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A GENERAL RESULT ON INFINITE DIVISIBILITY

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Using and refining a technique developed by O. Thorin, we prove:

THEOREM. *Let* $f(x) = C \cdot x^{\beta-1}h(x)$, $x > 0$, *be a probability density on* $(0, \infty)$. *Here* $\beta > 0$ *and h is continuous and satisfies* $h(0) = 1$. *Assume that h can be analytically continued to the whole complex plane cut along the negative real axis and assume that h satisfies some other regularity assumptions. If h is completely monotone on* $(0, \infty)$ *and if, for each fixed* $u > 0$, *the function* $h(uv(t))h(u/v(t))$, *where* $v(t) = t + 1 + (t^2 + 2t)^{\frac{1}{2}}$, *is completely monotone on* $(0, \infty)$, *then* $f(x)$ *is the density of a generalized gamma convolution and hence infinitely divisible.*

The theorem is applied to show the infinite divisibility of a rather large class of probability densities on $(0, \infty)$. In particular we show that a power with exponent of modulus > 1 of the ratio of two gamma distributed rv's has an infinitely divisible distribution.

1. Introduction and summary. The concept of infinite divisibility plays an important role in probability theory; see, e.g, Feller (1971). All infinitely divisible characteristic functions are obtained from the classical Lévy-Khintchine representation formula. However, this formula is usually of little help when one wants to verify that a given distribution is infinitely of little help when one wants to verify that a given distribution function is infinitely divisible.

Thorin (1977a, b) introduced the so-called generalized gamma convolutions, which trivially are infinitely divisible, and developed a technique that made it possible for him to show that both the Pareto and the lognormal distributions are generalized gamma convolutions. Further results were later obtained by Bondesson (1978), Goovaerts et al. (1977a, b, 1978), and Thorin (1978a, b).

To prove that a distribution on $[0, \infty)$ is a generalized gamma convolution one has to show that $\varphi'(-s)/\varphi(-s)$, where φ is the moment generating function (m.g.f.) of the distribution, is, apart from an additive constant, the Stieltjes transform of a nonnegative measure on $(0, \infty)$. That the validity of such a representation is a sufficient condition for infinite divisibility was independently of Thorin's work observed by Grosswald (1976), and used by him to show that the inverse of a gamma variable is infinitely divisible. Later Ismail and Kelker (1977), in particular, made use of Grosswald's observation to obtain further results. Like Goovaerts et al. and Grosswald they depended heavily on many (for a probabilist not very fascinating) known facts about certain special functions.

For a more detailed review of results obtained earlier by Thorin's (Grosswald's) method, see Section 5.

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In the present paper, the content of which has its roots in Thorin's significant pioneering contributions and also in the above-mentioned earlier paper by the present author, the infinite divisibility of a large class of probability densities on $(0, \infty)$ is established. See Theorem 2 in Section 4 and also Section 6. This class has the somewhat remarkable property that if X is a random variable (rv) with density in the class, then the class contains also the density of X^q for any real q, $|q| \geqslant 1$. The class includes the densities of gamma variables and ratios of two gamma variables and hence also the densities of powers with exponent of modulus $\geqslant 1$ of such variables. The result thus affirmatively answers questions asked independently by Bondesson (1978) and Ismail and Kelker (1977). These questions were partly answered by Thorin (1978b) who obtained the result on the infinite divisibility of powers of gamma variables.

In the proof of the general theorem (Theorem 1 in Section 3), facts from the theory of analytic functions will be used but no facts about special functions. When applying the theorem we shall use a crucial observation of Thorin (1978b), namely that, for $|\alpha| < 1$, the function $(v(t))^\alpha + (v(t))^{-\alpha}$, where $v(t) = t + 1 + (t^2 + 2t)^{\frac{1}{2}}$, has a completely monotone derivative in $t > 0$. Thorin proved this result by the aid of special functions.

2. Generalized gamma convolutions. For the readers' convenience we shall here quote a definition and two theorems due to Thorin (1977a, b) and make some comments.

DEFINITION (Thorin 1977a). A generalized gamma convolution is a distribution function $F(x)$ over $[0, \infty)$ such that, for $\operatorname{Re} s \leqslant 0$,

$$\varphi(s) = \int_{0-}^{\infty} e^{sx}\, dF(x) = \exp\left\{ as + \int_0^\infty \log\left\{ \frac{1}{1 - s/t} \right\} dU(t) \right\} \tag{2.1}$$

(where log stands for the branch which is real for negative s), where $a \geqslant 0$, and where $U(t)$ is right-continuous and nondecreasing and satisfies $U(0) = 0$, $\int_0^1 |\log t|\, dU(t) < \infty$, and $\int_1^\infty t^{-1}\, dU(t) < \infty$.

Note that the right-hand side of (2.1) is an analytic function in $C \setminus [0, \infty)$. It should also be remarked that a and U are uniquely determined by F. In fact, $a = \sup\{x;\ F(x) = 0\}$ by, e.g., Theorem 11.1.2 in Lukacs (1970). Further (2.1) implies that

$$\varphi'(s)/\varphi(s) = a + \int_0^\infty (t - s)^{-1}\, dU(t),$$

from which it is seen that the uniqueness of U follows from the uniqueness theorem for the Stieltjes transform; see, e.g., Widder (1946).

The generalized gamma convolutions are exactly those distributions on $[0, \infty)$ that can appear as (proper) weak limits of finite convolutions of gamma distributions and are thus infinitely divisible. This is more or less intuitively clear but also a consequence of the following "continuity theorem." ("Closure theorem" might have been a more appropriate name.)

CONTINUITY THEOREM (Thorin 1977b). *If a sequence* $(F_n)_{n=1}^{\infty}$ *of generalized gamma convolutions converges weakly to a* df F *as* $n \to \infty$, *then also* F *is a generalized gamma convolution and (with an obvious notation) for all finite continuity points* t *of* U

$$U(t) = \lim_{n\to\infty} U_n(t)$$

and

$$a = \lim_{A\to\infty} \lim_{n\to\infty}\left(a_n + \int_A^{\infty} t^{-1}\, dU_n(t)\right).$$

Note that the sequence $a_n = 0$, $U_n(t) = n \cdot \varepsilon(t - n/a)$, $n = 1, 2, \cdots$, where ε is a df degenerate at 0, in the limit gives a generalized gamma convolution degenerate at the point a.

REMARK 2.1. Every nondegenerate generalized gamma convolution F is absolutely continuous with continuous density for $x > 0$. For, if $0 < U(\infty) \leqslant 1$ and $a = 0$, then F is a scale mixture of the exponential distribution (see Thorin 1977b) and if $U(\infty) > 1$, then the characteristic function $\varphi(\mathrm{it})$ is absolutely integrable. For a more general result, see Remark 3.5 in Section 3.

REMARK 2.2. It is easy to see that a stable distribution on $(0, \infty)$ is a generalized gamma convolution. Thorin (1978a) extended the definition of generalized gamma convolution to include also distributions on $(-\infty, \infty)$ and showed that every stable distribution is an extended generalized gamma convolution.

The next theorem is a basic result for this paper. By C_+ we mean the complex s-plane cut along the positive real axis. We look upon C_+ as closed and denote its interior by C_+^0. We also use the convention that a real positive number always means a point on the upper side of the cut.

INVERSION THEOREM (Thorin 1977b). *Assume that* $\varphi(s) = \int_{0-}^{\infty} e^{sx}\, dF(x)$, $\operatorname{Re} s \leqslant 0$, *can be analytically continued to a zero-free function in* C_+^0. *Assume also that* φ *and* φ' *become continuous up to the boundary of* C_+ *(except possibly* φ' *at* $s = 0$) *and that* φ *has zero-free boundary values. Assume further that* $U(t) = \pi^{-1} \arg(\varphi(t))$ *is a nondecreasing function in* $t > 0$ (i.e., *that* $\varphi(t)$ *goes in the positive direction around the origin as* t *increases from* 0 *to* ∞) *and satisfies the conditions for the function* U *in the definition above. (The condition* $U(0) = 0$ *is trivially satisfied since* $\arg(\varphi(0)) = \arg(1) = 0$ *by definition.) Finally, assume that* $\varphi'(s)/\varphi(s) \to a \geqslant 0$ *uniformly as* $|s| \to \infty$, $s \in C_+$. *Then* F *is a generalized gamma convolution defined by* a *and* U.

The proof of the inversion theorem is very easy. Observe that $U'(t) = \pi^{-1}(d/dt) \operatorname{Im}[\log \varphi(t)] = \pi^{-1} \operatorname{Im}[\varphi'(t)/\varphi(t)]$ for $t > 0$; therefore it suffices to show that

$$\varphi'(s)/\varphi(s) = a + \frac{1}{\pi}\int_0^{\infty}(t - s)^{-1} \operatorname{Im}[\varphi'(t)/\varphi(t)]\, dt.$$

The result thus follows by an application of Cauchy's integral formula to $\varphi'(s)/\varphi(s) - a$ with a contour consisting of two straight lines on the two sides of the cut and of two circles centered at the origin, one with a very large radius and one with a very small radius.

Observe that $U(t) = \pi^{-1}\arg(\varphi(t))$ satisfies $\int_0^1 |\log t|\, dU(t) < \infty$ if there exists $\varepsilon > 0$ such that $\varphi'(t) = 0(t^{-1+\varepsilon})$ as $t \to 0$, $t > 0$.

3. The main theorem. We consider (probability) density functions on $(0, \infty)$ of the form

$$f_\beta(x) = C_\beta x^{\beta-1} h(x), \qquad x > 0,$$

where $\beta > 0$ and where C_β is a normalization constant. The following assumptions are made about the function h. By C_- is meant the whole complex z-plane cut along the negative real axis. Also C_- is looked upon as closed.

(1) The function h can be analytically continued to a function $h(z)$ in C_-^0.

(2) The function $h(z)$ is continuously differentiable up to the boundary of C_- and $h(0) = 1$. As an exception, $h'(z)$ is permitted to be discontinuous at $z = 0$ in C_-.

(3) For every admissible value of β there exists $\varepsilon > 0$ such that $h(z) = 0(|z|^{-(\beta+\varepsilon)})$ as $|z| \to \infty$, $z \in C_-$.

Note that if assumption (3) is satisfied for $\beta = \beta_0$, so it is for all $\beta \in (0, \beta_0]$. Observe also that since $\overline{h(\bar{z})}$ is an analytic function coinciding with $h(z)$ on the real positive axis, necessarily $h(\bar{z}) = \overline{h(z)}$, $z \in C_-$.

As a simple example of a function h satisfying the general assumptions above we may take

$$h(x) = (1 + cx^\alpha)^{-\gamma}, \qquad x > 0,$$

where $0 < \alpha < 1$, $\gamma > 0$, and $c > 0$. Clearly $x^{\beta-1}h(x)$ can be normalized to be a density function for all $\beta \in (0, \alpha\gamma)$.

The following theorem is the main result in this paper.

THEOREM 1. *Let h satisfy the assumptions* (1)–(3) *above; and let the following conditions hold.*

(A) *The function h is completely monotone on* $(0, \infty)$.

(B) *For every fixed $u > 0$, the function $h(uv(t))h(u/v(t))$, where $v(t) = t + 1 + (t^2 + 2t)^{\frac{1}{2}}$, is nonconstant and completely monotone on* $(0, \infty)$.

Then $f_\beta(x) = C_\beta x^{\beta-1}h(x)$, $x > 0$, is the density of a generalized gamma convolution with $a = 0$ and $U(\infty) = \beta$ and hence infinitely divisible.

REMARK 3.1. Note that $f_\beta(x)$ is completely monotone for $\beta \leqslant 1$; hence it follows immediately from the Goldie-Steutel theorem (see, e.g, Feller (1971), page 452 and Steutel (1973)) that infinite divisibility holds when $\beta \leqslant 1$. However, the most interesting cases occur when $\beta > 1$. Cf. Section 5.

PROOF. Following Thorin (1977b, 1978b), we introduce

$$f_{\beta,n}(x) = \frac{x^{n-1}}{(n-1)!}\int_0^\infty (n/y)^n \exp\{-nx/y\} f_\beta(y)\, dy, \qquad n = 1, 2, \cdots .$$

If X is an rv with density $f_\beta(x)$, then $f_{\beta,n}(x)$ is the density of $Y \cdot X$, where Y is gamma distributed with shape parameter n and scale parameter $1/n$. The m.g.f. of $f_{\beta,n}(x)$ is then given by

$$\varphi_{\beta,n}(s) = \int_0^\infty e^{sx} f_{\beta,n}(x)\,dx = \int_0^\infty \left(1 - \frac{sx}{n}\right)^{-n} f_\beta(x)\,dx, \qquad s \leqslant 0.$$

Clearly $\int_0^x f_{\beta,n}(y)\,dy \to \int_0^x f_\beta(y)\,dy$ weakly as $n \to \infty$. We shall show that, for $n > \beta$, the function $f_{\beta,n}(x)$ is the density of a generalized gamma convolution and hence, by the continuity theorem, so is $f_\beta(x)$.

Obviously $\varphi_{\beta,n}(s)$, $s \leqslant 0$, can be analytically continued to C_+^0. For, substituting $y = -sx$, we get

$$\varphi_{\beta,n}(s) = C_\beta(-s)^{-\beta} \int_0^\infty \left(1 + \frac{y}{n}\right)^{-n} y^{\beta-1} h(y/(-s))\,dy, \tag{3.1}$$

and the function on the right-hand side is (by the assumptions and a well-known theorem of Weierstrass) analytic in C_+^0. Since analogously

$$\varphi'_{\beta,n}(s) = C_\beta(-s)^{-(\beta+1)} \int_0^\infty \left(1 + \frac{y}{n}\right)^{-n-1} y^{\beta} h(y/(-s))\,dy, \tag{3.2}$$

the function $\varphi_{\beta,n}(s)$ is also (by the assumptions) continuously differentiable on the boundary of C_+ outside the origin. Substituting then in (3.1), for $s \in C_+ \setminus \{0\}$, $x = y/|s|$, we get

(3.3)

$$\varphi_{\beta,n}(s) = C_\beta \exp\{-i\beta \arg(-s)\} \int_0^\infty \left(1 + \frac{|s|x}{n}\right)^{-n} x^{\beta-1} h(x \exp\{-i \arg(-s)\})\,dx,$$

where $-\pi \leqslant \arg(-s) \leqslant \pi$. We have $\varphi_{\beta,n}(s) \to 1$ as $s \to 0$, $s \in C_+$. This follows from the fact that $\varphi_{\beta,n}(s) \to 1$ as $s \to 0$ along any differentiable path in C_+ through the origin (which is seen from (3.3) by calculus of residues) and from a general theorem on continuity of functions of two (or several) variables. Cf. the proof of Lemma 4.1 in Bondesson (1978).

From (3.3) we get in particular for $s > 0$

$$\varphi_{\beta,n}(s) = C_\beta \exp\{i\beta\pi\} \int_0^\infty \left(1 + \frac{sx}{n}\right)^{-n} x^{\beta-1} h(-x)\,dx. \tag{3.4}$$

Here we have used the convention that, for $s > 0$, $\varphi_{\beta,n}(s)$ denotes the value of $\varphi_{\beta,n}$ taken at the point s on the upper side of the cut along the positive real s-axis, and that, for $x > 0$, $h(-x)$ denotes the value of h at $-x$ on the upper side of the cut along the negative real axis.

From (3.1), (3.2), the dominated convergence theorem, and the assumption that $h(z)$ is continuous at the origin with $h(0) = 1$, it follows that for $n > \beta$

$\varphi'_{\beta,n}(s)/\varphi_{\beta,n}(s) =$

$$\begin{aligned} &-\frac{1}{s}\int_0^\infty \left(1 + \frac{y}{n}\right)^{-n-1} y^{\beta} h(-y/s)\,dy \Big/ \int_0^\infty \left(1 + \frac{y}{n}\right)^{-n} y^{\beta-1} h(-y/s)\,dy \\ &\sim -\frac{1}{s}\int_0^\infty \left(1 + \frac{y}{n}\right)^{-n-1} y^{\beta}\,dy \Big/ \int_0^\infty \left(1 + \frac{y}{n}\right)^{-n} y^{\beta-1}\,dy = -\frac{\beta}{s} \end{aligned} \tag{3.5}$$

as $|s| \to \infty$, $s \in C_+$. The asymptotic relation holds uniformly in C_+. Hence the last condition in Thorin's inversion theorem holds with $a = 0$ for $\varphi_{\beta, n}$.

We also easily find that

$$\varphi'_{\beta, n}(s) = c_{\beta, n}\varphi_{\beta+1, n+1}(s'), \qquad s \in C_+, \tag{3.6}$$

where $s' = (1 + n^{-1})s$ and $c_{\beta, n}$ is a positive constant.

In view of Thorin's inversion theorem, it is now certainly sufficient to prove that:

(a) for every $n \geq 1$, $\varphi_{\beta, n}(s)$ has no zeroes on the positive real axis;

(b) for every $n \geq 1$, $\arg(\varphi_{\beta, n}(s))$ is a nondecreasing function in $s > 0$;

(c) for $n > \beta$, $\varphi_{\beta, n}(s)$ has no zeroes in C_+;

(d) for $n > \beta$, $\arg(\varphi_{\beta, n}(+\infty)) < \infty$ and there exists $\varepsilon > 0$ such that $\varphi'_{\beta, n}(s) = O(s^{-1+\varepsilon})$ as $s \to 0$, $s > 0$.

If (*a*) holds, then we have, for $s > 0$,

$$\frac{d}{ds}\arg(\varphi_{\beta, n}(s)) = \operatorname{Im}\left[\varphi'_{\beta, n}(s)\,\overline{\varphi_{\beta, n}(s)}\,\right]/|\varphi_{\beta, n}(s)|^2.$$

Thus, to prove (a) and (b) it suffices to show that

$$\operatorname{Im}\left[\varphi'_{\beta, n}(s)\,\overline{\varphi_{\beta, n}(s)}\,\right] > 0, \qquad s > 0,$$

or equivalently, by (3.6),

$$\operatorname{Im}\left[\varphi_{\beta+1, n+1}(s')\,\overline{\varphi_{\beta, n}(s)}\,\right] > 0, \qquad s > 0. \tag{3.7}$$

From (3.4) we then find that we just have to show that

$$\Delta_{\beta, n}(s) = \int_0^\infty\int_0^\infty\left(1 + \frac{sx}{n}\right)^{-n-1}\left(1 + \frac{sy}{n}\right)^{-n} x^{\beta}y^{\beta-1}\operatorname{Im}\left[h(-x)\,\overline{h(-y)}\,\right]dx\,dy < 0.$$

Making the substitution (in a slightly more complicated form also used by Thorin (1977b, 1978b))

$$x = u \cdot v, \qquad y = u/v$$

and using the notation u' for su/n, we obtain

$$\Delta_{\beta, n}(s) = 2\int_0^\infty\int_0^\infty(1 + u'v)^{-n-1}\left(1 + \frac{u'}{v}\right)^{-n} u^{2\beta}\operatorname{Im}\left[h(-uv)\,\overline{h(-u/v)}\,\right]du\,dv$$

$$= 2\int_0^\infty u^{2\beta}\left(\int_0^\infty\left(1 + \frac{u'}{v}\right)\left((1 + u'v)\left(1 + \frac{u'}{v}\right)\right)^{-n-1}\operatorname{Im}\left[h(-uv)\,\overline{h(-u/v)}\,\right]dv\right)du.$$

As clearly $\chi(v) = ((1 + u'v)(1 + u'/v))^{-n-1}\operatorname{Im}[h(-uv)\overline{h(-u/v)}]$ satisfies $\chi(1/v) = -\chi(v)$ and hence $\int_0^\infty v^{-1}\chi(v)dv = 0$, the inner v-integral equals

$$\int_0^\infty\left((1 + u'v)\left(1 + \frac{u'}{v}\right)\right)^{-n-1}\operatorname{Im}\left[h(-uv)\,\overline{h(-u/v)}\,\right]dv$$

$$= (u')^{-n-1}\int_0^\infty\left(u' + \frac{1}{u'} + v + \frac{1}{v}\right)^{-n-1}\operatorname{Im}\left[h(-uv)\,\overline{h(-u/v)}\,\right]dv.$$

We now set $u' + 1/u' = 2(t + 1)$ and show that, for every fixed $u > 0$ and all $t > 0$,

$$\psi_n(t) = \int_0^\infty \left(2(t + 1) + v + \frac{1}{v}\right)^{-n-1} \operatorname{Im}\left[h(-uv)\overline{h(-u/v)}\right] dv < 0.$$

This is clearly sufficient for $\Delta_{\beta,n}(s) < 0$ to hold. We observe that

$$\psi_n(t) = \frac{(-1)^n 2^{-n}}{n!}\psi_0^{(n)}(t) \tag{3.8}$$

as the integral defining $\psi_n(t)$ is convergent also for $n = 0$. In fact, $\psi_0(t)$ can be evaluated by applying calculus of residues to the function

$$h(uz)h(u/z)\Big/\left(2(t + 1) - \left(z + \frac{1}{z}\right)\right), \qquad z \in C_-.$$

We integrate along a contour consisting of two straight lines on the two sides of the cut along the negative axis and of two circles centered at the origin, one with radius tending to infinity and the other with radius tending to zero. Observe that $1/z$ is on the lower side of the cut when z is on the upper. This explains the vanishing of the conjugate sign above. There are just two simple poles, namely $z = t + 1 \pm (t^2 + 2t)^{\frac{1}{2}}$. Note that their product is 1 and that $h(uz)h(u/z)$ thus takes the same value at the two poles. Setting $v(t) = t + 1 + (t^2 + 2t)^{\frac{1}{2}}$, we then get after some calculation

$$\psi_0(t) = -\pi h(uv(t))h(u/v(t)).$$

By condition (B), $-\psi_0(t)$ is nonconstant and completely monotone and hence the derivatives of $\psi_0(t)$ alternate between strictly positive and strictly negative values, i. e., by (3.8), $\psi_n(t) < 0$ for all $t > 0$. Thus (a) and (b) are proved.

We now turn to (c). First we note that

$$\operatorname{Im}\left[\varphi_{\beta,n}(s)\right] = \pi s\frac{(-1)^{n-1}}{n!}\left(\frac{n}{s}\right)^{n+1} f_\beta^{(n-1)}\left(\frac{n}{s}\right), \qquad s < 0, \tag{3.9}$$

as shown by Thorin (1978b). The formula can also be obtained by integrating the defining expression for $\varphi_{\beta,n}(s)$ by parts and using the inversion formula for the Stieltjes transform. As $h(x)$ and $x^{\beta-1}$, $0 < \beta \leqslant 1$, are both completely monotone so is $f_\beta(x)$ for $\beta \in (0, 1]$. The formula (3.9) thus shows that, for all $n \geqslant 1$,

$$\operatorname{Im}\left[\varphi_{\beta,n}(s)\right] \geqslant 0, \qquad s \geqslant 0, \qquad 0 < \beta \leqslant 1. \tag{3.10}$$

Let now $N_{\beta,n}$ stand for the number of zeroes of $\varphi_{\beta,n}(s)$ in C_+. Clearly, we may have $N_{\beta,n} = \infty$. Since $\varphi_{\beta,n}(s)$ takes on conjugate values for conjugate s-values and since $\arg(\varphi_{\beta,n}(s))$, $s \in C_+$, is continuous at the origin, we have by a standard formula from complex analysis

$$N_{\beta,n} = \lim_{R\to\infty}\left(\frac{1}{2\pi i}\int_{C(R,\bar{R})}\varphi'_{\beta,n}(s)/\varphi_{\beta,n}(s)\,ds + \frac{1}{\pi}\arg(\varphi_{\beta,n}(R))\right), \tag{3.11}$$

where $C(R, \bar{R})$ is a positively directed (at the point R cut) circle with center at the origin and radius R. Hence and by (3.5), for $n > \beta$,

$$N_{\beta,n} = -\beta + \frac{1}{\pi}\arg(\varphi_{\beta,n}(+\infty)). \tag{3.12}$$

The formula (3.10) shows that $0 \leqslant \arg(\varphi_{\beta,n}(+\infty)) \leqslant \pi$ for $0 < \beta \leqslant 1$; hence (3.12) implies that $N_{\beta,n} \leqslant -\beta + 1$ and thus $N_{\beta,n} = 0$ for $0 < \beta \leqslant 1$ and $n > \beta$.

The relation (3.7) (proved true) can be written

$$0 < \arg\big(\varphi_{\beta+1,n+1}(s')\,\overline{\varphi_{\beta,n}(s)}\,\big) < \pi, \qquad s > 0,$$

and hence

$$\arg(\varphi_{\beta+1,n+1}(+\infty)) \leqslant \pi + \arg(\varphi_{\beta,n}(+\infty)). \tag{3.13}$$

For any β we may write $\beta = \beta_0 + M$, where $0 < \beta_0 \leqslant 1$ and M is a nonnegative integer. It is thus sufficient to prove that, for $k = 0, 1, \cdots, M$ and $n > \beta_0 + k$, we have $N_{\beta_0+k,n} = 0$. We use induction on k to show this.

From the above we have $N_{\beta_0,n} = 0$ for $n > \beta_0$. Assume that $N_{\beta_0+k,n} = 0$ for $n > \beta_0 + k$ with $k < M$. Hence and by (3.12), for $n > \beta_0 + k$,

$$\arg(\varphi_{\beta_0+k,n}(+\infty)) = \pi(\beta_0 + k).$$

Together with (3.13) this relation yields

$$\arg(\varphi_{\beta_0+k+1,n+1}(+\infty)) \leqslant \pi + \pi(\beta_0 + k).$$

Using again (3.12), we then get

$$N_{\beta_0+k+1,n+1} \leqslant -(\beta_0 + k + 1) + 1 + (\beta_0 + k) = 0.$$

Hence $N_{\beta_0+k+1,n+1} = 0$ for $n > \beta_0 + k$ and thus $N_{\beta_0+k+1,n} = 0$ for $n > \beta_0 + k + 1$, i.e., the induction proof is complete. The condition (c) thus holds.

Turning to (d), we note that the validity of the first condition is immediate from the above. We have $\arg(\varphi_{\beta,n}(+\infty)) = \pi\beta$. By (3.2) the second condition is easily shown to be a consequence of assumption (3) (with the same ε). The details are omitted.

We have thus shown that $f_{\beta,n}(x)$ is the density of a generalized gamma convolution for $n > \beta$ and hence so is $f_\beta(x)$. To finish the proof we must show that the corresponding U satisfies $U(\infty) = \beta$. Obviously $a = 0$. From the analogues of (3.1) and (3.2) we conclude that the m.g.f. $\varphi_\beta(s)$ of $f_\beta(x)$ is continuously differentiable up to the boundary of C_+ outside the origin and moreover continuous at the origin. Since $\varphi_\beta(s)$ is the m.g.f. of a generalized gamma convolution (with $a = 0$), we have

$$\varphi_\beta(s) = \exp\left\{\int_0^\infty \log\left(\frac{1}{1 - s/t}\right)dU(t)\right\}, \quad s \in C_+^0,$$

and hence $\varphi_\beta(s)$ has no zeroes in C_+^0. Moreover, there is no zero on the cut. For, if $\varphi_\beta(s_0) = 0$ for $s_0 > 0$, then

$$\int_0^\infty \log\left|\frac{1}{1 - s/t}\right| dU(t) = \log|\varphi_\beta(s)| \to -\infty$$

as $C_+^0 \ni s \to s_0$. However, it is easy to see that the integral is bounded from below. The assertion follows. From the analogue of (3.5) we then get

$$\lim_{R\to\infty} \frac{1}{2\pi i} \int_{C(R,\bar{R})} \varphi_\beta'(s)/\varphi_\beta(s)ds = -\beta,$$

and hence the analogue of (3.11) with $N_\beta = 0$ yields $\arg(\varphi_\beta(+\infty)) = \pi\beta$. Finally, by Thorin's inversion theorem, we conclude that $U(\infty) = \beta$. □

REMARK 3.2. In the proof of (a) and (b) above we have used Thorin's (1977b, 1978b) method in a simplified form. The proof of (c) originates from the technique utilized in Bondesson (1978). Another proof of (c) is sketched in Remark 3.4 below.

REMARK 3.3. It should be emphasized that although we prove that $\varphi_\beta(s)$ has no zeroes for $s > 0$, this is done only when we know that $\varphi_\beta(s)$ is the m.g.f. of a generalized gamma convolution. The method of approximating $\varphi_\beta(s)$ by $\varphi_{\beta,n}(s)$ was originally introduced by Thorin (1977b) in his study of the lognormal distribution in order to avoid the complication that for this distribution $U(\infty) = \infty$. However, the method also makes it possible to avoid the problem of zeroes of the m.g.f. for $s > 0$. That motivates our use of the method. If a simple proof that $\varphi_\beta(s)$ has no zeroes for $s > 0$ can be found, the above proof can be made easier if one uses $\varphi_\beta(s)$ instead of $\varphi_{\beta,n}(s)$.

REMARK 3.4. It is not hard to show that, for $n \geqslant 1$ and $\beta \leqslant 1$, $f_{\beta,n}(x)$ is completely monotone and thus the Laplace transform of a nonnegative measure $dG_{\beta,n}(t)$ on $(0, \infty)$. Consequently, for $s \in C_+^0$, $\varphi_{\beta,n}(s) = \int_0^\infty (t-s)^{-1} dG_{\beta,n}(t)$, and hence

$$\operatorname{Im}[\varphi_{\beta,n}(s)] = \int_0^\infty \frac{\operatorname{Im} s}{|t-s|^2} dG_{\beta,n}(t).$$

Thus $\operatorname{Im}[\varphi_{\beta,n}(s)] \neq 0$ for $\operatorname{Im} s \neq 0$. Since moreover $\varphi_{\beta,n}(s)$ is strictly positive for $s \leqslant 0$, we have in an alternative way proved that $N_{\beta,n} = 0$ for $\beta \leqslant 1$. (In fact, we have shown that also $N_{1,1} = 0$; cf. the proof of Theorem 1.) Furthermore, we can also establish that $N_{\beta,n} = 0$ for $n \geqslant \beta$ and all β. For, the above result together with (a), (b), and (d) show that $\varphi_{\beta,n}(s)$ is the m.g.f. of a g.g.c. for $\beta \leqslant 1$. Hence $\varphi_{\beta,n}'(s)/\varphi_{\beta,n}(s) = \int_0^\infty (t-s)^{-1}\, dU_{\beta,n}(t)$. This representation shows that, for $\beta \leqslant 1$ and $n \geqslant 1$, $\varphi_{\beta,n}'(s) \neq 0$, $s \in C_+^0$. By the help of (3.6) and (a) we then conclude that $N_{\beta,n} = 0$ for $1 < \beta \leqslant 2$ and $n \geqslant 2$. The result wanted follows by iterations of this reasoning.

REMARK 3.5. It can be shown that every generalized gamma convolution with $a = 0$ and $U(\infty) = \beta_1$, $0 < \beta_1 < \infty$, has a density of the form $f(x) = x^{\beta_1 - 1}h_1(x)$, $x > 0$, where $h_1(x)$ is completely monotone; see Bondesson (1979a). Condition (A) of Theorem 1 is thus necessary.

REMARK 3.6. The result in Remark 3.5 provides us with a simple proof of the fact that $U(\infty) = \beta$ in Theorem 1. To see this, notice first that, by the continuity theorem for generalized gamma convolutions, we have for all finite continuity points T of U

$$U(T) = \lim_{n\to\infty} U_{\beta,n}(T) \leqslant \lim_{n\to\infty} U_{\beta,n}(\infty) = \lim_{n\to\infty} \frac{1}{\pi} \arg(\varphi_{\beta,n}(+\infty)) = \beta.$$

Hence $\beta_1 = U(\infty) \leqslant \beta$. Assume that $\beta_1 < \beta$. By Remark 3.5, $f_\beta(x) = x^{\beta_1 - 1}h_1(x)$ and hence $h_1(x) = C_\beta x^{\beta-\beta_1}h(x)$. It follows that $h_1(x) \to 0$ as $x\downarrow 0$. This is clearly a contradiction since $h_1(x)$ is completely monotone.

4. Applications. As an application of Theorem 1 in Section 3 we obtain:

THEOREM 2. *All density functions on* $(0, \infty)$ *of the form*

$$f(x) = C \cdot x^{\beta-1}\Pi_{j=1}^{M}\left(1 + \Sigma_{k=1}^{N_j} c_{jk}x^{\alpha_{jk}}\right)^{-\gamma_j}, \qquad x > 0, \tag{4.1}$$

where all the parameters are strictly positive and the α_{jk}*'s less than or equal to* 1, *are generalized gamma convolutions; consequently all densities (distributions) which are weak limits of densities of the form* (4.1) *are generalized gamma convolutions (and thus infinitely divisible) as well. In the nonlimit case the total spectral mass* $U(\infty)$ *of the generalized gamma convolution is* β.

PROOF. We may assume that the α_{jk}'s are strictly less than 1, for a density of the general form can be obtained as a weak limit of densities of the restricted form. To see that $U(\infty) = \beta$ also when some α_{jk} equals 1 we may use Remark 3.6. It is now easy to verify that the obvious function h satisfies the assumptions (1)–(3) in Section 3. For $z \in C_-$, writing $z = r\cdot e^{i\theta}$, $-\pi \leqslant \theta \leqslant \pi$, we see that $\operatorname{Im}[1 + \Sigma_{k=1}^{N_j} c_{jk}z^{\alpha_{jk}}] = \Sigma_{k=1}^{N_j} c_{jk}r^{\alpha_{jk}}\sin(\alpha_{jk}\theta)$ is strictly positive (negative) for $\theta > 0$ $(\theta < 0)$ and hence it follows, e.g., that $h(z)$ has no singularities in C_-^0. Now recall Theorem 1. Since the product $h = \Pi_{j=1}^{M} h_j$ of completely monotone functions h_j, $j = 1, \cdots, M$, is completely monotone as well and as clearly

$$\Pi_{j=1}^{M} h_j(uv)\Pi_{j=1}^{M} h_j(u/v) = \Pi_{j=1}^{M} h_j(uv)h_j(u/v),$$

it is furthermore no restriction to assume $M = 1$. In the sequel we therefore drop the index j. To abbreviate the formulae below we also set $0 = \alpha_0$ and $1 = c_0$. Obviously the function $\Sigma_{k=0}^{N} c_k x^{\alpha_k}$ has a completely monotone derivative and since the function $y^{-\gamma}$, $y > 0$, is completely monotone, so is $h(x) = (\Sigma_{k=0}^{N} c_k x^{\alpha_k})^{-\gamma}$ (see, e.g., Feller (1971), page 441). It remains to show that $h(uv(t))h(u/v(t))$, where $v(t) = t + 1 + (t^2 + 2t)^{\frac{1}{2}}$, is completely monotone on $(0, \infty)$ for every fixed $u > 0$. It suffices to verify that

$$g(t) = \left(\Sigma_{k=0}^{N} c_k(uv(t))^{\alpha_k}\right)\cdot\left(\Sigma_{k=0}^{N} c_k(u/v(t))^{\alpha_k}\right)$$

has a completely monotone derivative. We easily get

$$g(t) = \Sigma_{k=0}^{N} c_k^2 u^{2\alpha_k} + \Sigma_{0\leqslant k<j} c_k c_j u^{\alpha_k+\alpha_j}((v(t))^{\alpha_k-\alpha_j} + (v(t))^{\alpha_j-\alpha_k}).$$

By, e.g., formula (24) on page 197 in Erdélyi et al. (1954), we find that the derivative of $(v(t))^{\alpha} + (v(t))^{-\alpha}$, $0 < \alpha < 1$, is, apart from a positive constant factor, the Laplace transform of $e^{-y}K_\alpha(y)$, $y > 0$, where K_α is a modified Bessel function which happens to be nonnegative for $y > 0$. This observation,, which is due to Thorin (1978b), shows that $g(t)$ has a completely monotone derivative. □

COROLLARY 1. *If an rv X has a density of the form* (4.1), *the weak limit possibilities included, so has X^q for every real q, $|q| \geqslant 1$, and hence its distribution is a generalized gamma convolution and thus infinitely divisible.*

PROOF. Clearly it is sufficient to study the case when the density of X is strictly of the form (4.1). We have $f_{X^q}(x) = |q|^{-1}x^{1/q-1}f_X(x^{1/q})$. When the constant q is positive, the result wanted immediately follows. For q negative, the result follows by dividing the factor $x^{\beta-1}$ in (4.1) by $x^{\Sigma\gamma_j}$ and multiplying each of the other factors by x^{γ_j}, respectively. □

COROLLARY 2. *All densities of the form*

$$C \cdot x^{\beta-1} \exp\{-\Sigma_{k=1}^{N} c_k x^{\alpha_k}\}, \qquad x > 0, \tag{4.2}$$

where $c_k > 0$ and $|\alpha_k| \leqslant 1$, and their weak limits are generalized gamma convolutions and thus infinitely divisible.

PROOF. Clearly the density

$$C \cdot x^{\beta-1}\left(1 + \Sigma_{k=1}^{N_1} c_{1k} x^{\alpha_{1k}}\right)^{-\gamma_1}\left(1 + \Sigma_{k=1}^{N_2} c_{2k} x^{\alpha_{2k}}\right)^{-\gamma_2}, \qquad x > 0, \tag{4.3}$$

where $0 < \alpha_{1k} \leqslant 1$ and $-1 \leqslant \alpha_{2k} < 0$, can be rewritten to the form (4.1). Letting then γ_1 and γ_2 tend to infinity and letting c_{1k}, $k = 1, \cdots N_1$, and c_{2k}, $k = 1, \cdots, N_2$, tend to zero in an appropriate way, we get the result wanted. □

REMARK 4.1. Note that $\exp\{-\Sigma_{k=1}^{N} c_k x^{\alpha_k}\}$, where $|\alpha_k| \leqslant 1$, is not necessarily completely monotone and that its analytic continuation to C_- is not bounded if $|\alpha_k| > \frac{1}{2}$ for some k. Note also that if an rv X has a density of the form (4.2), so has X^q, $|q| \geqslant 1$.

REMARK 4.2. For a density $f(x)$ of the form (4.2) with some $\alpha_k < 0$ we have $U(\infty) = \infty$. This is a simple consequence of the result in Remark 3.5 and the fact that, for every $n > 0$, $x^{-n}f(x) \to 0$ as $x \to 0$. The same conclusion holds for the lognormal distribution; cf. Section 5 and Thorin (1977b).

5. Relation to earlier results. In this section some special cases of Theorem 2 and Corollary 2 are exposed. These special cases have been considered earlier by different authors using Thorin's (Grosswald's) method. In order to simplify the identification of the various distributions encountered we give some common names of these. The names are taken from Johnson and Kotz (1970).

Theorem 2 and Corollary 2 show that all the members of the families

$$C \cdot x^{\beta-1}(1 + cx^{\alpha})^{-\gamma}, \qquad x > 0, \quad 0 < \alpha \leqslant 1, \tag{5.1}$$

$$C \cdot x^{\beta-1}\exp\{-cx^{\alpha}\}, \qquad x > 0, \quad 0 < |\alpha| \leqslant 1, \tag{5.2}$$

$$C \cdot x^{\beta-1}\exp\{-(c_1 x + c_2 x^{-1})\}, \qquad x > 0, \quad -\infty < \beta < \infty, \tag{5.3}$$

where the natural restrictions are put on the unspecified parameters, are densities of generalized gamma convolutions and thus infinitely divisible. Setting in (5.2) $\beta = \sigma^{-2}(\mu + \alpha^{-1})$ and $c = \sigma^{-2}\alpha^{-2}$, where μ and σ ($\sigma > 0$) are constants, and letting α tend to zero, we obtain as a weak limit the density

(5.4) $$C \cdot x^{-1}\exp\{-(\log x - \mu)^2/(2\sigma^2)\}, \qquad x > 0.$$

Clearly (5.1) is the general form for the density of a power with exponent $1/\alpha$ of the ratio of two gamma variables. The so-called Burr distribution of Type XII appears as a special case. Setting $\alpha = 1$, we get the density of the ratio of two gamma variables. In particular the F-distribution is included in (5.1) for $\alpha = 1$. Permitting also a change of location, we see that (5.1) for $\alpha = 1$ gives the Pearson Type VI distribution. The Pareto distribution is obtained by setting $\alpha = 1$ and $\beta = 1$.

Obviously (5.2) corresponds to a power with exponent $1/\alpha$ of a gamma variable. Common names of this distribution for $\alpha > 0$ are generalized gamma distribution and Stacy distribution. The Weibull distribution appears as a special case. For $\alpha < 0$ the extreme value distribution of Type II is a special case and so is the stable distribution with index $\frac{1}{2}$.

The distribution (5.3) has been named generalized inverse Gaussian distribution by Barndorff-Nielsen (1977). Clearly the usual inverse Gaussian distribution is a special case. Finally (5.4) corresponds to a lognormal distribution.

The infinite divisibility of the Pareto distribution is (like the infinite divisibility of the densities (5.1)–(5.2) for certain combinations of the parameters) a consequence of the Goldie-Steutel theorem (cf. Remark 3.1) and was first pointed out by Steutel (1969) (without mentioning the name of Pareto). Thorin (1977a) proved that the Pareto distribution is a generalized gamma convolution (cf. Goovaerts et al. (1977a)). Grosswald (1976) proved that the inverse of a gamma variable is infinitely divisible and thus verified a conjecture made by Ismail and Kelker (1976). His proof was simplified independently by Ismail (1977) (though a correction is needed) and by Bondesson (1978) who also pointed out that the corresponding distribution is a generalized gamma convolution. In Bondesson (1978) it was further shown that the distributions of the square and the cube of a gamma variable are generalized gamma convolutions. Also Ismail and Kelker (1977) found that the square is infinitely divisible. Thorin (1978b) proved the general result for powers of a gamma variable. This result was conjectured independently by Bondesson (1978) and Ismail and Kelker (1977). It was noted in Bondesson (1978) that Thorin's (1977b) result that the lognormal distribution is a generalized gamma convolution is a limit case. The possible infinite divisibility of the lognormal distribution was mentioned as an open problem in the survey by Steutel (1973). The infinite divisibility of the ratio of two gamma variables was established independently by Goovaerts et al. (1978) and Ismail and Kelker (1977). Like Grosswald they all utilized many known facts about special functions. Also this problem was mentioned by Steutel (1973). The infinite divisibility of a power with exponent of modulus $\geqslant 1$ of the ratio of two gamma variables was conjectured by Ismail and Kelker (1977) and also mentioned as an open problem in Bondesson (1978).

Barndorff-Nielsen and Halgreen (1977) used Grosswald's (1976) result to prove that the generalized inverse Gaussian distribution (5.3) is infinitely divisible. A more probabilistic proof was given by Barndorff-Nielsen et al. (1978). In fact, they showed that, for $\beta < 0$, (5.3) is the density of a first passage time in a certain time-homogeneous diffusion process and thus infinitely divisible (cf. Kent (1978)). Independently of the present author, Halgreen (1979) later found that the generalized inverse Gaussian distribution is a generalized gamma convolution. Halgreen proposed cross references but then he does not refer to my paper. No explanation has been given to me.

It is of interest to note that Grosswald (1976) used the infinite divisibility of the inverse of a gamma variable to show that the t-distribution is infinitely divisible for all (even all nonintegral) degrees of freedom and thus solving a problem studied before by several authors. It was earlier noted by Ismail and Kelker (1976) that the t-distribution is obtained as a variance mixture of the normal distribution with mean zero with the distribution of the inverse of a gamma variable as variance mixing distribution. Such a variance mixture is infinitely divisible whenever the mixing distribution is infinitely divisible as noticed by Kelker (1971). In essentially the same way the infinite divisibility of the generalized inverse Gaussian distribution implies that Barndorff-Nielsen's (1977) so-called generalized hyperbolic distribution is infinitely divisible. Thorin (1978a) proved that a variance mixture of the normal distribution with a generalized gamma convolution as mixing distribution is an extended generalized gamma convolution; cf. Remark 2.1.

Let us finally mention a result obtained by the help of Thorin's method that is not an immediate consequence of Theorem 2: the distribution of the product of two independent gamma variables is a generalized gamma convolution. This was proved by Goovaerts et al. (1977b). However, it can be shown that in fact this result is a consequence of Theorem 2; see Section 6 and cf. the beginning of the proof of Theorem 1.

6. Further results (added in revision). Here some results in Bondesson (1979b) are briefly mentioned.

Let us denote the class of nondegenerate generalized gamma convolutions given in Theorem 2 by $\mathfrak{B}$. Somewhat surprisingly, $\mathfrak{B}$ *coincides* with the class of distributions with densities of the form

$$f(x) = C \cdot x^{\beta - 1} \prod_{j=1}^{M} (1 + c_j x)^{-\gamma_j}, \qquad x > 0,$$

and their nondegenerate weak limits. Further, $\mathfrak{B}$ precisely consists of those distributions on $(0, \infty)$ which have densities of the form

$$f(x) = C \cdot x^{\beta - 1} h_1(x) h_2(1/x), \qquad x > 0, \tag{6.1}$$

where $\beta \in R$ and where $h_1(x)$, $h_2(x)$ are Laplace transforms of generalized gamma convolutions. The corresponding measures $dU_1(t)$, $dU_2(t)$ can be chosen to be concentrated on $(1, \infty)$ and $[1, \infty)$, respectively, and are under this condition uniquely determined by $f(x)$.

It is usually not very difficult to determine whether or not a given density admits a representation of the form (6.1).

Notice that the class of functions (not densities) of the form (6.1) is closed with respect to multiplication. On the other hand, the class $\mathscr{B}$ is not closed with respect to convolution and hence $\mathscr{B}$ does not comprise all nondegenerate generalized gamma convolutions with $a = 0$. However, in a certain sense $\mathscr{B}$ is an optimal class of generalized gamma convolutions; for details see Bondesson (1979b).

The distribution of $X \cdot Y$, where X and Y are independent rv's and X is gamma and Y has a distribution belonging to $\mathscr{B}$, belongs to $\mathscr{B}$; this important property of the class $\mathscr{B}$ is established in Bondesson (1979c).

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REFERENCES

BARNDORFF-NIELSEN, O. (1977). Exponentially decreasing distributions for the logarithm of particle size. *Proc. Roy. Soc. Ser. A* **353** 401–419.

BARNDORFF-NIELSEN, O. and HALGREEN, C. (1977). Infinite divisibility of the hyperbolic and generalized inverse Gaussian distribution. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **38** 309–311.

BARNDORFF-NIELSEN, O., BLAESILD, P. and HALGREEN, C. (1978). First hitting time models for the generalized inverse Gaussian distribution. *Stochastic Processes Appl.* **7** 49–54.

BONDESSON, L. (1978). On infinite divisibility of powers of a Gamma variable. *Scand. Actuarial J.* **1978** 48–61.

BONDESSON, L. (1979a). On generalized gamma and generalized negative binomial convolutions, Part I. *Scand. Actuarial J.* **1979** 125–146.

BONDESSON, L. (1979b). On generalized gamma and generalized negative binomial convolutions, Part II. *Scand. Actuarial J.* **1979** 147–166.

BONDESSON, L. (1979c). On the infinite divisibility of products of powers of gamma variables. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **49** 171–175.

ERDÉLYI, A. et al. (1954). *Tables of Integral Transforms*, **1**. McGraw-Hill, New York.

FELLER, W. (1971). *An Introduction to Probability Theory and Its Applications*, **2**, 2nd ed. Wiley, New York.

GOOVAERTS, M. J., D'HOOGE, L. and DE PRIL, N. (1977a). On a class of generalized gamma convolutions, Part I. *Scand. Actuarial J.* **1977**, 21–30.

GOOVAERTS, M. J., D'HOOGE, L. and DE PRIL, N. (1977b). On the infinite divisibility of the product of two Γ-distributed stochastical variables. *Appl. Math. Computation* **3** 127–135.

GOOVAERTS, M. J., D'HOOGE, L. and DE PRIL, N. (1978). On the infinite divisibility of the ratio of two gamma-distributed variables. *Stochastic Processes Appl.* **7** 291–297.

GROSSWALD, E. (1976). The Student t-distribution of any degree of freedom is infinitely divisible. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **36** 103–109.

HALGREEN, C. (1979). Self-decomposability of the generalized inverse Gaussian distribution and hyperbolic distributions. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **47** 13–17.

ISMAIL, M. E. H. (1977). Bessel functions and the infinite divisibility of the Student t-distribution. *Ann. Probability* **5** 582–585.

ISMAIL, M. E. H. and KELKER, D. (1976). The Bessel polynomials and the Student t-distribution. *SIAM J. Math. Anal.* **7** 82–91.

ISMAIL, M. E. H. and KELKER, D. (1979). Special functions, Stieltjes transforms and infinite divisibility. *SIAM J. Math. Anal.* **10** 884–901.

JOHNSON, N. L. and KOTZ, S. (1970). *Continuous Univariate Distributions*, **1** *and* **2**. Houghton-Mifflin, Boston.

KELKER, D. (1971). Infinite divisibility and variance mixtures of the normal distribution. *Ann. Math. Statist.* **42** 802–808.

KENT, J. (1978). Some probabilistic properties of Bessel functions. *Ann. Probability* **6** 760–770.

LUKACS, E. (1970). *Characteristic Functions*, 2nd ed. Griffin, London.

STEUTEL, F. W. (1969). Note on completely monotone densities. *Ann. Math. Statist.* **40** 1130–1131.

STEUTEL, F. W. (1973). Some recent results in infinite divisibility. *Stochastic Processes Appl.* **1** 125–143.

THORIN, O. (1977a). On the infinite divisibility of the Pareto distribution. *Scand. Actuarial J.* **1977** 31–40.

THORIN, O. (1977b). On the infinite divisibility of the lognormal distribution. *Scand. Actuarial J.* **1977** 121–148.

THORIN, O. (1978a). An extension of the notion of a generalized gamma convolution. *Scand. Actuarial J.* **1978** 141–149.

THORIN, O. (1978b). Proof of a conjecture of L. Bondesson concerning infinite divisibility of powers of a gamma variable. *Scand. Actuarial J.* **1978** 151–164.

WIDDER, D. V. (1946). *The Laplace Transform*. Princeton Univ. Press.

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