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ON A STOPPED DOOB'S INEQUALITY AND GENERAL STOCHASTIC EQUATIONS

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An upper bound for $E(\sup_{0 \leqslant s < \tau} \|M_s\|^2)$, where M is a square integrable martingale and τ a stopping time is given in terms of $[M]_{\tau^-}$ and $\langle M \rangle_{\tau^-}$. Counter examples show that $4E(\langle M \rangle_{\tau^-})$, which is easily derived as an upper bound from a classical Doob's inequality, when τ is predictable or totally unaccessible, is no longer an upper bound in general. The obtained majoration is used to prove existence and uniqueness of strong solutions of a stochastic equation $dX_t = a(t, X)\, dZ_t$, where a is a functional, depending possibly on the whole past of X before t, and Z is a semimartingale. Our result thus extends to systems "with memory" recent results by Protter, Kazamaki, Doleans-Dade and Meyer.

Introduction. Recently P. E. Protter [12], N. Kazamaki [6], C. Doleans-Dade and P. A. Meyer [3] have considered stochastic integral equations of the type

$$X_t = \xi_0 + \int_0^t \alpha(s, X_{s^-})\, dZ_s$$

where $\alpha(s, X_{s^-})$ is a function of the position of process X immediately before s, and Z is a general semimartingale. Existence and unicity of the solutions were proved, when α satisfies proper Lipschitz-conditions (see [5] too). In this paper we consider the case, where $X \to \alpha(s, X)$ is a functional of the process X, which may depend on the whole past of the process before time s. The results of existence, unicity and nonexplosion of solutions here obtained in Section 4 generalize to our more general situation those of the above-mentioned authors.

These results, announced without proof in [9], will be compared with a theorem independently proved by M. Emery ([5]). The comparison will be made precise at the end. Let us say now that, in the case of driving terms which are real semimartingales, Emery's method extends C. Doleans-Dade and P. A. Meyer's method, while the technique used here rests on a powerful inequality, which is proved in the first part. This inequality, which is most helpful in "keeping control" of jumps of the stochastic driving term, makes it possible to reach immediately a wide important class of vector-valued stochastic driving terms which are not reducible to martingales and processes of bounded variation, as shown by a simple counter example given in subsection 4.5.

In working out fixed point-methods for our purpose, we met the problem of getting an upper bound of $E(\sup_{0 \leqslant s < \tau} |M_s|^2)$ in terms of $E(\langle M \rangle_{\tau^-})$ and $E([M]_{\tau^-})$, where M is a square integrable martingale and τ a stopping time. Although such an

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upper bound is trivially derived from a classical Doob's inequality, when τ is predictable or totally inaccessible, the problem is far from being so easy when τ is neither predictable nor totally inaccessible. Section 1 produces counter examples in this latter situation.

An inequality, which is valid for every stopping time τ, is derived in Section 2.

1. On a "stopped" Doob's inequality—counter-examples.

1.1. *Definitions and notations—problem.* In all this paper a stochastic basis: $(\Omega, (\mathcal{F}_t)_{t\in\mathbb{R}^+}, \mathcal{F}, P)$ is given: $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$ is an increasing, right continuous family of σ-subalgebras of a σ-algebra $\mathcal{F}$ of subsets of Ω, and P is a probability on $(\Omega, \mathcal{F})$. We make the usual completeness hypothesis: $\mathcal{F}$ is complete for P and every P-null set in $\mathcal{F}$ belongs to all $\mathcal{F}_t$'s.

An $\mathbb{H}$-valued process X, where $\mathbb{H}$ is a finite or infinite dimensional Banach space, is a mapping from $\mathbb{R}^+ \times \Omega$ into $\mathbb{H}$.

A stochastic process will be said to be *regular* if it is *adapted* to the filtration $(\mathcal{F}_t)_{t\in\mathbb{R}^+}$ (i.e.: X_t is $\mathcal{F}_t$-measurable for all t) and if its paths are right continuous and have left limits in every point $t \in \mathbb{R}^+$. (When X is real, by a limit we mean a limit in $\mathbb{R}$, *not* in $\overline{\mathbb{R}}$).

The process X is said to be *predictable* if it is (strongly) measurable for the σ-algebra $\mathcal{P}$ of predictable subsets of $\mathbb{R}^+ \times \Omega$: $\mathcal{P}$ is the σ-algebra generated by the class $\mathcal{R}$ of predictable rectangles, that is

$$\mathcal{R} := \{\,]s, t] \times F : s < t \in \mathbb{R}^+, F \in \mathcal{F}_s\}.$$

A process X is called a *P-null process*, if almost surely the paths: $t \to X(t, \omega)$ are identically zero functions. Two processes will be called *equivalent* if their difference is a P-null process.

When X is a regular process, we call ΔX the process:

$$\Delta X(t, \omega) = X(t, \omega) - \lim_{s\uparrow t;\, s<t} X(s, \omega).$$

For a stopping time τ we denote by $[\tau]$ the graph of τ.

We recall the following Doob-inequality for a real square integrable martingale M:

$$E\big(\sup_{0\leqslant s\leqslant t}|M_s|^2\big) \leqslant 4E|M_t|^2. \tag{1.1.1}$$

Considering a stopping time τ, and the stopped martingale $M_{.\wedge\tau}$ one can write:

$$E\big(\sup_{0\leqslant s\leqslant \tau}|M_s|^2\big) \leqslant 4E|M_\tau|^2 = 4E\langle M_\tau\rangle = 4E[M]_\tau \tag{1.1.2}$$

where $\langle M\rangle$ denotes the natural increasing process of $|M|^2$ and $[M]$ the quadratic variation of M (see, for example, [7]).

When the martingale M is continuous, an inequality of the type (1.1.2) insures us of the upper bound: $E(\sup_{0\leqslant s<\tau}|M_s|^2) \leqslant d$, when τ is the stopping time defined by $\tau = \inf\{t : \langle M\rangle_t \geqslant d\}$.

Unfortunately things are far from being so simple when M is discontinuous. A natural question is then the following: do the following inequalities hold for a square integrable martingale M and a stopping time τ?

(1.1.3) $$E\,\sup_{0\leqslant s<\tau}|M_s|^2 \leqslant 4E\langle M\rangle_{\tau^-};$$

(1.1.4) $$E\,\sup_{0\leqslant s<\tau}|M_s|^2 \leqslant 4E[M]_{\tau^-};$$

where $f(t^-)$ means $\lim_{s\uparrow t;\, s<t} f(s)$.

1.2. *Immediate answers and counter-examples.* Let us assume that $\langle M\rangle_{\tau^-} = \langle M\rangle_\tau$, which is the case when τ is totally unaccessible, or when M is continuous, then (1.1.3) is a trivial consequence of (1.1.2).

When τ is predictable we may consider an "announcing sequence" (τ_n) increasing towards τ with $\tau_n < \tau$ a.s. on $[\tau < \infty]$. In this case

$$E\big(\sup_{0\leqslant s<\tau}|M_s|^2\big) = \lim_n\uparrow E\big(\sup_{0\leqslant s\leqslant\tau_n}|M_s|^2\big) \leqslant \lim_n\uparrow E[M]_{\tau_n} = \lim_n\uparrow E\langle M\rangle_{\tau_n}.$$

Therefore (1.1.3) and (1.1.4) hold in this case.

The fact that (1.1.3) is true for predictable and totally unaccessible stopping times would seem to indicate that this inequality holds for a general stopping time τ. Unfortunately this is not the case, as shown by the following simple counter-examples, unless hypothesis of left quasicontinuity are put on the σ-algebras $(\mathcal{F}_t)$.

Counter-example to (1.1.3). We define: $\Omega = \{1, 2\}$, $\mathcal{F}_t = \{\varnothing, \Omega\}$ if $t < 1$ and $\mathcal{F}_t = \mathcal{P}(\Omega)$ if $t \geqslant 1$, $\tau = 1 + 1_{\{2\}}$, $P(1) = p_1 > 0$, $P(2) = p_2 = 1 - p_1 > 0$. This is a classical Dellacherie's example.

The predictable rectangles are of the form $]s, t] \times \Omega$ if $s < 1$ and $]s, t] \times F$, $F \in (\Omega)$ if $s \geqslant 1$. Therefore the traces of predictable sets on $[0, \tau]$ are of the form $[0, \tau] \cap \Pi^{-1}(\mathcal{B})$ where $\mathcal{B}$ is a Borel set of $\mathbb{R}^+$ and Π is the projection $\mathbb{R}^+ \times \Omega \to \mathbb{R}^+$. It follows that the graph $[\tau]$ of τ is included in the union $\{(1, 1), (1, 2)\} \cup \{(2, 2)\}$ of predictable graphs but is not itself predictable.

We consider then the martingale:

$$\begin{aligned} M(t) &:= 0 && \text{if } t < 1 \\ &:= \frac{1}{p_1}1_{\{1\}} - \frac{1}{p_2}1_{\{2\}} && \text{if } t \geqslant 1. \end{aligned}$$

It is easily computed:

$$\langle M\rangle_t = \left(\frac{1}{p_1} + \frac{1}{p_2}\right)1_{[1,\infty[}(t)$$

$$E\big(\sup_{s<\tau}|M_s|^2\big) = \frac{1}{p_2}$$

$$E(\langle M\rangle_{\tau^-}) = p_2\left(\frac{1}{p_1} + \frac{1}{p_2}\right) = \frac{1}{p_1}.$$

As the quotient p_1/p_2 may be chosen arbitrarily big, this is a counter-example to any inequality of the type (1.1.3) obtained by replacing the constant 4 by any other constant.

Counter-example to (1.1.4). In this example $\Omega = \{1, 2, \cdots, n\}$. The σ-algebras $\mathcal{F}_t$ are constant on $[k, k+1[$ and $\mathcal{F}_k$ is generated by the atoms: $\{1\}, \cdots, \{k\}$, $\{k+1, \cdots n\}$. The probability law P is defined by:

$$P\{1\} = q \qquad P\{2\} = (1-q)q \cdots P(n-1) = (1-q)^{n-2}q,$$
$$P\{n\} = (1-q)^{n-1}, \qquad\qquad 0 < q < 1.$$

The martingale M is constant on $[k, k+1[$ and:

$$M_0 = 0, \qquad M_{k+1} = M_k - \frac{1}{q} 1_{\{k+1\}} + \frac{1}{(1-q)} 1_{\{k+2, \cdots, n\}}.$$

The stopping time τ is defined through $\tau(\omega) = \omega$. It is seen as in the previous example, that τ is accessible but not predictable.

An easy calculation gives:

$$E[M]_\tau = \frac{1-q}{q} + 2\frac{(1-q)^2}{q} + \cdots + (n-2)\frac{(1-q)^{n-2}}{q} + (n-1)\frac{(1-q)^{n-2}}{q^2}$$

$$E(\sup_{0 \leqslant s < \tau} |M_s|^2) = \frac{1-q}{q} + 2^2\frac{(s-q)^2}{q} + \cdots + (n-2)^2\frac{(1-q)^{n-2}}{q} + (n-1)^2\frac{(1-q)^{n-2}}{q^2}.$$

When q tends to zero

$$\frac{E([M]_{\tau^-}}{E(\sup_{0 \leqslant s < \tau} |M_s|^2)} \sim \frac{1}{n-1}.$$

It is, therefore, possible to chose q and n in such a way that:

$$E \sup_{0 \leqslant s < \tau} |M_s'|^2 \geqslant CE[M]_{\tau^-}$$

where C is any positive constant.

2. Inequalities.

2.1. *Statement of the results.*

THEOREM 1. *Let M be a real or $\mathbb{H}$-valued ($\mathbb{H}$: Hilbert space) regular square integrable martingale. For every stopping time τ there exists a regular square integrable martingale W with the following properties*:

(i) $1_{[0, \tau[} W = 1_{[0, \tau[} M$
and therefore

$$1_{[0, \tau[}[W] = 1_{[0, \tau[}[M].$$

(ii) *The random measures $d\langle W \rangle$ and $d\langle M \rangle$ satisfy*:

$$d\langle W \rangle \leqslant d\langle M \rangle.$$

(iii) *For every positive predictable process* Y:

$$E(\int_{[0,\tau]} Yd[W]) = E(\int_{[0,\tau]} Yd\langle W\rangle) \leqslant E(\int_{[0,\tau[} Yd[W] + \int_{[0,\tau[} Yd\langle W\rangle).$$

If M *is purely discontinuous, so is* W.

THEOREM 2. *Let* M *be a real or* $\mathbb{H}$*-valued regular square integrable martingale. Then the following inequality holds for every stopping time* τ:

$$E(\sup_{0\leqslant t<\tau}\|M_t\|^2) \leqslant 4E(\langle M\rangle_{\tau^-} + [M^d]_{\tau^-})$$

where M^d *is the pure discontinuous part of the martingale* M *(see* [7] *or* [11]).

If the accessible part of $[\tau]$ *is previsible (in particular if the family* $(\mathcal{F}_t)$ *is quasi-left-continuous), then holds*:

$$E(\sup_{0\leqslant t<\tau}\|M_t\|^2) \leqslant 4E\langle M\rangle_{\tau^-}.$$

The last assertion of Theorem 2 is trivial as noticed in the beginning of subsection 1.2 above. As to the first part of Theorem 2 it is an easy consequence of Theorem 1. Let $M = M^c + M^d$ be the decomposition of M into its continuous and its discontinuous parts. (See [11] Chapter II), and let W be the process associated with M^d by Theorem 1. Then according to the classical Doob-inequality we get:

$$E(\sup_{0\leqslant t<\tau}\|M_t\|^2) = E(\sup_{0\leqslant t<\tau}\|M_t^c + W_t\|^2) \leqslant 4E\langle M^c + W\rangle_\tau.$$

As the martingales M^c and W are orthogonal, one may write:

$$E(\sup_{0\leqslant t<\tau}\|M_t\|^2) = 4E\langle M^c\rangle_\tau + 4E\langle W\rangle_\tau$$

and according to properties (ii) and (iii):

$$\begin{aligned} E(\sup_{0\leqslant t<\tau}\|M_t\|^2) &\leqslant 4E\{\langle M^c\rangle_{\tau^-} + \langle W\rangle_{\tau^-} + [W]_{\tau^-}\} \\ &\leqslant 4E\{\langle M^c\rangle_{\tau^-} + \langle M^d\rangle_{\tau^-} + [M^d]_{\tau^-}\} \\ &= 4E\{\langle M\rangle_{\tau^-} + \langle M^d\rangle_{\tau^-}\}. \end{aligned}$$

This proves Theorem 2 as a consequence of Theorem 1.

This argument shows that inequality in Theorem 2 can be slightly improved in the following way. Let (σ_n) be a denumerable family of predictable stopping times with disjoint graphs such that the predictable part of a family of stopping times carrying the jumps of M is included in $\cup_n[\sigma_n]$. The family (σ_n) is not uniquely determined, but the process $\Sigma_n \Delta M_{\sigma_n}.1_{[\sigma_n,\infty]}$ is, and it is a square integrable martingale, which we will call the *pure jump part* of M and will denote by M^j. As the process $\langle M - M^j\rangle$ is continuous the above argument for $M - M^d$ can be reproduced and gives immediately the following:

THEOREM 2′. *Let* M *be a real or* $\mathbb{H}$*-valued regular square integrable martingale. Then the following inequality holds for every stopping time* τ:

$$E(\sup_{0\leqslant t<\tau}\|M_t\|^2) \leqslant 4E(\langle M\rangle_{\tau^-} + [M^j]_{\tau^-})$$

where M^j *is the pure jump part of* M.

2.2. *Proof of Theorem* 1. The proof rests on two lemmas.

LEMMA 1. *Let* $(\Omega, \mathcal{F}, P)$ *be a probability space,* $\mathcal{G}$ *a sub-σ-algebra of* $\mathcal{F}$, A *an element of* $\mathcal{F}$ *and* $\mathcal{G}^*$ *the σ-algebra generated by* $\mathcal{G}$ *and* $\{A\}$. *For every* $\mathcal{G}^*$*-measurable (real or* $\mathbb{H}$*-valued) square integrable random variable* Z, *such that* $E(Z|\mathcal{G}) = 0$, *the following equalities (where* $A^* = \Omega - A$*) hold:*

(a) $E(1_A|\mathcal{G}).E(\|Z\|^2.1_A|\mathcal{G}) = E(1_{A^*}|\mathcal{G}).E(\|Z\|^2.1_{A^*}|\mathcal{G})$ a.s.;

(b) $E(1_{A^*}.\|Z\|^2) = E(1_A.E(\|Z\|^2|\mathcal{G}))$.

PROOF. (This very simple proof was mentioned to us by J. Jacod.) One may write $Z = 1_A X + 1_{A^*} Y$, where X and Y are $\mathcal{G}$-measurable. The condition $E(Z|\mathcal{G}) = 0$ gives:

$$E(1_A|\mathcal{G}).X = -E(1_{A^*}|\mathcal{G})Y. \tag{2.2.1}$$

As

$$E(\|Z\|^2.1_A|\mathcal{G}) = \|X\|^2.E(1_A|\mathcal{G})$$

and

$$E(\|Z\|^2.1_{A^*}|\mathcal{G}) = \|Y\|^2.E(1_{A^*}|\mathcal{G})$$

the inequality (2.2.1), which implies

$$\|X\|^2[E(1_A|\mathcal{G})]^2 = \|Y\|^2[E(1_{A^*}|\mathcal{G})]^2$$

gives immediately the formula (a) of Lemma 1.

As to formula (b) it is derived from formula (a) through the following chain of equalities:

$$\begin{aligned} E[1_A.E(\|Z\|^2|\mathcal{G})] &= E[1_A.E(\|Z\|^2.1_A|\mathcal{G})] + E[1_A.E(\|Z\|^2.1_{A^*}|\mathcal{G})] \\ &= E\{E(1_A|\mathcal{G}).E(\|Z\|^2.1_A|\mathcal{G})\} + E\{E(1_A|\mathcal{G}).E(\|Z\|^2.1_{A^*}|\mathcal{G})\} \\ &= E[\{E(1_{A^*}|\mathcal{G}).E(\|Z\|^2.1_{A^*}|\mathcal{G})\} + E(1_A|\mathcal{G}).E(\|Z\|^2.1_{A^*}|\mathcal{G})] \\ &= E\{E(\|Z\|^2.1_{A^*}|\mathcal{G})\} = E(\|Z\|^2.1_{A^*}). \end{aligned}$$

LEMMA 2. *Let* σ *be a predictable stopping time,* h *a real or* $\mathbb{H}$*-valued* $\mathcal{F}_\sigma$*-measurable square integrable random variable such that* $E(h|\mathcal{F}_{\sigma^-}) = 0$. *Let* τ *be a stopping time. We define* $R := \{\sigma \leqslant \tau, \sigma < \infty\}$, $A := \{\sigma < \tau\}$ $B := R - A$,

$$\begin{aligned} \sigma_A(\omega) &= \sigma(\omega) && \textit{if} \quad \omega \in A \\ &= +\infty && \textit{if} \quad \omega \notin A \\ \sigma_B(\omega) &= \sigma(\omega) && \textit{if} \quad \omega \in B \\ &= +\infty && \textit{if} \quad \omega \notin B. \end{aligned}$$

We denote by $\mathcal{F}^*_{\sigma^-}$ *the σ-algebra generated by* $\mathcal{F}_{\sigma^-}$ *and* A. *Then the processes:*

$$M := 1_R.1_{[\sigma,\infty[}.h$$

and

$$W := 1_{[\sigma_A,\infty[}h + 1_{[\sigma_B,\infty[}E(h|\mathcal{F}^*_{\sigma^-})$$

are regular square integrable martingales, for which properties (i) *to* (iii) *of Theorem* 1 *hold.*

PROOF. Using an announcing sequence (σ_n) of stopping times for σ we prove easily that:

$$R_n := \{\sigma_n < \tau\} \in \mathcal{F}_{\sigma_n} \cap \mathcal{F}_\tau$$

and, therefore,

$$R = \cap_n R_n \cap \{\lim_n \sigma_n < \infty\} \in \mathcal{F}_{\sigma^-}.$$

The hypothesis on h then implies that M is a martingale.

To prove that W is a martingale, it is enough to show that $M - W$ is a martingale. But

$$M - W = (h - E(h|\mathcal{F}^*_{\sigma^-}))1_{[\sigma_B, \infty[}. \tag{2.2.2}$$

From there on we show that $M - W$ is the regular version of the martingale $(E(\varphi|\mathcal{F}_t))_{t\in\mathbf{R}^+}$, where:

$$\varphi := 1_B(h - E(h|\mathcal{F}^*_{\sigma^-})) = 1_B.h - E(1_B.h|\mathcal{F}^*_{\sigma^-}).$$

Let (σ_n) be an announcing sequence for σ, as above. As $E(\varphi|\mathcal{F}_{\sigma^-}) = 0$ and $E(\varphi|\mathcal{F}_\sigma) = \varphi$ we have clearly $E(\varphi|\mathcal{F}_t) = 0$ for $t < \sigma$ and $E(\varphi|\mathcal{F}_t) = \varphi$ for $t \geq \sigma$. This proves that

$$E(\varphi|\mathcal{F}_t) = M_t - W_t \qquad \text{a.s. for all } t.$$

From the definition of M and W we get

$$\langle M\rangle = 1_R E(\|h\|^2|\mathcal{F}_{\sigma^-}).1_{[\sigma, \infty[}$$

and

$$\langle W\rangle = 1_R\{E(1_A.\|h\|^2|\mathcal{F}_{\sigma^-}) + E(1_B.\|E(h|\mathcal{F}^*_{\sigma^-})\|^2|\mathcal{F}_{\sigma^-})\}1_{[\sigma, \infty[}.$$

The one jump of $\langle W\rangle$ is carried by $[\sigma]$ and bounded by

$$1_R\{E(1_A.\|h\|^2|\mathcal{F}_{\sigma^-}) + E(1_B E(\|h\|^2|\mathcal{F}^*_{\sigma^-})|\mathcal{F}_{\sigma^-})\} = 1_R E(\|h\|^2|\mathcal{F}_{\sigma^-}).$$

Comparing with $\langle M\rangle$ we get immediately the inequality $d\langle W\rangle \leq d\langle M\rangle$ a.s., which is property (ii).

As M and W have jump only on $[\sigma]$ and coincide on $[\sigma_A]$, the property (i) is clear.

We have also for every predictable positive Y:

$$E(\textstyle\int_{[0,\tau]} Y d[W]) = E(1_B.Y_\sigma.\|E(h|\mathcal{F}^*_{\sigma^-})\|^2) + E(1_A Y_\sigma\|h\|^2).$$

By applying Lemma 1(b) with $Z = 1_R(Y_\sigma)^{\frac{1}{2}}E(h|\mathcal{F}^*_{\sigma^-})$ and $\mathcal{F}_{\sigma^-} = \mathcal{G}$, we get

$$E(\textstyle\int_{[0,\tau]} Y d[W]) = E(1_A Y_\sigma.\|h\|^2) + E\{1_A Y_\sigma E(\|E(h|\mathcal{F}^*_{\sigma^-})\|^2|\mathcal{F}_{\sigma^-})\}. \tag{2.2.3}$$

From the conditional Jensen's inequality we deduce:

$$\begin{aligned} E[1_R\|E(h|\mathcal{F}^*_{\sigma^-})\|^2|\mathcal{F}_{\sigma^-}] &= 1_R\{E[1_A\|E(h|\mathcal{F}^*_{\sigma^-})\|^2|\mathcal{F}_{\sigma^-}] + E[1_B\|E(h|\mathcal{F}^*_{\sigma^-})\|^2|\mathcal{F}_{\sigma^-}]\} \\ &\leq 1_R E(1_A.\|h\|^2|\mathcal{F}_{\sigma^-}) + E[1_B\|E(h|\mathcal{F}^*_{\sigma^-})\|^2|\mathcal{F}_{\sigma^-}] \\ &\leq \Delta\langle W\rangle_\sigma. \end{aligned}$$

This last inequality with (2.2.3) proves the formula (iii) of Theorem 1 for the martingales M and W of Lemma 2.

PROOF OF THEOREM 1. Let $[\tau] = [\tau_u] + [\tau_a]$ the decomposition of $[\tau]$ into its totally inaccessible and its accessible parts. If $\Delta M_{\tau_a} = 0$ a.s. the process $\langle M\rangle$ has no jump on the stopping time τ. Therefore

$$E(\textstyle\int_{[0,\tau]} Y[dM]) = E(\int_{[0,\tau]} Yd\langle M\rangle) = E(\int_{[0,\tau[} Yd\langle M\rangle)$$

and Theorem 1 is trivially true with $M = W$.

In the general situation we consider a sequence of predictable stopping times (τ_n), $n > 0$ with disjoint graphs, such that $[\tau_a] \subset \cup_n[\tau_n]$.

The process $S = \Sigma_n \Delta M_{\tau_n} 1_{[\tau_n, \infty[}$ is a square integrable martingale, and, denoting by N the martingale $M - S$ and by M^n the martingale $\Delta M_{\tau_n} 1_{[\tau_n, \infty[}$ we have

$$\textstyle\int Yd\langle M\rangle = \int Yd\langle N\rangle + \Sigma_n \int Yd\langle M^n\rangle \tag{2.2.4}$$

and

$$\textstyle\int Yd[M] = \int Yd[N] + \Sigma_n \int Yd[M^n] \tag{2.2.5}$$

for every positive predictable process Y.

To each M^n we associate the martingale W^n of Lemma 2. All martingales N and W^n are orthogonal and the series $N + \Sigma_n W^n$ converge in the space of square integrable martingales and we write W for the sum of the series. The properties (i) and (ii) of Theorem 1 follow immediately from the definition of W and Lemma 2 and we get for every positive predictable process:

$$\begin{aligned} E(\textstyle\int_{[0,\tau]} Yd\langle W\rangle) &= E(\textstyle\int_{[0,\tau]} Yd\langle N\rangle) + E(\Sigma_n \int_{[0,\tau]} Yd\langle M^n\rangle) \\ &\leqslant E(\textstyle\int_{[0,\tau[} Yd\langle N\rangle) + E(\Sigma_n(\int_{[0,\tau[} Yd\langle W^n\rangle + \int_{[0,\tau[} Yd[W^n]) \\ &\leqslant E(\textstyle\int_{[0,\tau[} Yd\langle W\rangle) + \int_{[0,\tau[} Yd[W]). \end{aligned}$$

This proves the property (iii).

It is clear from the definition of W, that this process is a pure jump martingale in the sense of Theorem 2′ as soon as $N = 0$.

3. A general stochastic equation—lemmas on semimartingales. The equation, which will be considered, will be written:

$$dX(t) = \mathbf{a}(t, X)\, dZ(t) \tag{3.1}$$

or in integral form

$$X(t) = \xi_0 + \textstyle\int_0^t \mathbf{a}(s, X)\, dZ(s) \tag{3.2}$$

where ξ_0 is an initial value. The process Z and the functional $\mathbf{a}$ of the process X will be described now. The hypothesis on Z and $\mathbf{a}$ will give, in particular, a meaning to the stochastic integral in (3.2), and the notion of solution of (3.2) will be given a precise definition.

3.1. *Predictable functionals of a process.* In (3.2) the functional $\mathbf{a}$ will be defined in such a way, that for a regular process X, $(\mathbf{a}(s, X) : s \in \mathbb{R}^+)$ is a predictable process.

We will write $\mathcal{Q}$ for the Boolean ring of subsets of $\mathbb{R}^+ \times \Omega$ generated by the family $\{]s, t] \times F;\ s < t \in \mathbb{R}^+,\ F \in \mathcal{F}_s\}$ of so-called predictable rectangles, and denote by $\mathcal{P}$ the σ-algebra of predictable subsets of $\mathbb{R}^+ \times \Omega$, which is by definition the σ-algebra generated by $\mathcal{Q}$.

In order to define the predictable functionals of a process, we introduce the following spaces.

Let $\mathbb{H}$ be an Hilbert-space, we will denote by $\mathbb{D}^{\mathbf{H}}$ the set of all the mappings from $\mathbb{R}^+$ into $\mathbb{H}$, which are regular (i.e., according to definitions in 1.1: are right continuous, have left limits in every $t \in \mathbb{R}^+$).

For every $t \in \mathbb{R}^+$ $\mathcal{D}_t^{\mathbf{H}}$ denotes the σ-algebra generated by the "cylinders" $\{f : f \in D^{\mathbf{H}}, f(s) \in B\}$ where $s \leqslant t$ and B is any Borel-set in $\mathbb{H}$.

We set $\tilde{\Omega}^{\mathbf{H}} := \Omega \times D^{\mathbf{H}}$ and we consider on $\tilde{\Omega}^{\mathbf{H}}$ the increasing family $(\tilde{\mathcal{F}}_t^{\mathbf{H}})_{t \in \mathbf{R}^+} := (\mathcal{F}_t \otimes \mathcal{D}_t^{\mathbf{H}})_{t \in \mathbf{R}^+}$ of σ-algebras.

Definition 1. The σ-algebra of subsets of $\tilde{\Omega}^{\mathbf{H}}$, generated by the family $\{]s, t] \times \tilde{F} : s \leqslant t \in \mathbb{R}^+, \tilde{F} \in \tilde{\mathcal{F}}_s^{\mathbf{H}}\}$ is called the σ-algebra of *predictable subsets of* $\tilde{\Omega}^{\mathbf{H}}$. It will be denoted by $\tilde{\mathcal{P}}^{\mathbf{H}}$. A mapping $\mathbf{a} : \tilde{\Omega}^{\mathbf{H}} \to \mathbb{K}$, where $\mathbb{K}$ is a Banach space, will be said a $\mathbb{K}$-valued predictable functional of the $\mathbb{H}$-valued regular processes, if it is a measurable mapping from $(\tilde{\Omega}^{\mathbf{H}}, \tilde{\mathcal{P}}^{\mathbf{H}})$ into K.

Remark 1. It should be noted that a predictable functional $\mathbf{a}$ is a nonanticipative functional of (t, f) in the following natural sense: let f and g be two regular mappings from $\mathbb{R}^+$ into $\mathbb{H}$, such that $f(s) = g(s)$ for every $s \leqslant t$. Then $\mathbf{a}(t, \omega, f) = \mathbf{a}(t, \omega, g)$. This follows immediately from the fact, that nonanticipative in this sense means only: adapted to the family $(\tilde{\mathcal{F}}_t^{\mathbf{H}})_{t \in \mathbf{R}^+}$ of σ-algebras.

Proposition 1. *Let K be a separable Banach space and X a regular $\mathbb{H}$-valued process. For every $\mathbb{K}$-valued predictable functional $\mathbf{a}$ the process*

$$(t, \omega) \to Y(t, \omega) := \mathbf{a}(t, \omega, X(\omega))$$

is a predictable K-valued process (we write $X(\omega)$ for the mapping $t \to X(t, \omega)$) and $Y(t, \omega)$ depends only on the values $X_s(\omega)$ $s < t$.

Proof. Because of the assumptions on $\mathbb{K}$ it is sufficient to prove the proposition for $\mathbb{K} = \mathbb{R}$. The classical monotone class-argument reduces then the proof to the case, where $\mathbf{a}$ is a functional of the following form:

$$\mathbf{a}(t, \omega, f) = 1_{]u, v]}(t).1_F(\omega).1_G(f)$$

with

$$u < v \in \mathbb{R}^+, \quad F \in \mathcal{F}_u, \quad G \in \mathcal{D}_u^{\mathbf{H}}.$$

If X is a regular $\mathbb{H}$-valued process, it is immediately clear that $1_G(X)$ is $\mathcal{F}_u$-measurable for every $G \in \mathcal{D}_u^{\mathbf{H}}$. In this case $(t, \omega) \to \mathbf{a}(t, \omega, X(\omega))$ is clearly the indicator-function of a predictable rectangle. Moreover $X_s'(\omega) = X_s(\omega)$ for every $s < t$ implies $\mathbf{a}(t, \omega, X'(\omega)) = \mathbf{a}(t, \omega, X(\omega))$. The proposition then follows.

EXAMPLE. Let α be a mapping from $\mathbb{R}^+ \times \Omega \times \mathbb{H}$ in $\mathbb{K}$, such that for every $t \in \mathbb{R}^+$ and $h \in \mathbb{H}$ the random variable $\alpha(t, . , h)$ is $\mathcal{F}_t$-measurable and $t \to \alpha(t, . , h)$ is left continuous. Then, denoting by $f(t^-)$ the left limit of f at point t, it follows from the left continuity of $t \to \alpha(t, \omega, f(t^-))$ that $(t, \omega, f) \to \mathbf{a}(t, \omega, f) := \alpha(t, \omega, f(t^-))$ defines a predictable functional of t and f. Therefore the process $(t, \omega) \to \alpha(t, \omega, X_{t^-}(\omega))$ is a predictable process for every regular process X.

This is the situation considered in [3] and [2], which thus appears as a special case of our situation here.

Let us remark that for every $f \in \mathfrak{D}^{\mathbf{H}}$ the mapping $t \to \sup_{s<t}\|f\|(s)$ is left continuous. The mapping $(t, f) \to \sup_{s<t}\|f\|(s)$ is, therefore, a predictable functional and there follows immediately the following lemma, which we state for easy later reference:

LEMMA 3. *If* $\mathbf{a}$ *is a predictable functional, the functional* $1_{\{(t,f)\,:\,\sup_{s<t}\|f(s)\| \leqslant d\}}\, \mathbf{a}$ *is for every* $d > 0$ *a predictable functional too.*

The same holds for $\mathbf{a}^d$, *where*

$$\mathbf{a}^d(t, \omega, f) := \mathbf{a}(t, \omega, f^d)$$

and

$$f^d(t) := \left(1 \wedge \frac{d}{\|f\|_t}\right) f(t), \text{ with } \|f\|_t := \sup_{0 \leqslant s < t} \|f(s)\|.$$

3.2. *Hypothesis on Z.* We recall the following definition (see P. A. Meyer [11]): a regular real or $\mathbb{G}$ valued process Z is called a semimartingale if it is the sum of a local martingale M and of a process V, the paths of which have bounded variation on any finite interval. There is no loss of generality in assuming $Z_0 = 0$.

If in the above decomposition the process V has locally integrable variation (i.e.: there exists an increasing sequence (τ_n) of stopping times such that $\lim_n \tau_n = +\infty$ and the variation of the paths of $V_{\tau_n \wedge .}$ on every interval $[0, s]$ is an integrable positive random variable), then the process Z is called a special semimartingale (see [11] Chapter 4).

The processes Z, which we will now consider, belong to a class of processes, which will be shown to include the semimartingales.

DEFINITION 2. Let σ be a stopping time. A real (resp.) $\mathbb{G}$ valued process Z will be said to satisfy the condition (*, $\mathbb{K}$) if the process $\int Y dZ$ is defined (see Remark 2 below) for every bounded predictable process Y with values in $\mathbb{K}$ (resp. in $\mathfrak{L}(\mathbb{G}; \mathbb{K})$), and if there exists a real increasing regular process Q, such that for every bounded predictable process Y with values in $\mathbb{K}$ (resp. $\mathfrak{L}(\mathbb{G}, \mathbb{K})$ and for every stopping time τ and σ, with $\tau \geqslant \sigma$, the following inequality holds:

$$(*) \qquad E\Big(\sup_{\sigma \leqslant t < \tau} \Big\|\int_{]\sigma, t]} Y_s dZ_s\Big\|^2 \leqslant E\Big\{\big[(Q_{\tau^-} - Q_\sigma) \vee 1\big] \int_{]\sigma, \tau[} \|Y_s\|^2 \, dQ_s\Big\}$$

REMARK 2. The process $\int YdZ$ is well defined when Z is a semimartingale. Every semimartingale satisfies (*), as will be stated in Proposition 3. It can be shown (see [8]), that, as soon as (*) holds for Z some Q and every indicator function Y of predictable rectangle $]s, t] \times F$, then the process $\int YdZ$ can be defined on $[\sigma, \tau[$ and satisfies (*) for the same Q and every predictable process Y such that the right member of (*) is finite.

REMARK 3. It should be noted that Q is not assumed to be integrable. This is an interesting feature of our method here. We will work, in fact, in stochastic intervals $[\sigma, \tau[$ such that $E[(Q_{\tau^-} - Q_\sigma)1_{[\tau>\sigma]}] < \infty$.

PROPOSITION 2. *Let Z be a process of the form $Z = M + V$, where M is a right continuous square integrable martingale and V a right continuous process with finite variation. Then Z satisfies the condition $(*, \mathbb{K})$ for every Hilbert space $\mathbb{K}$, with an associated process Q, which is integrable, if V has integrable variation.*

PROOF. If we apply the inequality of Section 2 to the martingale $(N_{t\vee\sigma} - N_\sigma)_{t\in\mathbf{R}^+}$, where $N = \int YdM$, we get:

$$E\left\{\sup_{\sigma\leqslant t<\tau}\|\textstyle\int_{]\sigma, t]} Y_s dZ_s\|^2\right\} \leqslant 4E\left\{\textstyle\int_{]\sigma,\tau[}\|Y_s\|^2 d([M]_s + \langle M\rangle_s)\right\}.$$

Let A be the variation of the paths of V. The Schwarz inequality gives then:

$$\begin{aligned} E\left\{\sup_{\sigma\leqslant t<\tau}\|\textstyle\int_{]\sigma,\tau]} Y_s dV_s\|^2\right\} &\leqslant E\left\{\sup_{\sigma\leqslant t<\tau}(A_t - A_\sigma)\textstyle\int_{]\sigma, t]}\|Y_s\|^2 dA_s\right\} \\ &\leqslant E\left\{(A_{\tau^-} - A_\sigma)\textstyle\int_{]\sigma,\tau[}\|Y_s\|^2 dA_s\right\}. \end{aligned}$$

If we set $Q = 8([M] + \langle M\rangle) + 2A$ we get immediately the inequality (*).

PROPOSITION 3. *Every semimartingale is a $(*, \mathbb{K})$ process. More precisely, for every $d > 0$ there exist two processes Z^d and V^d with $Z = Z^d + V^d$ and such that:*

(i)

$$V^d = \textstyle\sum_k \xi_k 1_{[T_k, \infty[}$$

where (T_k) is an increasing sequence of stopping times and the ξ_k's are $\mathcal{F}_{T_k}$-measurable random variables with $\|\xi_k\| \geqslant d$.

(ii) *Z^d is a $(*, \mathbb{K})$ process for every Hilbert space $\mathbb{K}$ with associated process Q having no jump greater than $16d^2 + 2d$.*

PROOF. We consider the well-measurable subset of $\mathbb{R}^+ \times \Omega : \{\|\Delta Z\| \geqslant d\} = \{(t, \omega) : \|Z(t, \omega) - \lim_{s\uparrow t} Z(s, t)\| \geqslant d\}$.

As every path of Z has a finite number of jumps $\geqslant d$ on any finite interval this set is the union of a denumerable family of graphs of an increasing sequence (T_n) of stopping times. Then we set:

$$V^d = \textstyle\sum_k \Delta Z_{T_k} 1_{[T_k, \infty[}.$$

The process $Z^d = Z - V^d$, which has bounded jumps, is a special semimartingale (see P. A. Meyer [1]). Therefore, there exists an increasing sequence (τ_n) of stopping times with $\lim\uparrow\tau_n = \infty$ and for each n a square integrable martingale

$M^{d,n}$ and a process $V^{d,n}$ with integrable variation A^{dn}, such that $Z^d_{\tau_n \wedge} = M^{d,n}_{\tau_n \wedge .} + V^{d,n}_{\tau_n \wedge .}$

We set:

$$Q := \Sigma_n 1_{]\tau_n, \infty[}\Big[8\big[M^{d,n+1}_{\tau_{n+1}\cdot}\big] + 8\langle M^{d,n+1}_{\tau_{n+1}\wedge .}\rangle + 2A^{d,n+1}_{\tau_{n+1}\wedge .}$$
$$-8\big[M^{d,n}_{\tau_n \wedge .}\big] - 8\langle M^{d,n}_{\tau_n \wedge .}\rangle - 2A^{d,n}_{\tau_n \wedge .}\Big].$$

As in the proof of Proposition 2 we get for every stochastic interval $[\sigma, \tau[$; and every n:

$$E\{\sup_{\sigma \leqslant t < (\tau \wedge \tau_n) \vee \sigma} \|\textstyle\int_{]\sigma, t]} Y_s dZ_s\|^2\} \leqslant \{[Q_{(\tau \wedge \tau_n) - \vee\sigma} - Q_\sigma]\textstyle\int_{]\sigma, (\tau \wedge \tau_n) \vee \sigma[}\|Y_s\|^2 dQ_s\}.$$

Letting n increase towards $+\infty$ we get property (*).

None of the processes $M^{d,n}$ and $V^{d,n}$, having jumps greater than d, the same holds for $[M^{d,n}]$ and $\langle M^{d,n}\rangle$. Therefore, no jump of Q is greater than $16d^2 + 2d$.

3.3. *Hypothesis on* **a**. The various hypotheses which will be made on **a**, beside predictability, are of Lipschitz type. For easy further reference we define:

DEFINITION 3. The functional **a** will be said to satisfy:

(L_1) if there exists $L > 0$ such that

(3.3.1) $$\|\mathbf{a}(t, \omega, f) - \mathbf{a}(t, \omega, f')\| \leqslant L \sup_{s<t}\|f(s) - f'(s)\|$$

for all $t \notin \mathbb{R}^+, f, f' \in D^{\mathbf{H}}$.

(L_2) if for every $d > 0$ there exists a constant $L_d > 0$ such that

(3.3.2) $$\|\mathbf{a}(t, \ , f) - \mathbf{a}(t, \ , f')\| \leqslant L_d \sup_{s<t}\|f(s) - f'(s)\|$$

for all $t \in \mathbb{R}^+$, and $f, f' \in \{g : g \in D^{\mathbf{H}}, \sup_{s<t}\|g(s)\| \leqslant d\}$.

(L_3) if there exist $C > 0$ such that

(3.3.3) $$\|\mathbf{a}(t, \omega, f)\| \leqslant C \sup_{0 \leqslant s < 0}[\|f(s)\| + 1] \text{ for all } t \in \mathbb{R}^+, f \in D^{\mathbf{H}}.$$

(L_4) if for every $d > 0$ there exist a constant $C_d > 0$ such that

(3.3.4) $$\|\mathbf{a}(t, \omega, f)\| \leqslant C_d \sup_{0 \leqslant s < t}[\|f(s)\| + 1]$$

for all $t \in \mathbb{R}^+$ and f such that $\sup_{0 \leqslant s < t}\|f(s)\| \leqslant d$.

4. Existence and unicity of strong solutions of equation (3.2). In all this section Z will be a $\mathbb{G}$-valued process ($\mathbb{G}$: Hilbert space) and **a** a predictable $\mathfrak{L}(\mathbb{G}; \mathbb{H})$-valued functional on $\Omega^{\mathbf{H}}$. ξ_0 is an $\mathbb{H}$-valued $\mathfrak{F}_0$-measurable random variable.

4.1. *Strong solutions.* In this paper we will consider only strong solutions of Equation 3.2. A process X defined on the open (resp. closed) stochastic interval $[0, \sigma[$ (resp. $[0, \sigma]$) is said to be a strong solution of (3.2) on $[0, \sigma[$ (resp. $[0, \sigma]$) with initial value ξ_0, if the process $(\int_0^t a(s, x)\, dZ_s)_t$ is well defined on $[0, \sigma[$ (resp. $[0, \sigma]$) as a regular process and differs from $X - \xi_0$ by a P-null process. (Following Proposition 1, X need only be given regular on $[0, \sigma[$ with left limit at σ, for $a(s, X)$ to exist on $[0, \sigma]$.)

Two solutions X and X' are said to be equal if they are defined on the same stochastic interval $[0, \sigma[$ and if the processes $X.1_{[0,\sigma[}$ and $X'1_{[0,\sigma[}$ are equivalent.

4.2. *Preliminary lemmas on existence.*

LEMMA 4. *Let us assume that Z is a $(*, \mathbb{H})$-process with associated process Q,* **a** *satisfies* (L_1) *and* (L_3) *with Lipschitz-constant L and constant C respectively. We suppose, moreover, Q bounded by $q > 0$ on $[0, \sigma[$ with $q < (1/L) \wedge 1$, and $\xi_0 \in L^2_{\mathbb{H}}(\Omega, \mathcal{F}_0, P)$. Then* (3.2) *admits a unique regular solution X on $[0, \sigma]$.*

PROOF. We consider the space $\Lambda(\sigma; \mathbb{H})$ of regular $\mathbb{H}$-valued processes X defined (up to an equivalence) on $[0, \sigma[$ with the norm:

$$\|X\|_{\Lambda,\sigma} = \left[E\left(\sup_{0 \leqslant t < \sigma}\|X_t\|^2\right)\right]^{\frac{1}{2}} < \infty.$$

We define the operator U on $\Lambda(\sigma; \mathbb{H})$ by:

$$(4.2.1) \qquad (UX)_t = \xi_0 + \textstyle\int_0^t \mathbf{a}(s, ., X)\, dZ_s.$$

This is possible because of (L_3) (see Remark 2). From the most classical fixed point theorem, the existence and unicity of the solution X on $[0, \sigma[$ will follow from the fact that U is a contraction.

But:

$$\|UX - UY\|^2_{\Lambda,\sigma} = E\left\{\sup_{0 \leqslant t < \sigma}\left\|\textstyle\int_{[0,t]}(\mathbf{a}(s, ., X) - \mathbf{a}(s, ., Y))\, dZ_s\right\|^2\right\}$$

and the $(*, \mathbb{H})$ condition gives:

$$\|UX - UY\|^2_{\Lambda,\sigma} \leqslant E(Q_{\sigma^-} \vee 1).\textstyle\int_{[0,\sigma[}\|\mathbf{a}(s, ., X) - \mathbf{a}(s, ., Y)\|^2\, dQ_s\}$$
$$\leqslant (q \vee 1)qL\|X - Y\|^2_{\Lambda,\sigma}.$$

Because of the assumption on q, U is a contraction.

Let ξ be a solution on $[0, \sigma[$. In view of the boundedness of **a**, the process $(\int_{[0,t]}a(s, ., \xi)\, dZ_s)_{0 \leqslant t < \sigma}$ has a.s. left limits, when t increases towards σ. It is then clear, because of the last assertion of Proposition 1, that the process X defined by:

$$X_t = \xi_t 1_{[t<\sigma]} + 1_{[t \geqslant \sigma]}\left(\lim_{s\uparrow\sigma}\xi_s + a(\sigma, .\xi)\,\Delta Z_\sigma\right)$$

is the unique solution of (3.2) on $[0, \sigma]$.

LEMMA 5. (*Extension principle for solutions*). *We assume, that Z is a $(*, \mathbb{H})$ process and the random functional* **a** *satisfies* L_2 *and* L_4. *Let ξ be a regular process, such that $1_{[0,\sigma]}\xi$ is a strong solution of* (3.2) *on* $[0, \sigma]$. *Then, if $P[\sigma < \infty] > 0$, there exists for each $\varepsilon > 0$ a stopping time τ and a regular process X on $[0, \tau]$, such that $P[\tau > \sigma] > P[\sigma < \infty] - \varepsilon$ and X is the unique strong solution of* (3.2) *on* $[0, \tau]$. *More precisely*

$$(4.2.2) \qquad X_t = \xi_{t \wedge \sigma} + \textstyle\int_{]\sigma, t \vee \sigma]}\mathbf{a}(s, ., X)\, dZ_s.$$

PROOF. We call $F_d = \left\{ \sup_{0 \leqslant s < \sigma} \|\xi_s\| \leqslant \frac{d}{2} \right\} \in \mathcal{F}_\sigma$. As $\lim_{s \uparrow \sigma;\, s < \sigma} \xi_s$ exists a.s. on $\sigma < \infty$ we can choose d in such a way that

$$P(F_d) > P[\sigma < \infty] - \varepsilon,$$

where $\varepsilon > 0$ is given.

We define next the stopping time

$$\tau_d' = \inf\left\{ t : t \geqslant \sigma, (Q_t \vee 1).(Q_t - Q_\sigma) \geqslant \frac{1}{4L_d} \right\}.$$

From the right continuity of Q we get

$$[\tau_d' > \sigma] = [\sigma < \infty]. \tag{4.2.3}$$

As $F_d \in \mathcal{F}_\sigma$ we may consider the stopping time

$$\tau_d'' := 1_{F_d} \tau_d' + 1_{CF_d} \sigma.$$

We consider next the subspace of $\Lambda(\tau_d'', \mathbb{H})$ (see proof of Lemma 4) consisting of those processes X, such that $1_{[0,\sigma]} X = 1_{[0,\sigma]} \xi$ and define the operator U on this subspace by:

$$UX_t = \xi_{t \wedge \sigma} + \textstyle\int_{]\sigma, \sigma \vee t]} \mathbf{a}^{2d}(s, .\,, X)\, dZ_s$$

where $\mathbf{a}^{2d}$ is the predictable functional of Lemma 3. Exactly the same reasoning as in Lemma 4 shows that U is a contraction and there exists a unique process X on $[0, \tau_d''[$ such that

$$X_t = \xi_{t \wedge \sigma} + \textstyle\int_{]\sigma, t \vee \sigma]} \mathbf{a}^{2d}(s, .\,, X)\, dZ_s \tag{4.2.4}$$

for $t < \tau_d''$.

Let us define now the stopping time τ:

$$\tau := \inf\{ t : t \geqslant \sigma \|X_t\| \geqslant 2d \} \wedge \tau_d''.$$

Because of the right continuity of X of the definition of τ_d'' and of (4.2.3) we get $[\tau > \sigma] \supset F_d$ and therefore $P[\tau > \sigma] > P[\sigma < \infty] - \varepsilon$. According to the definition of τ_d'' and τ, the inequality $\sup_{0 \leqslant s < t} \|X_s\| \leqslant 2d$ holds for every $(t, \omega) \in [\sigma, \tau[$ and therefore the processes X and $\xi_{.\wedge \sigma} + \int_{]\sigma, .\vee \sigma]} \mathbf{a}(s, .\,, x)\, dZ_s$ coincide on $[0, \tau[$.

As $\|\mathbf{a}(s, \omega, X)\| \leqslant C_{2d}$ for every $(s, \omega) \in [\sigma, \tau]$ the limit

$$X_{\tau^-} = \lim_{t \uparrow \tau;\, t < \tau} \textstyle\int_{]\sigma, \tau \vee \sigma]} \mathbf{a}(s, .\,, X)\, dZ_s$$

exists. As in the preceding Lemma we see that the process

$$\xi_{.\wedge \sigma} + \textstyle\int_{]\sigma, .\vee \sigma]} \mathbf{a}(s, .\,, X)\, dZ_s$$

defines also the unique solution of (3.2) on the closed stochastic interval $[0, \tau]$.

LEMMA 6. *Let us assume, that Z is a* $(*, \mathbb{H})$ *process, with associated increasing process Q, ξ_0 is any $\mathbb{H}$-valued random variable ($\mathcal{F}_0$-measurable) and* **a** *satisfies* (L_2) *and* (L_4). *Then $\forall\, \varepsilon > 0$ there exist a stopping time τ with $P[\tau > 0] \geqslant 1 - \varepsilon$ and a process X on* $[0, \tau]$ *such that X is the unique regular solution of* (3.2) *on* $[0, \tau]$.

PROOF. We choose first a $d > 0$ such that

$$P[\|\xi_0\| > d] < \varepsilon.$$

Applying Lemma 3 we substitute to **a** the function $\mathbf{a}^{2d}$, which satisfies (L_1) and (L_3), with the constant L_{2d} and C_{2d} respectively.
We define:

$$\sigma = \inf\left\{t : Q_t \geqslant \frac{1}{L_{2d}} \wedge 1\right\}.$$

Because of the right continuity of Q, $P[\sigma > 0] = 1$. We apply Lemma 4 to the equation

$$X_t = \xi_0 \wedge 2d + \int_0^t \mathbf{a}^{2d}(s, ., X)\, dZ_s.$$

But X is clearly a solution of (3.2) with initial condition ξ_0 on the stochastic interval $[0, \tau]$, where

$$\tau = \inf\{t : \|X_t\| > 2d\} \wedge \sigma.$$

Because of the right continuity of X

$$[\tau > 0] \supset \{\|\xi_0\| < 2d\} \quad \text{and therefore} \quad P[\tau > 0] \geqslant 1 - \varepsilon.$$

4.3. *Maximal solutions.*

DEFINITION 4. The couple (τ, X) where τ is a stopping time with $P[\tau > 0] > 0$ and X a process on $[0, \tau[$ is said to be a maximal solution of (3.2), if X is a regular process, which is a solution of (3.2) on $[0, \sigma[$ and if for any other couple (τ', X') with the same property the inequality $\tau' \geqslant \tau$ a.s. and the equality $1_{[0,\tau[}X = 1_{[0,\tau[}X'$ imply $\tau' = \tau$.

THEOREM 3. (*Existence and unicity of maximal solutions*). *Let Z be a* $(*, \mathbb{H})$ *process* (*this is the case if Z is any semimartingale*). *We assume conditions* L_2 *and* L_4 *on* **a**. *Then there exists a maximal solution* (τ, X) *of* (3.2.).

The couple (τ, X) *is unique in the following sense*: *for any other maximal solution* (τ', X') *one has* $\tau = \tau'$ a.s. *and* $X' - X$ *is a P-null process on* $[0, \tau[$.

Moreover, the stopping time τ of a maximal solution is predictable and on $[\tau < \infty]$ *holds*

$$\limsup_{t \uparrow \tau;\, t < \tau < \infty} \|X_t\| = +\infty.$$

PROOF. We consider the family $\mathbb{S}$ of couples (τ, X), where τ is a stopping time with $P[\tau > 0] > 0$, X is a regular process and is the unique solution of (3.2) on $[0, \tau[$; the set $\mathbb{S}$ is not empty according to Lemma 6. We will denote by τ_∞ the essential supremum of those stopping times τ and by (τ_n) an extracted increasing sequence such that $\tau_\infty = \lim_n \tau_n$ a.s. (The sequence may be indeed assumed to be increasing because of the unicity property: if (τ_1, X_1) and (τ_2, X_2) belong to $\mathbb{S}$, X_1 and X_2 are equivalent on $[0, \tau_1 \wedge \tau_2[$ and, therefore, X exists on $[0, \tau_1 \vee \tau_2[$ such that $(\tau_1 \vee \tau_2, X)$ belongs to $\mathbb{S}$).

Because of the unicity property the process X is well defined by $1_{[0,\tau_n[}X = 1_{[0,\tau_n[}X_n$ on $[0, \tau_\infty[$ and $(\tau_\infty, X) \in \mathbb{S}$. The couple (τ_n, X) is clearly the unique maximal solution.

We consider now the following family of stopping times:

$$\nu_n^d := \tau_n \wedge \inf\{t : \|X_t^n\| \geqslant d\}$$

$$\nu^d := \sup_n \nu_n^d.$$

We have clearly $\nu^d \leqslant \tau_\infty$ a.s. and we prove that:

$$P\big(\big[\nu^d = \tau_\infty\big] \cap \big[\tau_\infty < \infty\big]\big) = 0. \tag{4.3.1}$$

Let us assume indeed that $P[\nu^d = \tau_\infty < \infty] > 0$.

As the process X is bounded on $[0, \nu^d[$, the process $\int_0^t a(s, .X)\, dZs$ has a limit when $t \uparrow \nu^d$ $t < \nu^d$.

We can then extend X into a solution on $[0, \nu^d]$ by the already used procedure (see proof of Lemma 4).

By the extension Lemma 5 there would exist a (τ', X') with $\tau' \geqslant \nu^d$ and $P[\tau' > \nu^d = \tau_\infty] > 0$ such that $(\tau', X') \in \mathbb{S}$. This would mean $(\tau', X') \in \mathbb{S}$ and $P[\tau' > \tau_\infty] > 0$, contradicting the definition of τ_∞. This proves (4.3.1). But the definition of ν^d and the regularity of X show:

$$\tau_\infty = \lim_k \uparrow \nu^k.$$

This equality and (4.3.1) imply the predictability of τ_∞.

4.4 *Conditions for nonexplosion.*

THEOREM 4. *Let Z be a real* (resp. $\mathbb{G}$-*valued*) *semimartingale, and* **a** *an* $\mathbb{H}$-*valued* (resp. $\mathcal{L}(\mathbb{G}; \mathbb{H})$ *valued*) *predictable functional on* $\tilde{\Omega}^{\mathbb{H}}$. *We assume condition* (L_1) *and* (L_3) *for* **a**. *Then the unique maximal solution* (τ, X) *of* (3.2) *is such that* $\tau = +\infty$.

PROOF. Let us choose $q > 0$ with $q < (1/L) \wedge 1$, and, Q being an increasing process associated with Z, such that (*) holds, define recursively

$$\tau_0 = 0$$

$$\tau_{n+1} = \inf\{t : t > \tau_n, Q_t - Q_{\tau_n} \geqslant q\}.$$

We have

$$Q_{\tau_{n+1}^-} - Q_{\tau_n} \leqslant q \text{ for all } n.$$

Applying Lemma 4 we derive the existence of a unique solution on $[0, \tau_1]$.

Assuming that the solution is uniquely defined on $[0, \tau_n]$ we prove it is uniquely defined on $[0, \tau_{n+1}]$. As clearly $\lim_n \uparrow \tau_n = +\infty$ a.s. the theorem will follow.

To proceed with this induction we consider as in the proof of Lemma 4 the subspace of $\Lambda(\tau_{n+1}, \mathbb{H})$ consisting of those processes X, such that $1_{[0,\tau_n]}X = 1_{[0,\tau_n]}X^n$, where X^n is the already defined solution on $[0, \tau_n]$, and we define the operator U on this subspace by:

$$UX_t = X_{t\wedge\tau_n}^n + \int_{]\tau_n, \tau_n \vee t]} \mathbf{a}(s, ., X)\, dZs.$$

The existence and unicity of the solution X^{n+1} on $[0, \tau_{n+1}[$ follows from the contraction property of U and the unique extension of X^{n+1} to $[0, \tau_{n+1}]$ is as in Lemmas 4 and 5 defined through the formula:

$$X^{n+1}_{\tau_{n+1}} = X^{n+1}_{\tau^-_{n+1}} + \mathbf{a}(\tau_{n+1}, \cdot, X^{n+1})\Delta Z_{\tau_{n+1}},$$

where $\mathbf{a}(\tau_{n+1}, \cdot, X^{n+1})$ depends only on the values of X^{n+1} on $[0, \tau_{n+1}[$.

THEOREM 5. *Let Z be a real* (resp. *G-valued*) *semimartingale and* $\mathbf{a}$ *an $\mathbb{H}$-valued* (resp. *$(G; \mathbb{H})$-valued*) *predictable functional on $\tilde{\Omega}^{\mathbb{H}}$. We assume the conditions* (L_2) *and* (L_3) *for* $\mathbf{a}$. *Then the maximal solution (X, τ) of Theorem* 3 *is such that* $\tau = \infty$ a.s. (i.e.: *there is no explosion*).

PROOF. We have to prove that for every positive number α we have $P([\tau < \alpha]) = 0$. Assuming on the contrary that $P([\tau < \alpha]) = 2\delta > 0$ we will derive a contradiction.

Let Q be a process associated with Z, according to Proposition 3. There exists a positive number p and a stopping time τ'' such that

$$\tau'' := \inf\{t : Q_t \vee \|X_0\| > p\} \wedge \alpha$$

and

$$P([\tau'' > \tau] \cap [\tau < \alpha]) \geqslant 5.$$

We define $\tau' := \tau \wedge \tau''$, and the positive process on $[0, \tau[: X^*_t := \sup_{0 \leqslant s \leqslant t}\|X_s\|^2$. From Theorem 3 follows:

$$[\alpha > \tau' \geqslant \tau] \subset \{\lim_{t \uparrow \tau}\|X_t\| = \infty\}. \tag{4.4.1}$$

We will derive a contradiction with the assumption $\delta > 0$, by proving that the condition $P[\alpha > \tau' \geqslant \tau] \geqslant \delta > 0$ would have the following inequality as a consequence:

$$E(X^*_{\tau'^-}.1_{[\tau' < \alpha]}) \leqslant 2Cp^2 e^{(18c/\delta)p}. \tag{4.4.2}$$

This would contradict indeed the equality $E\{X^*_{\tau'^-}.1_{[\tau' < \alpha]}\} = \infty$, which follows from $\delta > 0$ and (4.4.1).

In order to prove (4.4.2), we define by induction the following sequences of stopping times (τ_n) and sets (F_n), writing $G := [\tau \leqslant \tau' < \alpha]$ for simplicity:

$$\sigma_1 := \inf\{t : t > 0, \|X_t\|^2 > 3\} \wedge \alpha$$

$$F_1 := [X^*_{\sigma_1} \geqslant 3]$$

and for $n \geqslant 1$:

$$\sigma_{n+1} := \inf\{t : t > \sigma_n, \|X_t\|^2 \geqslant 3^{n+1}\} \wedge (\sigma_n 1_{F_n} + \alpha 1_{CF_n})$$

$$F_{n+1} := [X^*_{\sigma_{n+1}} \geqslant 3^{n+1}].$$

It is clear from the definitions that $X^*_{\sigma_1} \geqslant 3$ on G and moreover:

(4.4.3) $$\sigma_n < \sigma \text{ and } X^*_{\sigma_n} \geqslant 3^{n-1} \text{ a.s. } on\ G;$$

(4.4.4) $$\lim_n \sigma_n = \tau' \text{ a.s. } on\ G;$$

(4.4.5) $$\sup_{\sigma_n < t \leqslant \sigma_{n+1}} \|X_t - X_{\sigma_n}\| \geqslant 2X^*_{\sigma_n} 1_{[\alpha > \sigma_{n+1} > \sigma_n]}.$$

From this last inequality (and the elementary inequality $(a+b)^2 \leqslant a^2 + 2b^2$ for $b \geqslant 2a$!) we get

$$1_{[\sigma_{n+1} < \alpha]} X^*_{\sigma_{n+1}} \leqslant 1_{[\sigma_n < \alpha]} X^*_{\sigma_n} + 2 \sup_{\sigma_n < t \leqslant \sigma_{n+1}} \|X_t - X_{\sigma_n}\|^2.$$

Writing $x_n := E[1_{[\sigma_n < \alpha]} X^*_{\sigma_{n+1}}]$, the assumptions on the process Z and $\mathbf{a}$ then imply for the sequence (x_n):

$$\begin{aligned} x_{n+1} &\leqslant x_n + 2E\left\{\sup_{\sigma_n < t \leqslant \sigma_{n+1}} \|\textstyle\int_{]\sigma_{n+1}, t]} \mathbf{a}(s, ., x)\, dZ_s\|^2\right\} \\ &\leqslant x_n + 2E\left\{\left[(Q_{\sigma_{n+1}} - Q_{\sigma_n}) \vee 1\right] \textstyle\int_{]\sigma_n, \sigma_{n+1}]} \|\mathbf{a}(s, ., x)\|^2\, dQ_s\right\} \\ &\leqslant x_n + 2CE\left\{(Q_{\sigma_{n+1}} - Q_{\sigma_n})\left(1 + \sup_{0 \leqslant 1 < \sigma_{n+1}} \|X_s\|^2\right)\right\} \\ &\leqslant x_n + 2C(1 + 3^{n+1})(q_{n+1} - q_n) \end{aligned}$$

if we write q_n for $E(Q_{\sigma_n})$.

But according to (4.4.3) and the assumption $\delta > 0$, we have $x_n \geqslant 3^{n-1}\delta$, so that the last inequality gives:

$$x_{n+1} \leqslant \left(1 + 18\frac{C}{\delta}(q_{n+1} - q_n)\right)x_n + 2C(q_{n+1} - q_n).$$

As Q and X_0 are bounded by p on $[0, \tau'[$, this implies

$$x_{n+1} \leqslant 2Cq_{n+1} p e^{(18C/\delta) q_{n+1}} \leqslant 2Cp^2 e^{(18C/\delta)p}.$$

Because of (4.4.4) inequality (4.4.2) follows, and therefore the contradiction.

4.5 *Comparison with other methods and results and conclusion..*

(1) We proved in Section 3, Proposition 2, that the class of Hilbert-valued processes which satisfy the (*)-inequality (Definition 2) includes semimartingales. Here is an important usual example of an L^2-valued continuous deterministic process which is not a semimartingale: consider the "spectral" (nonrandom) process Z, defined by $Z_t = 1_{]0,t[} \in L^2(\mathfrak{R}_+)$. The function $t \to Z_t$ is, as is well known, of unbounded variation on any interval $]0, \alpha]$, $\alpha > 0$, and has a zero "martingale part". It satisfies, however, the condition (*) with process $Q : Q_t := t$. Processes of this type are to be met frequently as soon as infinite-dimensional-valued processes are considered. The techniques tied to the bounded variation of the "nonmartingale part" of a semimartingale are not extendable to this case.

(2) In [5] M. Emery considered the following situation: Z is a real semimartingale and F is a mapping of cadlag processes into cadlag processes, with a nonanticipative property and satisfying a Lipschitz condition, which is essentially condition (L_1) above. If Ω is the canonical space of paths, then Emery's situation is

contained in ours. When Ω is not the canonical space, these inclusions do not hold anymore, which leads to the following remark.

(3) We assumed the functional **a** to be defined path-by-path, because such a functional is given in most physical problems. But it is clear that a more abstract form can be given to the theorem, involving a functional $\tilde{\mathbf{a}}$ mapping cadlag process into locally bounded predictable processes, adapted in the following sense:

$$X_s = X_s' \quad \text{for all} \quad s < t \Rightarrow (\tilde{a}X)_t = (\tilde{a}X')_t$$

and satisfying the lipschitz condition:

(L_2') $\|\tilde{a}X - \tilde{a}X'\|_t \leqslant L_b(t).(\sup_{s<t}\|X_s - X_s'\|)$ for all $b > 0$, all t and all couples X and X' of cadlag processes such that $\sup_{s<t}\|X_s\| \leqslant b$ and $\sup_{s<t}\|X_s'\| \leqslant b$.

The results and proofs in such an abstract situation would be pure rewording of what has been done in this paper.

(4) Since the writing of the manuscript of this paper, the authors have developed stability results, which are in the process of being published.

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