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THE BROWNIAN ESCAPE PROCESS[1]

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Let X be the Brownian motion process in $\mathbb{R}^d$, $d \geqslant 3$ with $X(0) = 0$. Let L_r be the last exit time of X from the ball of radius r centered at the origin. Then (L_r) has independent increments and we compute the distribution of L_r. When $d = 3$ this yields a simple proof of a recent result of Pitman.

Let $X = (X_t)$ be the standard Brownian motion process in $\mathbb{R}^d$. If $r > 0$, define

$$L_r = \sup\{t: |X_t| \leqslant r\}. \tag{1}$$

We shall assume that $d \geqslant 3$. Then almost surely $|X_t| \to \infty$ as $t \to \infty$, and so L_r is finite almost surely. If $P = P^0$ is the law of the Brownian motion starting from the origin we shall call the process $(L_r, r \geqslant 0; P)$ the *Brownian escape* process. Clearly $r \to L_r$ is strictly increasing and $L_0 = 0$.

Recently Pitman [5] has shown that when $d = 3$, (L_r, P) is a stable subordinator of index $\frac{1}{2}$ and rate $2^{\frac{1}{2}}$; that is, (L_r, P) has stationary independent increments and the Laplace transform of its distribution is given by

$$E(e^{-\alpha L_r}) = e^{-r(2\alpha)^{\frac{1}{2}}}. \tag{2}$$

Pitman derives this as a corollary of a deep result connecting one-dimensional Brownian motion and the three-dimensional Bessel process. The purpose of this note is to give a simple direct proof of Pitman's result and to study the escape process when $d > 3$.

Here is our main result.

Theorem. *Let $d \geqslant 3$. Then the process (L_r, P) is continuous in probability and has independent increments. If $r > 0$, L_r has a continuous density given by $(t > 0)$*

$$P[L_r \in dt] = r^{d-2}\left[2^{(d-2)/2}\Gamma\left(\frac{d-2}{2}\right)t^{d/2}\right]^{-1} e^{-r^2/2t}\, dt. \tag{3}$$

The escape process has stationary increments if and only if $d = 3$.

Remark. If $d = 3$, (3) is the familiar density of the stable subordinator whose Laplace transform is given by (2). Although (L_r) does not have stationary increments when $d > 3$, it is trying in some average sense to "look like" a stable

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subordinator of index $\frac{1}{2}$. More precisely we shall see that

(4) $$\int_0^\infty E(e^{-\alpha L_r})\, dr = \left[c(d)(2\alpha)^{\frac{1}{2}}\right]^{-1}$$

(5) $$c(d) = \frac{\Gamma\left(\frac{d-2}{2}\right)}{\pi^{\frac{1}{2}}\Gamma\left(\frac{d-1}{2}\right)}.$$

Of course, (4) is exactly what one would obtain if (L_r) were a stable subordinator of index $\frac{1}{2}$ and rate $c(d) \cdot 2^{\frac{1}{2}}$.

PROOF. Using the scaling property of the Brownian motion (i.e., if $c > 0$, $c^{-\frac{1}{2}}X(ct)$ is again a Brownian motion), it is immediate that L_r has the same distribution as $r^2 L_1$ under $P = P^0$. Since L_r is increasing this implies that $r \to L_r$ is continuous at each fixed r almost surely, and hence it is continuous in probability.

To see that (L_r) has independent increments we define $R_t = |X_t|$. Then $(R_t; t \geqslant 0)$ is a continuous strong Markov process on $\mathbb{R}^+ = [0, \infty)$ (it is the d-dimensional Bessel process) and

(6) $$L_r = \sup\{t\colon R_t \leqslant r\}$$

is the "last exit" time of (R_t) from the interval $[0, r]$. Let $(\mathcal{G}_t)_{t \geqslant 0}$ denote the usual (completed) fields of the diffusion (R_t). A by now standard result (see, e.g., Theorem 37 of [1]) states that under $P = P^0$ the post L_r process $(R_{L_r+t})_{t>0}$ is conditionally independent of $\mathcal{G}_{L_r}$ given R_{L_r}. But by the continuity of the paths $R_{L_r} = r$ almost surely P and so in the present case $(R_{L_r+t})_{t>0}$ is *independent* of $\mathcal{G}_{L_r}$. Here $\mathcal{G}_{L_r}$ is the usual field associated with a last exit time. If $r < s$, then

(7) $$L_s - L_r = \sup\{t > 0\colon R_{L_r+t} \leqslant s\},$$

and so $L_s - L_r$ is independent of $\mathcal{G}_{L_r}$. Also $R_{t \wedge L_r}$ is $\mathcal{G}_{t \wedge L_r}$ measurable. But $t \wedge L_r \leqslant L_r$ and is $\mathcal{G}_{L_r}$ measurable, and consequently $\mathcal{G}_{t \wedge L_r} \subset \mathcal{G}_{L_r}$. If $a < r$, then

$$L_a = \sup\{t\colon R_{t \wedge L_r} \leqslant a\},$$

and so each L_a with $a \leqslant r$ is $\mathcal{G}_{L_r}$ measurable. As a result the process (L_r, P) has independent increments.

Before coming to the proof of (3) we introduce the process

(8) $$F_t = \inf\{|X_s|\colon s > t\}, \qquad t \geqslant 0.$$

Clearly $t \to F_t$ is increasing and continuous and $\lim_{t\to\infty} F_t = \infty$ almost surely. Note that $F_t = F_0 \circ \theta_t$ and $F_t = \inf\{|X_s|\colon s \geqslant t\}$. If $0 < a < b$, let $T_{ab} = \inf\{t\colon |X_t| \notin (a, b)\}$. It is well known and easily checked that if $a < |x| < b$, then

(9) $$P^x\left[|X(T_{ab})| = b\right] = \left[\left(\frac{b}{a}\right)^{d-2} - 1\right]^{-1}\left[\left(\frac{b}{a}\right)^{d-2} - \left(\frac{b}{|x|}\right)^{d-2}\right].$$

Letting $b \to \infty$ in (9) we obtain

$$(10) \qquad P^x[F_0 > a] = 1 - \left(\frac{a}{|x|}\right)^{d-2} \qquad 0 < a < |x|.$$

We are now prepared to prove (3). Clearly

$$P^0[L_r < t] = P^0[F_t > r] = E^0\{P^{X(t)}[F_0 > r]\},$$

and using (10) we have

$$P^0[L_r < t] = (2\pi t)^{-d/2} \int_{|x|>r} e^{-|x|^2/2t}\left[1 - \left(\frac{r}{|x|}\right)^{d-2}\right] dx$$

$$= \frac{2}{\Gamma(d/2)(2t)^{d/2}} \int_r^\infty u[u^{d-2} - r^{d-2}]e^{-u^2/2t}\, du$$

$$= \left[\Gamma\left(\frac{d}{2}\right)\right]^{-1}\left\{\int_{r^2/2t}^\infty v^{(d-2)/2}e^{-v}\, dv - r^{d-2}(2t)^{1-d/2}e^{-r^2/2t}\right\}.$$

Differentiating with respect to t we obtain (3).

Let us denote the density in (3) by $g_d(r, t)$. Consulting a table of Laplace transforms one finds

$$(11) \qquad \int_0^\infty e^{-\alpha t} g_3(r, t)\, dt = e^{-r(2\alpha)^{\frac{1}{2}}}$$

$$(12) \quad \int_0^\infty e^{-\alpha t} g_d(r, t)\, dt = \left[2^{(d-4)/2}\Gamma\left(\frac{d-2}{2}\right)\right]^{-1}\left(r(2\alpha)^{\frac{1}{2}}\right)^{(d-2)/2} K_{(d-2)/2}\left(r(2\alpha)^{\frac{1}{2}}\right)$$

where K_ν is the usual modified Bessel function of the third kind. Of course, when $d = 3$, (12) reduces to (11). Also, if $d = 5$, (12) reduces to $(1 + r(2\alpha)^{\frac{1}{2}})e^{-r(2\alpha)^{\frac{1}{2}}}$. Either calculating from scratch using (10), or by using (27) on page 51 of [2], we obtain (4) and (5) by integrating (12) in r over $(0, \infty)$. Therefore if (L_r) has *stationary* independent increments, then one must have

$$\int_0^\infty e^{-\alpha t} g_d(r, t)\, dt = e^{-rc(d)(2\alpha)^{\frac{1}{2}}}$$

and one easily checks (compare derivatives at $\alpha = 0$) that this contradicts (12) if $d > 3$. This completes the proof of the theorem.

REMARK. Since any increment of a process with independent increments that is continuous in probability has an infinitely divisible distribution, the density $g_d(r, t)$ is infinitely divisible. Also if $0 \leqslant a < b$, the increment $L_b - L_a$ is a positive random variable having an infinitely divisible distribution whose Laplace transform, in view of (12), is given by

$$(13) \qquad \frac{b^\nu K_\nu\left(b(2\alpha)^{\frac{1}{2}}\right)}{a^\nu K_\nu\left(a(2\alpha)^{\frac{1}{2}}\right)}, \qquad \nu = \frac{d-2}{2}, d \geqslant 3.$$

One can invert this explicitly when $d = 5$, but the resulting density does not seem to be particularly interesting. On the other hand, starting with the Bessel diffusion

Y with generator $\frac{1}{2}(D^2 + ((2\nu + 1)/x)D)$ on $x \geqslant 0$ where $D = d/dx$, the same argument shows that if

$$f_\nu(t) = \left[2^\nu \Gamma(\nu) t^{\nu+1}\right]^{-1} e^{-1/2t}, \qquad t > 0, \tag{14}$$

then $a^{-2} f_\nu(t/a^2)$ is the density of the escape process $L_a = \sup\{t\colon Y_t \leqslant a\}$. In particular f_ν is infinitely divisible for $\nu > 0$, and (13) is the Laplace transform of an infinitely divisible distribution for $\nu > 0$. The fact that f_ν is an infinitely divisible density may also be obtained as a limiting case of a recently announced result of Ismail and Kelker on the infinite divisibility of the F-distribution. See [4].

It is interesting to compare these results to the familiar first passage results. Let $T_r = \inf\{t\colon |X_t| \geqslant r\}$. Then using the strong Markov property (T_r, P) is an increasing process with independent increments for $d \geqslant 1$. Using a standard martingale argument, or the generator if we deal with the Bessel process (see, e.g., [3]), one has

$$E^0(e^{-\alpha T(r)}) = \left(r(2\alpha)^{\frac{1}{2}}\right)^\nu \left[2^\nu \Gamma(\nu + 1) I_\nu\left(r(2\alpha)^{\frac{1}{2}}\right)\right]^{-1} \tag{15}$$

where $\nu = (d-2)/2$, $d \geqslant 1$ or $\nu \geqslant -\frac{1}{2}$. Here I_ν is the modified Bessel function of the first kind. If $a < b$ the increment $T_b - T_a$ has an infinitely divisible distribution whose Laplace transform is given by

$$\frac{b^\nu I_\nu\left(a(2\alpha)^{\frac{1}{2}}\right)}{a^\nu I_\nu\left(b(2\alpha)^{\frac{1}{2}}\right)}. \tag{16}$$

If $d = 1$, (15) reduces to $(\cosh r(2\alpha)^{\frac{1}{2}})^{-1}$. It is interesting to note that the densities for the last exit (escape) process L_r have simple expressions while those of the first passage process T_r seem to be more complicated. For example, when $d = 1$, the density of T_r may be expressed in terms of theta functions.

The expression (3) for the density of L_r is also an easy consequence of formula (13) in [6].

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