The Annals of Probability
1979, Vol. 7, No. 6, 1051–1055

A MARTINGALE INEQUALITY FOR THE SQUARE AND MAXIMAL FUNCTIONS[1]

BY LOUIS H. Y. CHEN

University of Singapore and Stanford University

An inequality for certain random sequences more general than martingales or nonnegative submartingales is proved. Three special cases are deduced, one of which generalizes and refines a result of Austin. As an application of the inequality, the special cases are used to give new proofs of Burkholder's $L \log L$ and L_p (for $1 < p \leqslant 2$) inequalities for the square function of a martingale or a nonnegative submartingale.

1. Introduction and notation. An inequality involving a class of functions is proved for random sequences (or nonnegative random sequences) $f = (f_1, f_2, \cdots)$ whose terms are integrable and which satisfy the condition

$$E(f_{n+1}|f_n) = (\text{or} \geqslant) f_n \qquad \text{a.s. for} \quad n \geqslant 1. \tag{1.1}$$

Three special cases (Corollary 2.1) are deduced, one of which generalizes and refines a result of Austin (1966). As an application of the inequality, the special cases are used to give new proofs of Burkholder's (1966) $L \log L$ and L_p (for $1 < p \leqslant 2$) inequalities for the square function of a martingale or a nonnegative submartingale.

Random sequences whose terms are integrable and which satisfy (1.1) are more general than weak martingales (or weak submartingales) which in turn are more general than martingales (or submartingales). Weak martingales (or weak submartingales) were first defined in Nelson (1970) (see also Berman (1976)). They are random sequences $f = (f_1, f_2, \cdots)$ such that f_n is integrable and $E(f_n|f_m) = (\text{or} \geqslant) f_m$ a.s. for $1 \leqslant m \leqslant n$ and $n \geqslant 1$.

Although the inequality and the special cases in this paper are proved for random sequences (or nonnegative random sequences) satisfying (1.1), they are also new for martingales (or nonnegative submartingales). Also, Corollary 2.1 is quite surprising, since there exist random sequences which are L_p-bounded ($1 \leqslant p < \infty$), satisfy (1.1) and diverge a.s. (see, for example, Starr (1965)).

Throughout this paper, unless otherwise stated, $f = (f_1, f_2, \cdots)$ will denote a random sequence (or a nonnegative random sequence) defined on a probability space such that f_n is integrable for $n \geqslant 1$ and such that (1.1) holds. As usual $f_0 \equiv 0$. The difference sequence of f will be denoted by $d = (d_1, d_2, \cdots)$. Also

$$S_n(f) = (\Sigma_{i=1}^n d_i^2)^{\frac{1}{2}},\ S(f) = \sup_{1 \leqslant n < \infty} S_n(f),$$
$$f_n^* = \sup_{1 \leqslant i \leqslant n} |f_i|,\ f^* = \sup_{1 \leqslant n < \infty} f_n^*$$

Received November 18, 1977.

[1]Research supported in part by Office of Naval Research Grant N00014-76-C-0475.

AMS 1970 *subject classification.* Primary 60G45.

Key words and phrases. Martingale inequality, maximal function, square function, weak martingale.

and

$$\|f\|_p = \sup_{1 \leqslant n < \infty} \|f_n\|_p \quad \text{for} \quad 1 \leqslant p < \infty.$$

2. The main results. We first derive an identity.

LEMMA 2.1. *Let φ be a differentiable function whose derivative φ' is an indefinite integral of φ'' such that $\varphi(0) = \varphi'(0) = 0$, φ'' is a nonnegative and even function, and such that for $n \geqslant 1$, $f_n\varphi'(f_n)$ is integrable. Define $K_i(x) = (d_i - x)^+$ if $x \geqslant 0$ and $= (d_i - x)^-$ if $x < 0$, $i \geqslant 1$. Then for $n \geqslant 1$, $\varphi(f_n)$ is integrable, and*

$$E\varphi(f_n) = (or \geqslant)\Sigma_{i=1}^n E\textstyle\int_{-\infty}^{\infty}\varphi''(f_{i-1} + x)K_i(x)dx. \tag{2.1}$$

Furthermore, for $i \geqslant 1$,

$$\textstyle\int_{-\infty}^{\infty}K_i(x)dx = \frac{1}{2}d_i^2. \tag{2.2}$$

PROOF. Since the proof of (2.2) is easy, we omit it here. Since $\varphi(0) = \varphi'(0) = 0$, we have $\varphi'(x) = \int_0^x\varphi''(t)dt$ and $\varphi(x) = \int_0^x\varphi'(t)dt$. It follows that φ' is an odd function and φ an even function. Therefore

$$\begin{aligned} 0 \leqslant \varphi(x) &= \textstyle\int_0^{|x|}\varphi'(t)dt = |x|\varphi'(|x|) - \int_0^{|x|}t\varphi''(t)dt \\ &\leqslant x\varphi'(x). \end{aligned} \tag{2.3}$$

The integrability of $\varphi(f_n)$ then follows from that of $f_n\varphi'(f_n)$. We also need the integrability of $d_i\varphi'(f_{i-1})$ for $i \geqslant 1$. Since $\varphi'' \geqslant 0$, φ' is nondecreasing. Therefore

$$|f_i|\varphi'(|f_{i-1}|)I(|f_{i-1}| \leqslant |f_i|) \leqslant |f_i|\varphi'(|f_i|)$$

and

$$|f_i|\varphi'(|f_{i-1}|)I(|f_{i-1}| > |f_i|) \leqslant |f_{i-1}|\varphi'(|f_{i-1}|).$$

Hence

$$\begin{aligned} |d_i\varphi'(f_{i-1})| &= |d_i|\varphi'(|f_{i-1}|) \\ &\leqslant |f_i|\varphi'(|f_{i-1}|) + |f_{i-1}|\varphi'(|f_{i-1}|) \\ &\leqslant |f_i|\varphi'(|f_i|) + 2|f_{i-1}|\varphi'(|f_{i-1}|). \end{aligned}$$

This implies the integrability of $d_i\varphi'(f_{i-1})$. We now derive (2.1). The left hand side of (2.1) is equal to

$$\begin{aligned} &\Sigma_{i=1}^n E[\varphi(f_i) - \varphi(f_{i-1})] \\ &\quad = (\text{or} \geqslant)\Sigma_{i=1}^n E[\varphi(f_i) - \varphi(f_{i-1}) - d_i\varphi'(f_{i-1})] \\ &\quad = \Sigma_{i=1}^n E\{\textstyle\int_0^{d_i}\int_0^y\varphi''(f_{i-1} + x)dxdy\}I(d_i \geqslant 0) \\ &\qquad + \Sigma_{i=1}^n E\{\textstyle\int_{d_i}^0\int_y^0\varphi''(f_{i-1} + x)dxdy\}I(d_i < 0). \end{aligned} \tag{2.4}$$

Now $\varphi'' \geqslant 0$. So we may reverse the order of the double integration in (2.4). *By this, the extreme right hand side of* (2.4) *yields*

$$\Sigma_{i=1}^n E\textstyle\int_{-\infty}^{\infty}\varphi''(f_{i-1} + x)K_i(x)dx$$

and the lemma is proved.

In the case where f is a martingale or a nonnegative submartingale, let τ be a stopping time. By replacing f in (2.1) by the stopped martingale or nonnegative submartingale f^τ, we obtain

$$E\varphi(f_n^\tau) \geqslant \Sigma_{i=1}^n EI(\tau \geqslant i)\int_{-\infty}^{\infty}\varphi''(f_{i-1} + x)K_i(x)dx \tag{2.5}$$

where equality holds in the martingale case. If the differences of f are mutually independent with zero means, $\varphi(x) = x^2$, and $\tau = \inf\{n : |f_n| > a\}$ where $a > 0$, then (2.5) immediately yields Kolmogorov's two inequalities in the proof of the three series theorem.

THEOREM 2.1. *Let ψ' be a nonnegative and even function which is nonincreasing on* $[0, \infty)$, *and let* $\psi(x) = \int_0^x \psi'(t)dt$. *Then*

$$ES^2(f)\psi'(f^*) \leqslant 2 \sup_n E|f_n|\psi(|f_n|). \tag{2.6}$$

PROOF. There is nothing to prove if the right-hand side of (2.6) is infinite. So we assume it to be finite. Let $K_i(x)$ be as in Lemma 2.1. It is not difficult to see that, for $i \geqslant 1$, $f_{i-1} + x$ lies between f_{i-1} and f_i on $\{x : K_i(x) > 0\}$. Now let $\varphi'' = \psi'$, $\varphi'(x) = \int_0^x\varphi''(t)dt = \psi(x)$ and $\varphi(x) = \int_0^x\varphi'(t)dt$. Then the integrability condition in Lemma 2.1 is satisfied and the lemma immediately yields

$$ES_n^2(f)\psi'(f_n^*) \leqslant 2E\varphi(f_n) \leqslant 2E|f_n|\psi(|f_n|) \leqslant 2 \sup_n E|f_n|\psi(|f_n|)$$

where the second inequality follows from (2.3). By letting $n \to \infty$ and applying Fatou's lemma, the theorem is proved.

We now deduce from (2.6) three special cases.

COROLLARY 2.1. *We have*

$$E\frac{S^2(f)}{1 + f^{*2}} \leqslant \pi\|f\|_1; \tag{2.7}$$

$$E\frac{S^2(f)}{1 + f^*} \leqslant 2 \sup_n E|f_n| \log(1 + |f_n|); \tag{2.8}$$

$$E\frac{S^2(f)}{f^{*2-p}} \leqslant \frac{2}{p-1}\|f\|_p^p, \qquad 1 < p \leqslant 2. \tag{2.9}$$

PROOF. For (2.7), let $\psi'(x) = (1 + x^2)^{-1}$; and for (2.8), let $\psi'(x) = (1 + |x|)^{-1}$. For (2.9), we first let $\psi'(x) = (a + |x|)^{p-2}$ where $a > 0$ and then let $a \downarrow 0$.

The inequality (2.7) generalizes and refines a result of Austin (1966) who proved that the square function of an L_1-bounded martingale is square integrable on any set where the maximal function is bounded above. Also Corollary 2.1 is quite surprising since the sequences f in (2.7)–(2.9) can diverge a.s. To construct such f, let $f_1 = h_{\theta_1}$ and $f_n = h_{\theta_n}/\Pi_{i=1}^{n-1}\cos(\theta_{i+1} - \theta_i)$ for $n \geqslant 2$, where h_{θ_n} is as defined in Starr (1965).

3. Applications. In this section we use Corollary 2.1 to give new proofs of Burkholder's $L \log L$ and L_p (for $1 < p \leqslant 2$) inequalities for the square function of a martingale or a nonnegative submartingale. These inequalities were first proved by Burkholder (1966). Since then different proofs have been given (see, for example, Gordon (1972), Burkholder (1973), Chao (1973) and Garsia (1973)).

THEOREM 3.1. *Let $f = (f_1, f_2, \cdots)$ be a martingale or a nonnegative submartingale. Then*

$$ES(f) \leqslant 2\left(\frac{e}{e-1}\right)^{\frac{1}{2}}\left[1 + \sup_n E|f_n| \log^+|f_n|\right]. \tag{3.1}$$

PROOF. We shall use the following inequality which dates back to Young (1913). It can also be found in Doob (1953).

$$a \log^+ b \leqslant a \log^+ a + be^{-1} \text{ for } a \geqslant 0 \text{ and } b \geqslant 0. \tag{3.2}$$

Replacing f_i by $\lambda^{-1}f_i$ in (2.8) of Corollary 2.1 where $\lambda = Ef^*$, we obtain

$$E\frac{S^2(f)}{\lambda + f^*} \leqslant 2 \sup_n E|f_n| \log(1 + \lambda^{-1}|f_n|) \tag{3.3}$$

which by (3.2) is less than or equal to

$$2 \sup_n\left[E|f_n| \log^+|f_n| + (\lambda e)^{-1}(\lambda + E|f_n|)\right] \leqslant 2\left[1 + \sup_n E|f_n| \log^+|f_n|\right].$$

Now applying the Cauchy-Schwarz inequality to

$$ES(f) = E\left(\frac{S^2(f)}{\lambda + f^*}\right)^{\frac{1}{2}}(\lambda + f^*)^{\frac{1}{2}}$$

and using (3.3) and the following inequality of Doob (1953) for submartingales,

$$\|f^*\|_1 \leqslant \left(\frac{e}{e-1}\right)\left[1 + \sup_n E|f_n| \log^+|f_n|\right],$$

we obtain (3.1). This proves the theorem.

THEOREM 3.2. *Let $f = (f_1, f_2, \cdots)$ be a martingale or a nonnegative submartingale. Then for $1 < p \leqslant 2$,*

$$\|S(f)\|_p \leqslant 2^{\frac{1}{2}} p^{\frac{1}{2}} q \|f\|_p \tag{3.4}$$

where $p^{-1} + q^{-1} = 1$.

PROOF. Applying Hölder's inequality to

$$\|S(f)\|_p^p = E\left(\frac{S^2(f)}{f^{*2-p}}\right)^{\frac{1}{2}p} (f^{*p})^{1-\frac{1}{2}p},$$

we obtain

$$\|S(f)\|_p^p \leqslant \left(E\frac{S^2(f)}{f^{*2-p}}\right)^{\frac{1}{2}p} (\|f^*\|_p^p)^{1-\frac{1}{2}p}$$

which by (2.9) of Corollary 2.1 is less than or equal to

$$\left(\frac{2}{p-1}\|f\|_p^p\right)^{\frac{1}{2}p}(\|f^*\|_p^p)^{1-\frac{1}{2}p}.$$

This together with the following inequality of Doob (1953) for submartingales,

$$\|f^*\|_p \leqslant q\|f\|_p \qquad \text{for} \quad 1 < p < \infty, p^{-1} + q^{-1} = 1,$$

imply

$$\|S(f)\|_p \leqslant \left(\frac{2}{p-1}\right)^{\frac{1}{2}} q^{1-\frac{1}{2}p}\|f\|_p \leqslant 2^{\frac{1}{2}}p^{\frac{1}{2}}q\|f\|_p.$$

This proves the theorem.

The absolute constants in (3.1) and (3.4) seem to be the lowest ever obtained.

Acknowledgment. The author wishes to thank the referee for suggesting the example in Starr (1965).

REFERENCES

[1] Austin, D. G. (1966). A sample function property of martingales. *Ann. Math. Statist.* **37** 1396–1397.

[2] Berman, Jeremy (1976). An example of a weak martingale. *Ann. Probability* **4** 107–108.

[3] Burkholder, D. L. (1966). Martingale transforms. *Ann. Math. Statist.* **37** 1494–1504.

[4] Burkholder, D. L. (1973). Distribution function inequalities for martingales. *Ann. Probability* **1** 19–42.

[5] Chao, Jia-Arng (1973). A note on martingale square functions. *Ann. Probability* **1** 1059–1060.

[6] Doob, J. L. (1953). *Stochastic Processes*. Wiley, New York.

[7] Garsia, Adriano M. (1973). *Martingale Inequalities*: *Seminar Notes on Recent* Benjamin, Reading, Massachusetts.

[8] Gordon, Louis (1972). An equivalent to the martingale square function inequality. *Ann. Math. Statist.* **43** 1927–1934.

[9] Nelson, Paul I. (1970). A class of orthogonal series related to martingales. *Ann. Math. Statist.* **41** 1684–1694.

[10] Starr, Norton (1965). On an operator limit theorem of Rota. *Ann. Math. Statist.* **36** 1864–1866.

[11] Young, W. H. (1913). On a certain series of Fourier. *Proc. London Math. Soc.* **11** 357–366.

Department of Mathematics
University of Singapore
Bukit Timah Road
Singapore 10