The Annals of Probability
1979, Vol. 7, No. 5, 887–892

ON LINEARITY OF s-PREDICTORS

BY DIETER LANDERS AND LOTHAR ROGGE

University of Cologne and University of Konstanz

Let $(\Omega, \mathcal{A}, P)$ be a probability space and $\mathcal{B} \subset \mathcal{A}$ be a σ-field. Let s with $1 < s < \infty$ be fixed. If $f \in L_s(\Omega, \mathcal{A}, P)$ and $\mathcal{B} \subset \mathcal{A}$ is a σ-field, the unique element $g_f \in L_s(\Omega, \mathcal{B}, P)$ such that $\|f - g_f\|_s = \inf\{\|f - h\|_s : h \in L_s(\Omega, \mathcal{B}, P)\}$ is called the s-predictor of f relative to the s-norm and the σ-field $\mathcal{B}$. Such g_f exists and is uniquely determined. The mapping $P_s^{\mathcal{B}} : f \to g_f$ is called a prediction operator. The prediction operator is not necessarily a linear operator. The problem is to characterize the σ-fields $\mathcal{B}$ in terms of $P|\mathcal{A}$ for which $P_s^{\mathcal{B}}$ is a linear operator. We show that, for a fixed σ-field $\mathcal{B}$, the prediction operators $P_s^{\mathcal{B}}$ are linear for all s or for no $s \neq 2$. We give a necessary and sufficient condition for the linearity of s-predictors in terms of conditional expectations only. If moreover regular conditional probabilities given $\mathcal{B}$ exist, the s-predictors are linear if and only if the regular conditional probabilities of $P|\mathcal{A}$ given $\mathcal{B}$ consist only of measures concentrated on at most two points. Furthermore we obtain a simple criterion that s-prediction coincides with the usual conditional expectation (i.e., with 2-prediction): the conditional expectations of indicator functions may only assume the values 0, $\frac{1}{2}$ and 1.

1. Introduction and notations. If $P|\mathcal{A}$ is a probability measure (p-measure) let $L_s(\Omega, \mathcal{A}, P)$ be the Banach space of equivalence classes of functions which are integrable in the sth mean ($s \geqslant 1$). The spaces $L_s(\Omega, \mathcal{A}, P)$, $1 < s < \infty$, are uniformly convex. Hence if $S \subset L_s(\Omega, \mathcal{A}, P)$, $1 < s < \infty$, is a closed linear subspace and $f \in L_s(\Omega, \mathcal{A}, P)$ there exists a uniquely determined element $g_f \in S$ such that

$$\|f - g_f\|_s = \inf\{\|f - h\|_s : h \in S\}.$$

The mapping $P_s^S : L_s(\Omega, \mathcal{A}, P) \to S$ defined by $P_s^S : f \to g_f$ is a prediction operator. The problem treated by both Andô [1] and Rao [4] was to find conditions on S in order that P_s^S be linear for any subspace S. We consider here the particular subspace $S = L_s(\Omega, \mathcal{B}, P)$ and treat the linearity of P_s^S which we denote by $P_s^{\mathcal{B}}$. We believe that our conditions for this special case are much easier to verify than the conditions given in Andô [1] or Rao [4]. Moreover our methods are completely different from those of Andô [1] and Rao [4]. The prediction operator has the following properties as can be seen from [2], page 114:

(i) $P_s^{\mathcal{B}}(\alpha f) = \alpha P_s^{\mathcal{B}} f$, for all $\alpha \in \mathbb{R}$;

(ii) $f \to P_s^{\mathcal{B}} f$ is continuous in the sth norm;

(iii) $P_s^{\mathcal{B}} f$ can be characterized as that $\mathcal{B}$-measurable function $g \in L_s$ which

Received August 22, 1977; revised July 7, 1978.

AMS 1970 *subject classifications.* Primary 60G25; secondary 46E30, 47H15.

Key words and phrases. Conditional, expectation, s-prediction, uniformly convex spaces, regular conditional probabilities.

fulfills

$$P_2^{\mathfrak{B}}\left((f-g)^{\underline{s-1}}\right) = 0,$$

where $y^{\underline{s}} := |y|^s \operatorname{sign} y$ is defined for each $y \in \mathbb{R}$ and $s > 0$.

The 2-prediction coincides with the conditional expectation which is usually denoted by $E(f|\mathfrak{B})$ or $E^{\mathfrak{B}}f$. Hence in our notation $E^{\mathfrak{B}}f$ is denoted by $P_2^{\mathfrak{B}}f$. If A is a set and 1_A the corresponding indicator function, we write $P_s^{\mathfrak{B}}A$ instead of $P_s^{\mathfrak{B}}1_A$. For $s = 2$, $P_2^{\mathfrak{B}}A$ is the usual conditional probability of A given $\mathfrak{B}$. In general we do not distinguish between a function and the corresponding equivalence class.

DEFINITION. The p-measure $P|\mathfrak{A}$ is $\mathfrak{B}$-conditional atomar if for all $A_1, A_2 \in \mathfrak{A}$ with $A_1 \subset A_2$ the implication $P_2^{\mathfrak{B}}(A_2)(\omega) < 1 \Rightarrow [P_2^{\mathfrak{B}}(A_1)(\omega) = 0$ or $P_2^{\mathfrak{B}}(A_1)(\omega) = P_2^{\mathfrak{B}}(A_2)(\omega)]$ holds P-a.e., i.e., iff

$$P\{\omega : P_2^{\mathfrak{B}}(A_2)(\omega) < 1, 0 < P_2^{\mathfrak{B}}(A_1)(\omega) < P_2^{\mathfrak{B}}(A_2)(\omega)\} = 0$$

Whether a σ-field $\mathfrak{B} \subset \mathfrak{A}$ is conditional atomar or not depends on $\mathfrak{B}$ and $P|\mathfrak{A}$. In Criterion 5 we give a necessary and sufficient condition for $\mathfrak{B}$-conditional atomarity in terms of regular conditional probabilities.

For the sake of completeness we cite at first a lemma of [3] which we need for the proof of our results.

Let $\Delta_n := \{(x_1, \cdots, x_n) \in [0,1]^n : \sum_{i=1}^n x_i = 1\}$. The proof of the following lemma uses the property (iii).

LEMMA 1. *Let $P|\mathfrak{A}$ be a p-measure, $\mathfrak{B} \subset \mathfrak{A}$ a σ-field and $1 < s < \infty$. Then for each simple function $f = \sum_{i=1}^n \alpha_i 1_{A_i}$, where $A_i \in \mathfrak{A}$, $i = 1, \cdots, n$ are disjoint, $\sum_{i=1}^n A_i = \Omega$ and $\alpha_1 \leqslant \cdots \leqslant \alpha_n$ there exists a continuous function*

$$H^{(s)}_{\alpha_1, \cdots, \alpha_n} : \Delta_n \to \mathbb{R}$$

such that

$$P_s^{\mathfrak{B}}f = H^{(s)}_{\alpha_1, \cdots, \alpha_n}(P_2^{\mathfrak{B}}A_1, \cdots, P_2^{\mathfrak{B}}A_n).$$

$H^{(s)}_{\alpha_1, \cdots, \alpha_n}(x_1, \cdots, x_n)$ is the unique solution y of $\sum_{i=1}^n (\alpha_i - y)^{\underline{s-1}} x_i = 0$. Put $r := 1/(s-1)$ and $\varphi_s(x) := x^r/(x^r + (1-x)^r)$. Then $\varphi_s(x) = H^{(s)}_{0,1}(1-x, x)$, whence $P_s^{\mathfrak{B}}A = \varphi_s(P_2^{\mathfrak{B}}A)$.

2. The results. We prove now that $P|\mathfrak{A}$ is conditional atomar iff the prediction operator $P_s^{\mathfrak{B}}$ is linear for some s with $1 < s < \infty$, $s \neq 2$.

THEOREM 2. *Let $P|\mathfrak{A}$ be a p-measure, $\mathfrak{B} \subset \mathfrak{A}$ a sub-σ-field and $1 < s < \infty$, $s \neq 2$. Then $P_s^{\mathfrak{B}}|L_s(\Omega, \mathfrak{A}, P)$ is linear if and only if $P|\mathfrak{A}$ is $\mathfrak{B}$-conditional atomar.*

PROOF. (i) Let $P|\mathfrak{A}$ be $\mathfrak{B}$ conditional atomar. Since $P_s^{\mathfrak{B}}: L_s(\Omega, \mathfrak{A}, P) \to L_s(\Omega, \mathfrak{B}, P)$ is a homogeneous operator, see (i), we have only to prove that $P_s^{\mathfrak{B}}$ is additive. Let $f = \sum_{i=1}^n \alpha_i 1_{A_i}$ with disjoint $A_i \in \mathfrak{A}$, $i = 1, \cdots, n$, $\sum_{i=1}^n A_i = \Omega$ and $n \geqq 2$ be given. We show at first that

$$(1) \qquad P_s^{\mathfrak{B}}f = \sum_{i=1}^n \alpha_i P_s^{\mathfrak{B}}A_i.$$

To prove this, we may assume w.l.o.g. that $\alpha_1 \leqslant \alpha_2 \cdots \leqslant \alpha_n$. Then according to our Lemma 1:

$$P_s^{\mathfrak{B}} f = H^{(s)}_{\alpha_1, \cdots, \alpha_n}(P_2^{\mathfrak{B}} A_1, \cdots, P_2^{\mathfrak{B}} A_n)$$

and

$$P_s^{\mathfrak{B}}(A_i) = \varphi_s(P_2^{\mathfrak{B}} A_i) \qquad \text{for } i = 1, \cdots, n.$$

Hence (1) is equivalent to

$$H^{(s)}_{\alpha_1, \cdots, \alpha_n}(P_2^{\mathfrak{B}} A_1, \cdots, P_2^{\mathfrak{B}} A_n) = \Sigma_{i=1}^n \alpha_i \varphi_s(P_2^{\mathfrak{B}} A_i) \tag{2}$$

where we assume that $P_2^{\mathfrak{B}} A_i$ are chosen in such a way that

$$\Sigma_{i=1}^n P_2^{\mathfrak{B}} A_i(\omega) = 1 \qquad \text{everywhere.} \tag{3}$$

At first we prove that for P-a.a. $\omega \in \Omega$ there exist $i_\omega, j_\omega \in \{1, \cdots, n\}$, $i_\omega \neq j_\omega$ such that

$$P_2^{\mathfrak{B}} A_{i_\omega}(\omega) + P_2^{\mathfrak{B}} A_{j_\omega}(\omega) = 1. \tag{4}$$

Assume that this is false; then $P(C) > 0$ where

$$C := \cap_{i \neq j}\{\omega \in \Omega : P_2^{\mathfrak{B}} A_i(\omega) + P_2^{\mathfrak{B}} A_j(\omega) < 1\}.$$

According to (3) for each $\omega \in \Omega$ there exists $l \in \{1, \cdots, n\}$ such that $P_2^{\mathfrak{B}} A_l(\omega) > 0$. If $\omega \in C$, then $P_2^{\mathfrak{B}} A_l(\omega) < 1$, hence according to (3) there exists $k \neq l$, $k \in \{1, \cdots, n\}$ such that

$$P_2^{\mathfrak{B}} A_k(\omega) > 0$$

whence

$$C \subset \cup_{l \neq k}\{\omega \in \Omega : P_2^{\mathfrak{B}} A_l(\omega) > 0, P_2^{\mathfrak{B}} A_k(\omega) > 0\}.$$

Therefore there exists a pair l, k with $l \neq k$ such that $P(D) > 0$ where

$$D := C \cap \{\omega \in \Omega : P_2^{\mathfrak{B}} A_l(\omega) > 0, P_2^{\mathfrak{B}} A_k(\omega) > 0\}.$$

Put $E_1 := A_l$, $E_2 := A_l + A_k$, then $E_1, E_2 \in \mathcal{Q}$ and $E_1 \subset E_2$. According to the definition of D we have for almost all $\omega \in D$

$$P_2^{\mathfrak{B}} E_2(\omega) = P_2^{\mathfrak{B}} A_l(\omega) + P_2^{\mathfrak{B}} A_k(\omega) < 1$$

and

$$0 < P_2^{\mathfrak{B}} A_l(\omega) = P_2^{\mathfrak{B}} E_1(\omega) < P_2^{\mathfrak{B}} A_l(\omega) + P_2^{\mathfrak{B}} A_k(\omega) = P_2^{\mathfrak{B}} E_2(\omega).$$

Hence $P\{\omega : P_2^{\mathfrak{B}} E_2(\omega) < 1, 0 < P_2^{\mathfrak{B}} E_1(\omega) < P_2^{\mathfrak{B}} E_2(\omega)\} > 0$ contradicting the conditional atomarity of $\mathfrak{B}$ with respect to $P|\mathcal{Q}$. Therefore (4) is proven.

To prove (2) it suffices according to (4) to show that for all $i, j \in \{1, \cdots, n\}$, $i \neq j$ and all $x \in [0, 1]$

$$H^{(s)}_{\alpha_1, \cdots, \alpha_n}(x_1, \cdots, x_n) = \alpha_i \varphi_s(x) + \alpha_j \varphi_s(1 - x)$$

if $x_i = x$, $x_j = 1 - x$ and $x_\nu = 0$ elsewhere.

Let w.l.o.g. $i = 1, j = 2$. Using that $H^{(s)}_{\alpha_1, \cdots, \alpha_n}(x_1, \cdots, x_n)$ is the unique solution y of $\sum_{i=1}^n (\alpha_i - y)^{\underline{s-1}} x_i = 0$, one has to show

$$(\alpha_1 - [\alpha_1\varphi_s(x_1) + \alpha_2\varphi_s(1-x_1)])^{\underline{s-1}} x_1 + (\alpha_2 - [\alpha_1\varphi_s(x_1) + \alpha_2\varphi_s(1-x_1)])^{\underline{s-1}} (1-x_1) = 0.$$

This follows by a little computation from the definition of φ_s. Hence we have proven (1).

From (1) we obtain that $P_s^{\mathfrak{B}}$ is additive on the system of $\mathfrak{A}$-measurable simple functions:

If f, g are two $\mathfrak{A}$-measurable simple functions, then there exists a representation of f and g such that

$$f = \sum_{i=1}^n \alpha_i 1_{A_i}$$
$$g = \sum_{i=1}^n \beta_i 1_{A_i}$$

where $A_i \in \mathfrak{A}$, $i = 1, \cdots, n$ are disjoint and $\sum_{i=1}^n A_i = \Omega$. From (1) we obtain that

$$P_s^{\mathfrak{B}}(f+g) = P_s^{\mathfrak{B}} f + P_s^{\mathfrak{B}} g,$$

i.e., $P_s^{\mathfrak{B}}$ is additive on the system of all $\mathfrak{A}$-measurable simple functions. The additivity of $P_s^{\mathfrak{B}}$ on $L_s(\Omega, \mathfrak{A}, P)$ follows now from continuity condition (ii) of the introduction.

(ii) In the converse direction we shall even show, if $P_s^{\mathfrak{B}}|\mathfrak{A}$ is additive for some $s \neq 2$, i.e., $P_s^{\mathfrak{B}}(A_1 + A_2) = P_s^{\mathfrak{B}} A_1 + P_s^{\mathfrak{B}} A_2$ for disjoint $A_1, A_2 \in \mathfrak{A}$, then $P|\mathfrak{A}$ is $\mathfrak{B}$-conditional atomar.

Let $A_1, A_2 \in \mathfrak{A}$ with $A_1 \subset A_2$ be given. Put $A_3 = A_2 - A_1$. If $P_s^{\mathfrak{B}}|\mathfrak{A}$ is additive we have

$$P_s^{\mathfrak{B}} A_2 = P_s^{\mathfrak{B}} A_1 + P_s^{\mathfrak{B}} A_3 \qquad P-\text{a.e.}$$

and hence according to our Lemma 1

$$\text{(5)} \quad \varphi_s(P_2^{\mathfrak{B}} A_1 + P_2^{\mathfrak{B}} A_3) = \varphi_s(P_2^{\mathfrak{B}} A_2) = \varphi_s(P_2^{\mathfrak{B}} A_1) + \varphi_s(P_2^{\mathfrak{B}} A_3) \qquad P-\text{a.e.}$$

We have to show

$$P\{\omega : (P_2^{\mathfrak{B}} A_1 + P_2^{\mathfrak{B}} A_3)(\omega) < 1, P_2^{\mathfrak{B}} A_1(\omega) > 0, P_2^{\mathfrak{B}} A_3(\omega) > 0\} = 0.$$

Hence $P|\mathfrak{A}$ is according to (5) $\mathfrak{B}$-conditional atomar if we prove

$$\text{(6)} \qquad x_1, x_2 \geqq 0, x_1 + x_2 < 1, \varphi_s(x_1 + x_2) = \varphi_s(x_1) + \varphi_s(x_2)$$

implies $x_1 = 0$ or $x_2 = 0$.

We consider at first the case $s < 2$ and prove

$$\text{(7)} \qquad \varphi_s(x_1 + x_2) > \varphi_s(x_1) + \varphi_s(x_2)$$

for $0 < x_1, x_2$ and $x_1 + x_2 < 1$. This implies (6).

To prove (7), differentiate φ_s. We have

$$\varphi_s'(x) = \frac{rx^{r-1}(1-x)^{r-1}}{\left[x^r + (1-x)^r\right]^2} \qquad \text{for} \quad x \in (0, 1)$$

Since $s < 2$ and hence $r > 1$ the numerator function is monotone increasing and the denominator function is monotone decreasing over $(0, \frac{1}{2}]$. Hence

(8) $\varphi_s'(x),\ x \in \left(0, \frac{1}{2}\right]$ is monotone increasing for $s < 2$.

As $\varphi_s(x) + \varphi_s(1-x) = 1$ we have $\varphi_s'(x) = \varphi_s'(1-x)$. Using these facts it is easy to prove (7).

We consider now the case $s > 2$ and hence $r < 1$. Then $\varphi_s'(x),\ x \in (0, \frac{1}{2}]$ is monotone decreasing. This yields similarly

(9) $\varphi_s(x_1 + x_2) < \varphi_s(x_1) + \varphi_s(x_2)$ for $0 < x_1, x_2$ and $x_1 + x_2 < 1$,

and hence (6) holds also in this case.

As can be seen from the proof (ii) of Theorem 2 $P_s^{\mathfrak{B}}$ is superadditive for $1 < s < 2$ and subadditive for $2 < s < \infty$.

Though the proof of Theorem 2 shows that additivity of $P_s^{\mathfrak{B}}$ extends from the indicator functions, for instance, to the class ϕ of all test functions—i.e., all $\mathfrak{A}$ measurable functions with values in $[0, 1]$—this does not hold true in general for subadditivity or superadditivity instead of additivity. One can easily show, using conditions (i) and $P_s^{\mathfrak{B}}1 = 1$, that $P_s^{\mathfrak{B}}$ is linear on ϕ—and hence by the proof of Theorem 2 on $L_s(\Omega, \mathfrak{A}, P)$ for all $s > 1$—if $P_{s_0}^{\mathfrak{B}}$ is subadditive on all test functions or superadditive on all test functions (or on all nonnegative simple functions) for some s_0 with $1 < s_0 < \infty$ and $s_0 \neq 2$.

From Theorem 2 and the remarks thereafter one easily obtains

COROLLARY 3. *Let $P|\mathfrak{A}$ be a p-measure and $\mathfrak{B} \subset \mathfrak{A}$ a σ-field. Then the following conditions are equivalent*:

(i) $P_s^{\mathfrak{B}}|\ L_s(\Omega, \mathfrak{A}, P)$ *is linear for all* $s > 1$,
(ii) $P_s^{\mathfrak{B}}|\mathfrak{A}$ *is additive for some* $s > 1 (s \neq 2)$,
(iii) $P_s^{\mathfrak{B}}|\phi$ *is subadditive for some* $s > 1 (s \neq 2)$,
(iv) $P_s^{\mathfrak{B}}|\phi$ *is superadditive for some* $s > 1 (s \neq 2)$,
(v) $P|\mathfrak{A}$ *is* $\mathfrak{B}$*-conditional atomar.*

As a further application we obtain necessary and sufficient conditions under which the s-predictors are the 2-predictors, i.e., the usual conditional expectations.

COROLLARY 4. *Let $P|\mathfrak{A}$ be a p-measure and $\mathfrak{B} \subset \mathfrak{A}$ be a σ-field. Then the following conditions are equivalent*:

(i) $P_2^{\mathfrak{B}}A \in \{0, \frac{1}{2}, 1\}$ P-a.e. *for each* $A \in \mathfrak{A}$,
(ii) $P_s^{\mathfrak{B}}|\mathfrak{A} = P_2^{\mathfrak{B}}|\mathfrak{A}$ *for some* $s > 1 (s \neq 2)$,
(iii) $P_s^{\mathfrak{B}} = P_2^{\mathfrak{B}}$ *on* $L_s \cap L_2$ *for all* $s > 1$.

PROOF. We have $P_s^{\mathfrak{B}}A = \varphi_s(P_2^{\mathfrak{B}}A)$ according to our Lemma 1. If $s \neq 2$, then $\varphi_s(x) = x$ iff $x \in \{0, \frac{1}{2}, 1\}$.

If (i) is fulfilled, then $P_2^{\mathfrak{B}}A = \varphi_s(P_2^{\mathfrak{B}}A) = P_s^{\mathfrak{B}}A$ P-a.e. Hence $P_s^{\mathfrak{B}}|\mathcal{C} = P_2^{\mathfrak{B}}|\mathcal{C}$, whence $P_s^{\mathfrak{B}}|\mathcal{C}$ is additive. According to Corollary 3, $P_s^{\mathfrak{B}}|L_s(\Omega, \mathcal{C}, P)$ is therefore linear and coincides with $P_2^{\mathfrak{B}}$ on $\mathcal{C}$ and hence on all $\mathcal{C}$-measurable simple functions. Using a continuity argument (see (ii) of the introduction), one obtains $P_s^{\mathfrak{B}} = P_2^{\mathfrak{B}}$ on $L_s \cap L_2$. Therefore (i) implies (iii).

Since (iii) $\Rightarrow$ (ii) trivially, it remains to show (ii) $\Rightarrow$ (i). Assume to the contrary that there exists $A \in \mathcal{C}$ with

$$P\left(\left\{\omega : P_2^{\mathfrak{B}}A(\omega) \notin \left\{0, \tfrac{1}{2}, 1\right\}\right\}\right) > 0.$$

Since $P_s^{\mathfrak{B}}A = P_2^{\mathfrak{B}}A$ we obtain according to our Lemma 1

$$\varphi_s\left(P_2^{\mathfrak{B}}A(\omega)\right) = P_2^{\mathfrak{B}}A(\omega)$$

on a set of positive measure with $P_2^{\mathfrak{B}}A(\omega) \notin \{0, \frac{1}{2}, 1\}$. Therefore there would exist an $x \neq 0, \frac{1}{2}, 1$ with $\varphi_s(x) = x$.

Now we will give a criterion for $\mathfrak{B}$-conditional atomarity in terms of regular conditional probabilities which can be easily proven.

CRITERION 5. Let P be a p-measure on a countably generated σ-field $\mathcal{C}$ on Ω and $\mathfrak{B} \subset \mathcal{C}$ be a sub-σ-field. Assume that there exists a regular conditional probability $R : \mathcal{C} \times \Omega \to [0, 1]$ for $P|\mathcal{C}$, given $\mathfrak{B}$. Then $P|\mathcal{C}$ is $\mathfrak{B}$-conditional atomar iff for P-almost all $\omega \in \Omega$ the p-measure $A \to R(A, \omega)$, $A \in \mathcal{C}$, is concentrated on at most two atoms of $\mathcal{C}$.

Acknowledgment. We thank the referee for improving the presentation.

REFERENCES

[1] ANDÔ, T. (1966). Contractive projections in L_p-spaces. *Pacific J. Math.* **17** 391–405.

[2] ANDÔ, T. and AMEMIYA, I. (1965). Almost everywhere convergence of prediction sequence in $L_p(1 < p < \infty)$. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **4** 113–120.

[3] LANDERS, D. and ROGGE, L. (1978). Connection between the different L_p-predictions with applications. *Z. Wahrscheinlichkeitstheorie und verw. Gebiete* **45** 169–173.

[4] RAO, M. M. (1974). Inference in stochastic processes IV: Predictors and projections. *Indian J. Statist.* Ser. A **36** 63–120.

MATHEMATISCHES INSTITUT
DER UNIVERSITÄT KÖLN
WEYERTAL 86-90
D-5000 KÖLN 41
WEST GERMANY

FACHBEREICH STATISTIK
UNIVERSITÄT KONSTANZ
D-775 KONSTANZ
POSTFACH 5560
WEST GERMANY