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HITTING SPHERES WITH BROWNIAN MOTION

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Let X_t be standard Brownian motion in R^d starting at fixed X_0, and let T be the hitting time for a sphere or concentric spherical shell. Explicit formulas, in terms of a natural Laplace-Gegenbauer transform, are obtained for the joint distribution of T and X_T.

1. Introduction. Let X_t be a standard d-dimensional Brownian motion with nonrandom starting point X_0. When $d \geqslant 2$ we seek explicit formulas which will determine the joint distributions of the first time $T \leqslant \infty$ and place X_T (which is only defined when T is finite) where X_T hits a sphere centered at the origin, either from the inside or from the outside, or exits from the region bounded by concentric spheres.

For the Brownian motion X_t we let $x = |X_0|$, $\theta_t = \angle X_0 0 X_t$ if $x \neq 0$, $\theta_t = \angle u 0 X_t$ for an arbitrary but fixed nonzero vector u, in case $x = 0$. By rotational symmetry we need only determine the joint distribution of T and θ_T. For this we need only determine the expectations $E_x(e^{-sT}\cos n\theta_T)$, $s > 0$, $n = 0, 1, 2, \cdots$, but for $d \geqslant 3$ it turns out to be more convenient first to compute a "Laplace-Gegenbauer" transform defined in the following way.

Let C_n^h be the Gegenbauer polynomial of (exact) degree n and order $h > 0$ (Watson (1944) page 50), so that $C_n^h(t)$ is the coefficient of z^n in the Maclaurin expansion of $(1 - 2tz + z^2)^{-h}$; for $h = 0$ it is customary to take $C_0^0 = 1$, $C_n^0 = \lim_{h \to 0} h^{-1} C_n^h = 2T_n/n$, T_n the nth Čebyšev polynomial: $T_n(\cos\theta) = \cos n\theta$. Note that $C_n^h(1) = (-1)^n \binom{-2h}{n}$ for $h \neq 0$, $C_n^0(1) = 2/n$ for $n \neq 0$, and therefore $C_n^h(1)$ never vanishes. For each h the polynomials C_k^h, $0 \leqslant k \leqslant n$, have the same linear span as $1, x, \cdots, x^n$; in particular $\cos n\theta$ is a linear combination of the $C_k^h(\cos\theta)$.

Let $h = (d - 2)/2$, and for $s > 0$, $n = 0, 1, 2, \cdots$ set up the processes

$$Y_n(s, t) = e^{-st} C_n^h(\cos\theta_t)/C_n^h(1).$$

(In two dimensions $Y_n(s, t)$ reduces to $e^{-st}\cos n\theta_t$.) It will simplify statements of the results to set $v = (2s)^{\frac{1}{2}}$, and to write E for E_x. Then

(1) Theorem. *In each case the joint distribution of T and θ_T is uniquely determined by the values of the expectations $E(Y_n(s, T))$, $n = 0, 1, 2, \cdots$, $s > 0$.*

Let I and K be the Bessel functions "of purely imaginary argument" (Watson (1944) pages 77–78).

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For the interior problem we have

(2) THEOREM. *If* $0 \leqslant x < a$ *and* T *is the hitting time for the sphere* $\Sigma^{d-1}(0, a)$ *of radius* a, *center* 0, *then for* $x > 0$.

$$E(Y_n(s, T)) = (a/x)^h I_{n+h}(vx)/I_{n+h}(va), \tag{3}$$

while for $x = 0$ *only the case* $n = 0$ *yields a nonzero expectation, which is given by*

$$E(e^{-sT}) = (va)^h / \{2^h \Gamma(h + 1) I_h(va)\}, \tag{4}$$

the hitting point X_T *being uniformly distributed over the sphere independently of* T.

For the exterior problem we have

(5) THEOREM. *If* $x > b$ *and* T *is the hitting time for* $\Sigma^{d-1}(0, b)$ *then*

$$E(Y_n(s, T)) = (b/x)^h K_{n+h}(vx)/K_{n+h}(vb). \tag{6}$$

(In (6) it is understood that $Y_n(s, T) = 0$ when $T = \infty$, even though θ_T is undefined then.)

For a concentric shell the result is

(7) THEOREM. *If* $a < x < b$ *and* T *is the hitting time for the shell* $\Sigma^{d-1}(0, a) \cup \Sigma^{d-1}(0, b)$ *then*

$$E(Y_n(s, T); |X_T| = a) = (a/x)^h (I^b K^x - I^x K^b)/(I^b K^a - I^a K^b), \tag{8}$$

in which abbreviations such as $I^x = I_{n+h}(vx)$ *have been introduced*;

$$E(Y_n(s, T); |X_T| = b) = (b/x)^h (I^a K^x - I^x K^a)/(I^a K^b - I^b K^a), \tag{9}$$

so that (9) is obtained from (8) by interchanging a and b.

Specializations to known results are the following: for the interior problem setting $n = 0$ gives

$$E(e^{-sT}) = (a/x)^h I_h(vx)/I_h(va),$$

as noted by Doob (1955). Again for the interior problem, letting $s \to 0$ ($v \to 0$) yields

$$E\big(C_n^h(\cos \theta_T)\big) = (x/a)^n C_n^h(1),$$

which, in effect, determines the density of X_T on $\Sigma^{d-1}(0, a)$ as the classical Poisson kernel; cf. Stein and Weiss (1971), pages 145, 150. There is an analogous formula for the exterior problem.

For the exterior problem in dimension two we obtain for $n = 0$,

$$E(e^{-sT}) = K_0(vx)/K_0(vb),$$

as given by Spitzer (1958), and in all dimensions $d \geqslant 2$ letting $n = 0$ and $s \to 0$ recovers the classical expression $P(T < \infty) = (b/x)^{d-2}$.

The same specializations in the shell problem give the well-known formulas

$$\begin{aligned} P(|X_T| = a) &= (\log b - \log x)/(\log b - \log a), & d = 2, \\ &= (x^{2-d} - b^{2-d})/(a^{2-d} - b^{2-d}), & d > 2. \end{aligned}$$

Finally, we may note that letting $b \to \infty$ in (8) recovers (6) (with a replacing b), and letting $a \to 0$ in (9) yields (3) (with b in place of a).

An abstract of these results appears in Wendel (1978).

2. Measures on spheres. The results of this section are more or less known, but are included for ease of reference. See especially Gangolli (1964), Bingham (1972), and Kent (1977), together with their references.

Let $\mu(.)$ be a finite measure on the unit sphere Σ^{d-1}. We say that μ is *axially symmetric* (AS) with axis $\xi \in \Sigma^{d-1}$ in case $\mu(RB) = \mu(B)$ for each Borel set B and each orthogonal transformation R that leaves ξ fixed; AS functions are defined similarly. For such a measure we define its Gegenbauer transform $\hat{\mu}_n$ by

$$C_n^h(1)\hat{\mu}_n = \int_{\Sigma^{d-1}} C_n^h(\cos\theta)\mu(d\eta), \qquad n = 0, 1, 2, \cdots,$$

where θ is the angle $\angle \xi 0 \eta$.

(In dimension two, i.e., on Σ^1, an axially symmetric measure is invariant under reflection across a diameter, *even* with respect to angles measured from the diameter. Then the function $\hat{\mu}$ is just the cosine transform,

$$\hat{\mu}_n = \int_0^{2\pi} \cos n\theta \mu(d\theta).)$$

(10) THEOREM. *The* AS *measure* μ *is uniquely determined by its transform* $\hat{\mu}_n$.

This follows from the fact (cf. Stein & Weiss (1971), page 149) that linear combinations of Gegenbauer polynomials of $\cos\theta$ are uniformly dense in the space of continuous AS functions.

Let ν be another finite measure on Σ^{d-1}, also AS with axis ξ; for $\eta \in \Sigma^{d-1}$ let R be a rotation carrying η to ξ and set $\nu_\eta(B) = \nu(RB)$ for Borel B. It is easy to check that ν_η is well defined and that $\nu_\xi = \nu$. Clearly the family $\{\nu_\eta\}$ is invariant in the sense that $\nu_\eta(B) = \nu_{R\eta}(RB)$ for all η, R, B; consequently ν_η is AS with axis η. All invariant families $\{\nu_\eta\}$ arise in this way. We note also that $\nu_\eta(B)$ is a measurable function of $\eta \in \Sigma^{d-1}$.

We define the convolution $\rho = \mu * \nu$ by

$$\rho(B) = \int_{\Sigma^{d-1}} \mu(d\eta)\nu_\eta(B). \tag{11}$$

Then ρ is AS with axis ξ and for its transform we have

$$\hat{\rho} = \hat{\mu}\hat{\nu} \tag{12}$$

cf. Gangolli (1964), page 218. In two dimensions this is a well-known fact about cosine transforms of even measures. For a direct proof in higher dimensions we appeal to the fact that the $C_n^h(\cos\theta)$ are zonal harmonics. Concretely this means that for $\xi, \eta, \zeta \in \Sigma^{d-1}$, η not parallel to ζ,

$$\text{ave } C_n^h(\cos \angle \xi 0 \zeta') = C_n^h(\cos \angle \xi 0 \eta) C_n^h(\cos \angle \eta 0 \zeta)/C_n^h(1),$$

where the average is taken with respect to ζ' running over the sphere Σ^{d-2} formed by intersecting Σ^{d-1} with the hyperplane through ζ perpendicular to η. (Watson (1944), page 369. When $\eta \neq \pm \zeta$ the sphere Σ^{d-2} degenerates to the point η, and the formula is seen to hold trivially.) Then

$$\begin{aligned} C_n^h(1)\hat{\rho}_n &= \int C_n^h(\cos \angle \xi 0 \zeta)\rho(d\zeta) \\ &= \int C_n^h(\cos \angle \xi 0 \zeta)\int \mu(d\eta)\nu_\eta(d\zeta) \\ &= \int \mu(d\eta)\int C_n^h(\cos \angle \xi 0 \zeta)\nu_\eta(d\zeta). \end{aligned}$$

Because ν_η is AS with axis η we may replace ζ in the last cosine by $R\zeta = \zeta'$, where R is an arbitrary orthogonal transformation leaving η fixed. If we then perform the averaging described above we obtain (12).

3. Proofs. Theorem (1) is an immediate consequence of (10). Alternatively, as noted in the introduction the joint distribution is determined by the expectations $E(e^{-sT}\cos n\theta_T)$, $s > 0$, $n = 0, 1, 2, \cdots$; these are linear combinations of the quantities $E(Y_n(s, T))$.

Let $P(\xi, \eta, t)$ be the density at η of a Brownian particle starting at $X_0 = \xi \in R^d$, and let $p(\xi, \eta, s)$ be its Laplace transform,

(13) $$p(\xi, \eta, s) = \int_0^\infty e^{-st}P(\xi, \eta, t)\, dt = 2^{-h}\pi^{-h-1}v^h|\xi - \eta|^{-h}K_h(v|\xi - \eta|),$$

$s > 0$, $\xi \neq \eta$, Watson (1944), (15) page 183.

Let B and C be Borel sets of the unit sphere $\Sigma^{d-1} = \Sigma^{d-1}(0, 1)$ and $(0, \infty)$ respectively. For T the hitting time of the sphere $\Sigma^{d-1}(0, a)$ let $F(\xi, a, B, C) = P\{X_T \in aB \text{ and } T \in C|X_0 = \xi\}$, with Laplace transform $f(\xi, a, B, s) = \int_0^\infty e^{-st}F(\xi, a, B, dt)$. (The definitions of F, f do not exclude that T may be infinite with positive probability.) If $|\zeta| > a > |\xi|$ then the strong Markov property and continuity of paths gives the relation

(14) $$p(\xi, \zeta, s) = \int_{\Sigma^{d-1}} f(\xi, a, d\eta, s)p(a\eta, \zeta, s);$$

letting $|\zeta| = z > a$ we replace ζ by $z\zeta$, where now $\zeta \in \Sigma^{d-1}$, and rewrite (14) as

(15) $$p(\xi, z\zeta) = \int_{\Sigma^{d-1}} f(\xi, a, d\eta)p(a\eta, z\zeta),$$

where for simplicity the transform variable s has been suppressed.

Holding $a > 0$, $\eta \in \Sigma^{d-1}$, $z > a$, and s fixed we can regard $p(a\eta, z\zeta)$ as the density on Σ^{d-1} of a measure $\nu_\eta(.)$ on Borel sets. Clearly the family of such measures is invariant. For fixed ξ the measure $\mu(.) = f(\xi, a, .)$ is AS with axis ξ, because of the rotational symmetry of Brownian motion. Therefore (15) is an instance of the equation (11). In order to apply (12) we obtain the required Gegenbauer transforms by combining (13) with Macdonald's addition theorem for the function K_h (cf. Watson (1944), (8) page 365). First we find that for $|\xi| = x < z$, $|\zeta| = 1$,

$$\begin{aligned} p(\xi, z\zeta, s) &= \pi^{-d/2}\Gamma(h)x^{-h}z^{-h}\Sigma_0^\infty(n + h)I_{n+h}(vx)K_{n+h}(vz)C_n^h(\cos \angle \xi 0 \zeta), \quad d \geqslant 3 \\ &= \pi^{-1}\{I_0(vx)K_0(vz) + 2\Sigma_0^\infty I_n(vx)K_n(vz)\cos(n \angle \xi 0 \zeta)\}, \quad d = 2. \end{aligned}$$

Then orthogonality of the Gegenbauer polynomials for fixed η (Watson (1944), page 367) and uniform convergence of the series on Σ^{d-1} yield the transform

$$\hat{\rho}_n = \hat{\rho}_n(\xi, s) = c_{n,h} x^{-h} z^{-h} I_{n+h}(vx) K_{n+h}(vz), \ |\xi| = x < z,$$

and similarly

$$\hat{\nu}_n = c_{n,h} a^{-h} z^{-h} I_{n+h}(va) K_{n+h}(vz),$$

where the $c_{n,h}$ are positive constants whose precise form is of no importance. Solving (12) we obtain

$$\hat{\mu}_n = \hat{f}_n(x, a, s) = (a/x)^h I_{n+h}(vx)/I_{n+h}(va),$$

which proves (3). (The notation for the arguments of $\hat{f}_n$ is intended to underscore the fact that the dependence on the vector ξ is only through its length $|\xi| = x$.) Equation (4) follows by letting $x \to 0$; this completes the proof of (2). The proof of (6) is entirely similar, except that we work from the outside of the sphere of radius b towards the inside.

In order to prove (8) and (9) we set up the measures

$$_aF(\xi, b, B, C) = P\{X_T \in bB \text{ and } T \in C | X_0 = \xi\}$$

$$_bF(\xi, a, B, C) = P\{X_T \in aB \text{ and } T \in C | X_0 = \xi\}$$

where $a < |\xi| = x < b$, B and C are Borel subsets of Σ^{d-1} and $(0, \infty)$ respectively, and the notation of presubscripts is suggested by that of Chung (1967) for "taboo" states in Markov chains. These measures have Laplace transforms (s suppressed) $_af(\xi, b, B)$, $_bf(\xi, a, B)$.

Consider a Brownian path which starts at ξ and first hits the outer sphere $\Sigma^{d-1}(0, b)$ in the region bB, whether or not it earlier hits the inner sphere. Taking these two possibilities into account and using the strong Markov property we can write

$$f(\xi, b, B) = {}_af(\xi, b, B) + \int_{\Sigma^{d-1}} {}_bf(\xi, a, d\eta) f(a\eta, b, B),$$

where f is the function introduced early in this section. Interchanging the roles of outer and inner we have similarly

$$f(\xi, a, B) = {}_bf(\xi, a, B) + \int_{\Sigma^{d-1}} {}_af(\xi, b, d\eta) f(b\eta, a, B).$$

Taking Gegenbauer transforms we obtain

$$\hat{f}_n(x, b) = {}_a\hat{f}_n(x, b) + {}_b\hat{f}_n(x, a)\hat{f}_n(a, b) \tag{16}$$

$$\hat{f}_n(x, a) = {}_b\hat{f}_n(x, a) + {}_a\hat{f}_n(x, b)\hat{f}_n(b, a),$$

where again we have adopted a notation emphasizing the dependence of the transforms only on the lengths of the relevant vectors.

Solving the system (16) with the aid of the known values of $\hat{f}_n(\ldots)$ from (3) and (6) yields the expressions (8), (9). That the system (16) is nonsingular, the denominator of (8), (9) is not zero, follows from the nonvanishing of the Wronskian of I_h, K_h (Watson (1944), (19) page 80.)

4. Remarks and acknowledgments. I am grateful to my colleague C. T. Shih for suggesting this problem area by asking for the joint distribution of the exit time and place for a planar Brownian motion started inside a circle. In general the interior problem can also be solved by exploiting the exponential martingale

$$M_t = \exp\{-v^2t/2 + v(u \cdot X_t)\}, \qquad v > 0, u \in \Sigma^{d-1},$$

a Bessel-Gegenbauer expansion (Watson (1944), (3) page 369) for $\exp(z \cos \phi)$, and the averaging process used above to prove (12); we omit the details. No such attack succeeded for the exterior problem, and results of Robbins and Siegmund (1973) and Lai (1974) suggest that none is possible. The method actually used was motivated in part by the treatment in Darling and Siegert (1953) of hitting time problems for linear diffusions on semiinfinite or finite intervals.

When $n = 0$ the expressions (3), (6), (8), (9) give the complete monotonicity and infinite divisibility of certain quotients of modified Bessel functions with arguments proportional to $s^{\frac{1}{2}}$. This overlaps with recent results of Ismail and Kelker (1978) and Kent (1978); moreover, the present Theorem (2) is the drift-free case of the more general formula (6.1) in the latter reference. I express my appreciation to them for preprints. Finally, it is a pleasure to thank R. M. Dudley and W. E. Pruitt for corrections and significant improvements to this paper.

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