbb{N}_0. \tag{2.10}$$

Clearly, $d_n \leqslant 0$ for $n \in \mathbb{N}$ iff $r_0 \leqslant 1$. Now let $r_0 > 1$, and suppose that

(2.11)

$$d_1 > 0,\ \ d_2 \geqslant 0, \cdots, d_{n_1} \geqslant 0,\ \ d_{n_1+1} < 0, \cdots, d_{n_1+m} =: d_{n_2} \leqslant 0,\ \ d_{n_2+1} > 0.$$

Then we have, putting $p_{n-j} = 0$ if $j > n$,

$$\begin{aligned} p_{n_1-j} &\leqslant p_{n_2-j} \qquad j = m+1, m+2, \cdots \\ p_{n_1-j} &\leqslant p_{n_1} \qquad j = 1, 2, \cdots, m. \end{aligned} \tag{2.12}$$

From (2.10) and (2.11) we have

$$(n_1+1)d_{n_1+1} = (r_0-1)p_{n_1} - \Sigma_{j=0}^{n_1-1}\lambda_j p_{n_1-j-1} < 0, \tag{2.13}$$

$$(n_2+1)d_{n_2+1} = (r_0-1)p_{n_2} - \Sigma_{j=0}^{n_2-1}\lambda_j p_{n_2-j-1} > 0. \tag{2.14}$$

As $\Sigma_0^{m-1}\lambda_j p_{n_2} \leqslant \Sigma_0^{m-1}\lambda_j p_{n_2-j-1}$, from (2.14) it follows that ($r_0 = r_n + \Sigma_{j=0}^{n-1}\lambda_j$)

$$(r_m - 1)p_{n_2} > \Sigma_{j=m}^{n_1-1}\lambda_j p_{n_2-j-1}. \tag{2.15}$$

But, from (2.12) and (2.15) we obtain

$$\Sigma_{j=0}^{n_1-1}\lambda_j p_{n_1-j-1} \leqslant \Sigma_{j=0}^{m-1}\lambda_j p_{n_1} + \Sigma_{j=m}^{n_1-1}\lambda_j p_{n_2-j-1}$$
$$< p_{n_1}(r_0 - r_m) + p_{n_2}(r_m - 1) < p_{n_1}(r_0 - 1),$$

which contradicts (2.13). It follows that (2.11) is impossible.

COROLLARY 2.4. *A discrete self-dec distribution* $(p_n)_0^\infty$ *is unimodal; it is nonincreasing* iff $r_0 = p_1/p_0 \leqslant 1$. *Equivalently, an* inf div *distribution on* $\mathbb{N}_0$ (*with* $p_0 > 0$) *is unimodal if* r_n (cf. (1.5)) *is nonincreasing; it is nonincreasing* iff *in addition* $r_0 \leqslant 1$.

REMARK 1. In Theorem 2.3 the r_n are not supposed to be all nonnegative, i.e., we seem to find a sufficient condition for unimodality of more general sequences than inf div distributions. For nonnegative p_n, however, r_n nonincreasing implies $r_n \geqslant 0$ ($n \in \mathbb{N}_0$).

REMARK 2. Theorem 2.3 could be used to give a slightly simpler proof of the unimodality of continuous self-dec distributions on $(0, \infty)$, as any such distribution is the limit of discrete self-dec distributions. This procedure amounts to a more drastic discretization than the one used in [7].

3. Discrete stability. The set of distributions on $\mathbb{R}$ that are (strictly) stable with exponent γ is the subset of the set of self-decomposable distributions with rv's X satisfying (cf. [2], page 171)

$$(s + t)^{1/\gamma}X = s^{1/\gamma}X_1 + t^{1/\gamma}X_2 \qquad s, t > 0, \tag{3.1}$$

in distribution, where X_1 and X_2 are independent and distributed as X. We rewrite (3.1) as

$$X = \alpha X_1 + (1 - \alpha^\gamma)^{1/\gamma}X_2 \qquad 0 < \alpha < 1. \tag{3.2}$$

Now replacing αX_1 by $\alpha \circ X_1$ as defined in (2.3), and similarly for the other term, we obtain the discrete analogue of (3.2). In terms of p.g.f.'s we then have

(3.3)
$$P(z) = P(1 - \alpha(1 - z))P\big(1 - (1 - \alpha^\gamma)^{1/\gamma}(1 - z)\big) \qquad |z| \leqslant 1; \alpha \in (0, 1),$$

and we give the following definition.

DEFINITION 3.1. A p.g.f. P (with $0 < P(0) < 1$) is called (strictly) discrete stable with exponent $\gamma > 0$ if it satisfies (3.3).

From (3.3) it follows that

$$\frac{1 - P\big(1 - (1 - \alpha^\gamma)^{1/\gamma}(1 - z)\big)}{(1 - \alpha)(1 - z)} = \frac{P(1 - \alpha(1 - z)) - P(z)}{(1 - \alpha)(1 - z)P(1 - \alpha(1 - z))} \to \frac{P'(z)}{P(z)}$$

as $\alpha \uparrow 1$. Putting $(1 - \alpha^\gamma)^{1/\gamma} = u$, this means that

$$\frac{1 - P(1 - u(1 - z))}{(u(1 - z))^\gamma} \to \gamma^{-1}(1 - z)^{1-\gamma}\frac{P'(z)}{P(z)} \qquad u \downarrow 0, \tag{3.4}$$

and with $z = 0$,

$$\frac{1 - P(1 - u)}{u^\gamma} \to \frac{p_1}{\gamma p_0} \qquad u \downarrow 0. \tag{3.5}$$

Combining (3.4) and (3.5) we conclude that

$$\frac{P'(z)}{P(z)} = \frac{p_1}{p_0}(1 - z)^{\gamma - 1} \qquad z \in [0, 1). \tag{3.6}$$

As $P'(1) > 0$ (possibly infinite) unless $P(0) = 1$, from (3.6) we see that $0 < \gamma \leqslant 1$.
Integrating (3.6) we obtain

$$P(z) = P_\gamma^\lambda(z) := \exp\{-\lambda(1 - z)^\gamma\} \qquad |z| \leqslant 1; \lambda > 0, \tag{3.7}$$

by analytic continuation. As any P satisfying (3.7) satisfies (3.3), we have now proved

THEOREM 3.2. *Discrete stable* p.g.f.'*s* (i.e., *satisfying* (3.3)) *only exist for* $\gamma \in (0, 1]$, *and all stable* p.g.f.'*s with exponent* γ *are given by* (3.7).

REMARK. The discrete stable p.g.f.'s are quite similar to the Laplace transforms $\exp(-\lambda\tau^\gamma)$ of the stable distributions on $(0, \infty)$ (cf. [2], page 448). Rather curiously, the Poisson distribution replaces the degenerate one, i.e., we have

COROLLARY 3.3. *The Poisson distribution is discrete stable with exponent one.*
Further, as in the continuous case, we have by (3.3) and (2.1)

COROLLARY 3.4. *A discrete stable distribution is discrete self-decomposable, and hence unimodal.*

REMARK. If we define a p.g.f. P to be in the domain of (discrete) attraction of a stable p.g.f. P_γ if there exist α_n such that

$$\lim_{n\to\infty}\{P(1 - \alpha_n + \alpha_n z)\}^n = P_\gamma(z),$$

then it follows that all distributions with finite first moment are attracted by the Poisson distribution: take $\alpha_n = n^{-1}$. A general theory of attraction could easily be developed. However, as for $\gamma \in (0, 1)$ we have $P_\gamma(1 - \tau) = \exp(-\tau^\gamma)$, and for every finite $\tau \geqslant 0$

$$P^n(1 - \alpha_n\tau) = \{E\exp(X\log(1 - \alpha_n\tau))\}^n \sim \{E\exp(-\alpha_n\tau X)\}^n \qquad n \to \infty,$$

$X \in \mathbb{N}_0$ is in the domain of discrete attraction of P_γ^λ iff it is in the domain of attraction of $\exp(-\lambda\tau^\gamma)$ (cf. remark following Theorem 3.2).

4. Concluding remarks. We were led to consider equation (2.1) by first considering a more formal analogue of (1.1), viz. (cf. [4])

$$P(z) = \frac{P(\alpha z)}{P(\alpha)} P_\alpha(z) \qquad |z| \leqslant 1, \alpha \in (0, 1). \tag{4.1}$$

This equation can be treated in the same way as (2.1), and it turns out that one has

THEOREM 4.1. *A* p.g.f. P, *with* $P(0) > 0$, *satisfies* (4.1) iff *it is infinitely divisible*, i.e., (cf. (1.3)), iff *it is compound Poisson.*

Defining $\alpha * X$ (in distribution) by its p.g.f. $1 - \alpha + \alpha P(z)$, or by

$$\alpha * X = \Sigma_1^N X_j,$$

with N as in (2.3), we may consider the equation $X = \alpha * X' + X_\alpha$, or in terms of p.g.f.'s

$$P(z) = \{1 - \alpha + \alpha P(z)\} P_\alpha(z) \qquad |z| \leqslant 1; \alpha \in (0, 1), \tag{4.2}$$

to obtain

THEOREM 4.2. *A* p.g.f. P, *with* $P(0) > 0$, *satisfies* (4.2) iff *it is compound geometric.*

Equation (1.1) can be handled in a similar fashion, avoiding the use of triangular arrays, and one finds in exactly the same way: φ satisfies (1.1) iff (this seems to be new)

$$\varphi(t) = \exp \textstyle\int_0^t h(u) u^{-1}\, du \qquad t \in \mathbb{R},$$

where $\exp(h(u))$ is an inf div characteristic function. To prove this, however, one needs to know that φ' exists in $\mathbb{R} \setminus \{0\}$, and is such that $t\varphi'(t) \to 0$ as $t \to 0$. No such complication arises in the case of distributions on $[0, \infty)$ if one uses Laplace transforms instead of ch.f.'s.

Corollary 3.3 seems to suggest that the distribution of a sum of i.i.d. random variables with only a first moment should be approximated by a discrete stable Poisson distribution rather than by a stable degenerate distribution. If higher moments exist, a normal approximation would, of course, be preferable.

It might be possible to develop a theory of *discrete* limiting distributions for maxima of i.i.d. random variables in $\mathbb{N}_0$. This will be investigated later.

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