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MONOTONICITY OF AN INTEGRAL OF M. KLASS[1]

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For each value of β, $0 < \beta < 2$, the integral

$$\int_{-\infty}^{\infty}\{1 - \exp(-x^{-2}\sin^2 tx)\}|t|^{-1-\beta}dt$$

decreases monotonically as a function of x, $x > 0$. This result is useful in approximating the absolute βth moment of the sum of zero mean i.i.d. random variables.

Let $0 < \beta < 2$; define $g\colon \mathbb{R} \to \mathbb{R}$ by

$$g(x) = \int_{-\infty}^{\infty}\{1 - \exp(-x^{-2}\sin^2 tx)\}|t|^{-1-\beta}dt \qquad \text{for} \quad x \neq 0$$

and by

$$g(0) = \lim_{x\to 0} g(x) = \int_{-\infty}^{\infty}(1 - e^{-t^2})|t|^{-1-\beta}dt.$$

It is easy to check that g is continuously differentiable for $x \neq 0$.

The following result was conjectured by M. Klass (1978):

THEOREM. *$g'(x) \leqslant 0$ for all $x > 0$.*

REMARKS. Actually, Klass worked with the integral

$$\int_{-\infty}^{\infty}\left\{1 - \exp\frac{\cos tx - 1}{x^2}\right\}|t|^{-1-\beta}dt;$$

the double angle formula $\cos 2\theta - 1 = -2\sin^2\theta$ shows that his integral is our $2^{-\beta/2}g(2^{-\frac{1}{2}}x)$. He gave a proof of the theorem in the special case $\beta = 1$ and verified it by computer for many other values of β. He used the present result to derive high precision bounds on $E|S_n|^\beta$, where S_n is the sum of i.i.d. random variables.

PROOF. It is clear from the definition of g as an integral that $0 < g(x) < g(0)$ for $x > 0$, and that $\lim_{x\to\infty} g(x) = 0$. Hence $g'(x) < 0$ for certain values of x arbitrarily close to 0 and for certain other values of x arbitrarily large. A Laplace transform argument below will show that the set of x for which $g'(x) \leqslant 0$ is an interval; taken together these observations certainly imply that $g'(x) \leqslant 0$ for all $x > 0$.

The Laplace transform argument is easiest done under the change of variables $s = x^{-2}$. Let

$$(*) \qquad k(s) = \int_{-\infty}^{\infty}(1 - e^{-s\sin^2 t})|t|^{-(1+\beta)}dt;$$

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then $k(s) = g(s^{-\frac{1}{2}})s^{\beta/2}$ so if $x > 0$, $g'(x) \leqslant 0$ if and only if $h(s) = s^{\beta/2}(d/ds)k(s)s^{-\beta/2}) \geqslant 0$.

The claim is that the set of $s > 0$ such that $h(s) \geqslant 0$ forms an interval. Actually, more is true: $h(s)$ has at most one sign change in $(0, \infty)$. This follows because $h(s)$ is (a constant multiple of) the Laplace transform of a certain function $f_Z - (\beta/2)f_{UZ}$ defined below, which itself has at most one sign change in $(0, \infty)$. (By the "variation diminishing property of the Laplace transform" (Karlin, 1968) Laplace transforming cannot increase the number of sign changes of a function.) The gist of the proof consists in exhibiting $f_Z - (\beta/2)f_{UZ}$, showing that its Laplace transform is a constant multiple of the function $h(s)$, and finally, showing that $f_Z - (\beta/2)f_{UZ}$ has at most one sign change.

By definition

$$h(s) = s^{\beta/2}\frac{d}{ds}(k(s)s^{-\beta/2}) = k'(s) - \frac{\beta}{2s}k(s).$$

It is clear from (*) that $k(0) = 0$, so

$$\begin{aligned} h(s) &= k'(s) - \frac{\beta}{2s}\textstyle\int_0^s k'(\sigma)d\sigma \\ &= k'(s) - \frac{\beta}{2}Ek'(sU), \end{aligned}$$

where U is a random variable uniformly distributed on [0, 1]. Examination of (*) shows that $k'(s)$ is a Laplace transform:

$$k'(s) = \textstyle\int_{-\infty}^{\infty}\sin^2 t|t|^{-(1+\beta)}e^{-s\sin^2 t}dt.$$

Let T be a random variable (independent of U) with density function $(1/c)\sin^2 t|t|^{-(1+\beta)}$, where $c = \int_{-\infty}^{\infty}\sin^2 t|t|^{-(1+\beta)}dt$. Then $k'(s)$ is just c times the Laplace transform of $Z = \sin^2 T$, that is,

$$k'(s) = c \cdot E(e^{-sZ}).$$

Let f_Z be the density function of the random variable Z and let f_{UZ} be the density function of the random variable UZ. $f_Z(t)$ and $f_{UZ}(t)$ both vanish if t is outside the range [0, 1]. Then

$$\begin{aligned} h(s) &= c\left(Ee^{-sZ} - \frac{\beta}{2}Ee^{-sUZ}\right) \\ &= c\textstyle\int_0^1\left(f_Z(z) - \frac{\beta}{2}f_{UZ}(z)\right)e^{-sz}dz, \end{aligned}$$

as promised.

All the remains to be proven is that the function $f_Z - (\beta/2)f_{UZ}$ has at most one sign change in (0, 1).

To see this define the random variable T' to be the mod π residue of T, so $P(0 \leqslant T' < \pi) = 1$ and $P(\exists n \in \mathbb{Z}$ such that $T' = T + n\pi) = 1$. The density function of T' is equal to 0 for $t < 0$ and $t > \pi$, and for $0 < t < \pi$ is given by

$(1/c)\sin^2 t\phi(t)$ where

$$\phi(t) = \Sigma_{n=-\infty}^{\infty}|t - n\pi|^{-1-\beta}.$$

Calculus, change of variables, and the fact that $\phi(t) = \phi(\pi - t)$, yields

$$A\left(f_Z(z) - \frac{\beta}{2}f_{UZ}(z)\right) = \tan t\phi(t) - \beta\int_t^{\pi/2}\phi(u)du = l(t), \qquad \text{say},$$

where $A > 0$ is some suitable constant and $t = \arcsin z^{\frac{1}{2}}$ for $0 < z < 1$, i.e., for $0 < t < \pi/2$. We prove $l'(t) \geqslant 0$ on $(0, \pi/2]$; this shows $f_Z - \frac{\beta}{2}f_{UZ}$ is monotone on $(0, 1]$.

Differentiating, we obtain

$$l'(t) = \sec^2 t\phi(t) + \tan t\phi'(t) + \beta\phi(t);$$

we show this is positive by arguing term by term in the summation defining ϕ. Since

$$\phi'(t) = \Sigma_{n=-\infty}^{\infty}\frac{-(1+\beta)}{|t - \pi n|^{2+\beta}}\cdot \operatorname{sgn}(t - \pi n)$$
$$= -(1+\beta)\Sigma_{n=-\infty}^{\infty}\frac{1}{|t - \pi n|^{1+\beta}}\frac{1}{t - \pi n},$$

the nth term in $l'(t)$ is

$$\frac{1}{|t - \pi n|^{1+\beta}}\left\{\sec^2 t + \beta - (1+\beta)\frac{\tan t}{t - \pi n}\right\}.$$

We argue that the quantity in braces is $\geqslant 0$ for each t in $(0, \pi/2]$ and each integer n.

Since $\sec^2 t$ and $\tan t$ are periodic with period π, it suffices to show that

$$(**) \qquad \sec^2 w + \beta \geqslant (1+\beta)\frac{\tan w}{w}\cdot$$

for all w of the form $w = t - \pi n$, where n is an integer and $0 < w \leqslant \pi/2$.

By periodicity, and since $\tan w \geqslant 0$ for all our w's, it suffices to check the "worst" case $0 < w \leqslant \pi/2$. Further, $\tan w \geqslant w$ for each such w, so the function $\sec^2 w + \beta - (1+\beta)(\tan w/w)$ decreases as β increases. Thus if the inequality (**) is verified for $\beta = 2$ it is automatically true for all $\beta < 2$. When $\beta = 2$, (**) reduces to

$$w(1 + 2\cos^2 w) \geqslant 3\cos w \sin w$$

for all $w \in (0, \pi/2]$; the double angle formulae imply that this is equivalent to

$$\lambda(\theta) = \theta(2 + \cos\theta) - 3\sin\theta \geqslant 0$$

for all $\theta \in (0, \pi]$. Now,

$$\lambda'(\theta) = 2 - 2\cos\theta - \theta\sin\theta,$$
$$\lambda''(\theta) = \sin\theta - \theta\cos\theta,$$

and

$$\lambda'''(\theta) = \theta \sin \theta \geqslant 0 \text{ in } (0, \pi].$$

Further, $\lambda(0) = \lambda'(0) = \lambda''(0) = 0$, so by Taylor's theorem, if $0 < \theta \leqslant \pi$,

$$\lambda(\theta) = \int_0^\theta \lambda'''(t) \frac{(\theta - t)^2}{2} dt \geqslant 0. \qquad \square$$

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