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MARTINGALES WITH GIVEN ABSOLUTE VALUE[1]

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A construction is given for a martingale having a given right continuous strictly positive submartingale as its absolute value. The existence of a martingale whose absolute value has the same distribution as the given submartingale was shown by Gilat.

1. Introduction. A striking theorem of Gilat (1977) states, in the continuous parameter case, that given a nonnegative right continuous submartingale $(X_t)_{t\geqslant 0}$, there exists a right continuous martingale $(M_t)_{t\geqslant 0}$, possibly defined on a different probability space, such that $|M_t|$ and X_t have the same distribution. We give here a construction of (M_t) on an augmentation of the probability space for (X_t) in the special case where X is strictly positive. The result states in the case where X_t is continuous that if $X_t = Y_t + A_t$ is the Doob-Meyer decomposition of (X_t) into a martingale (Y_t) and a predictable increasing process (A_t), then M_t is obtained by a Poisson randomization of the sign of X_t, with $dA_t/2X_t$ as the intensity of sign changes.

2. The submartingale $X_t = t$. The particular case $X_t = t$ involves the basic process of sign changes needed in the next section. Let $(Z_t)_{-\infty<t<\infty}$ denote a right continuous stationary homogeneous Markov process with states ± 1, indexed by the entire real line, and having $P\{Z_t = 1\} = \frac{1}{2}$ for all real t, and with jump rate equal to $\frac{1}{2}$ so that $P\{Z_{t+h} \neq Z_t\} \sim h/2$ as $h \searrow 0$. The set of jump times for (Z_t) is then a stationary Poisson point process with intensity equal to $\frac{1}{2}$. The transition function for (Z_t) is given by

$$P\{Z_{s+h} = \varepsilon | Z_s\} = (1 + \varepsilon e^{-h} Z_s)/2 \qquad (\varepsilon = \pm 1, h > 0). \tag{2.1}$$

(2.2) LEMMA. *Let $W_t = tZ_{\log t}$ for $t > 0$ and $W_0 = 0$. Then $(W_t)_{t\geqslant 0}$ is a right continuous martingale with $|W_t| = t$ for all $t \geqslant 0$. Moreover, if (V_t) is a right continuous martingale on an arbitrary probability space with $|V_t| = t$ for all t, then the process (V_t) has the same distribution as (W_t).*

PROOF. It is obvious that $(W_t)_{t\geqslant 0}$ is right continuous and that $|W_t| = t$. For this proof only, let $\mathcal{F}_s = \sigma\{Z_t : t \leqslant s\}$. The martingale property of (W_t) is equivalent, after a change of variables, to the identity, for $s < t$

$$E\{Z_t | \mathcal{F}_s\} = e^{s-t} Z_s.$$

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Since $Z_t = \pm 1$, this last identity is equivalent to

$$P\{Z_t = 1|\mathcal{F}_s\} = \big(1 + e^{-(t-s)}Z_s\big)/2. \tag{2.3}$$

That (2.3) holds is evident from (2.1).

As to the uniqueness assertion, one sees that if $(V_t)_{t\geqslant 0}$ is a martingale over $(\Omega', (\mathcal{F}_t'), P')$ with $|V_t| = t$ then $U_t = e^{-t}V(e^t)$, $(t \in \mathbb{R})$, is a process taking values ± 1, $E'U_t = e^{-t}E'V(e^t) = e^{-t}E'V_0 = 0$, and for $s < t$, $E'\{U_t|\mathcal{F}'_{\exp s}\} = e^{-t}V(e^s) = e^{-(t-s)}U_s$. Using the fact that $U_t = \pm 1$ then gives, as in (2.3),

$$P'\{U_t = 1|\mathcal{F}'_{\exp s}\} = \big(1 + e^{-(t-s)}U_s\big)/2$$

so that, from (2.1), (U_t) is a homogeneous Markov process with transition function the same as that of (Z_t), and the fact that $E'U_t = 0$ shows that it is stationary, completing the proof.

3. Strictly positive case. We assume given a right continuous submartingale $(X_t)_{t\geqslant 0}$ satisfying the hypothesis

$$X_t > 0 \quad \text{and} \quad X_{t-} > 0 \quad \text{for all} \quad t \geqslant 0. \tag{3.1}$$

We augment the probability space underlying (X_t) by means of the usual product space construction so that one may assume that the augmented space $(\Omega, \mathcal{F}, P)$ carries a copy of the process (Z_t) described in Section 2 and that the processes (X_t) and (Z_t) are independent. As in Section 2, W_t denotes the process $tZ_{\log t}$.

Since X_t and X_{t-} never vanish, one has the multiplicative decomposition

$$X_t = Q_t I_t \tag{3.2}$$

where Q_t is a positive right continuous martingale with $Q_0 = X_0$ and I_t is a right continuous predictable increasing process with $I_0 = 1$ (see [2]). In terms of the Doob-Meyer decomposition $X_t = Y_t + A_t$, the process I_t^{-1} is the exponential, in the Stieltjes integral sense, of $-dA_t/(Y_{t-} + A_t)$. That is, I_t satisfies $dI_t/I_{t-} = dA_t/(Y_{t-} + A_t)$. Of course, $(Y_{t-} + A_t)$ is just the predictable projection of (X_t). We avoid mentioning, whenever possible, the filtration of $(\Omega, \mathcal{F}, P)$ over which a process is a martingale. It is to be understood then that the minimal σ-fields are to be used, after adjoining null sets and making them right continuous.

(3.3) PROPOSITION. *Let X be a positive right continuous submartingale satisfying* (3.1) *with multiplicative decomposition given by* (3.2). *Then $M_t = Q_t W(I_t) = X_t Z(\log I_t)$ is a right continuous martingale whose absolute value is X_t on the augmented space.*

PROOF. Note first that it suffices to give a proof under the additional hypothesis that (X_t) is of class(D), for if (X_t) is replaced by $X_{t\wedge n}$, $(X_{t\wedge n})$ is of class(D) and its multiplicative decomposition is just $X_{t\wedge n} = Q_{t\wedge n} I_{t\wedge n}$; applying the theorem to $X_{t\wedge n}$ shows that $Q_{t\wedge n} W(I_{t\wedge n})$ is a martingale with absolute value $X_{t\wedge n}$, and this is obviously enough for the general case.

Assume now that X is of class(D) and for $\varepsilon > 0$, let τ_s^ε denote the right continuous increasing process inverse to the strictly increasing predictable process

$I_t^\varepsilon = I_t + \varepsilon t$. That is, $\tau_s^\varepsilon = \inf\{t : I_t^\varepsilon > s\}$. Then τ_s^ε is, in fact, continuous, finite valued, and one has $\tau^\varepsilon(I_t^\varepsilon) = t$ for all $t \geqslant 0$.

Since (X_t) is of class(D), X_∞ exists and closes the submartingale. In the decomposition (3.2), (Q_t) is then a uniformly integrable martingale which we close at ∞. For each $\varepsilon > 0$, the process $(Q(\tau_s^\varepsilon))_{s \geqslant 1}$ is a right continuous positive uniformly integrable martingale which is independent of the martingale $(W_s)_{s \geqslant 1}$. The process $(Q(\tau_s^\varepsilon)W_s)_{s \geqslant 1}$ is, therefore, a right continuous martingale with absolute value $sQ(\tau_s^\varepsilon)$.

For each fixed t, I_t^ε is a stopping time for $(Q(\tau_s^\varepsilon)W_s)_{s \geqslant 1}$ (or at least for an obvious filtration over which it is a martingale). As s varies over $[1, I_t^\varepsilon]$, we have $0 \leqslant \tau_s^\varepsilon \leqslant t$, and because $sQ(\tau_s^\varepsilon) \leqslant I^\varepsilon(\tau_s^\varepsilon)Q(\tau_s^\varepsilon) = X(\tau_s^\varepsilon) + \varepsilon\tau_s^\varepsilon Q(\tau_s^\varepsilon) \leqslant X(\tau_s^\varepsilon) + \varepsilon t Q(\tau_s^\varepsilon)$, the martingale $Q(\tau_s^\varepsilon)W_s$, stopped at I_t^ε, is uniformly integrable. It follows by optional sampling that the process $t \to Q(\tau^\varepsilon(I_t^\varepsilon))W(I_t^\varepsilon) = Q_t W(I_t^\varepsilon)$ is a right continuous martingale. As $\varepsilon \searrow 0$, $I_t^\varepsilon \searrow I_t$ and since $|Q_t W(I_t^\varepsilon)| = Q_t I_t^\varepsilon = X_t + \varepsilon t Q_t$, one may appeal to uniform integrability to conclude that $Q_t W(I_t)$ is a right continuous martingale with absolute value $Q_t I_t = X_t$.

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