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RATES OF ESCAPE OF INFINITE DIMENSIONAL BROWNIAN MOTION

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In this paper the analogue in infinite dimensions of the Erdös-Dvoretzky rate of escape test for finite dimensional Brownian motion is proved. Some examples are constructed which exhibit the essential differences between the finite and infinite dimensional cases and which suggest several conjectures and problems.

Introduction. Let $(V, \|\cdot\|)$ be a real separable infinite dimensional Banach space and let $X = \{X(t); t \geqslant 0\}$ be a Brownian motion on V with $X(0) = 0$. (See Section 3 for the definition.) If X is genuinely d-dimensional with $d \geqslant 3$ (see Remark 2) then $P[\lim_{t\uparrow\infty}\|X(t)\| = \infty] = 1$, as one may easily show, see Section 3. If also $d < \infty$, then for any positive function $h = h(t)\downarrow 0$ as $t\uparrow\infty$

$$(1.1)\quad P\Big[\|X(t)\| \leqslant t^{\frac{1}{2}}h(t) \quad \text{i.o.} \quad as\ t\uparrow\infty\Big]$$
$$= 0 \text{ or } 1 \text{ according as } \Sigma_k h^{d-2}(2^k) \text{ converges or diverges}$$

(i.o. = infinitely often.) See Dvoretzky and Erdös (1951). The main purposes here are (a) to find the appropriate extension of (1.1) to the case $d = \infty$ (Theorem 2), and (b) to exhibit some examples which illustrate the essential differences between the cases $d < \infty$ and $d = \infty$. These examples also motivate some conjectures and suggestions for further research. (See Section 6). In Section 2 we prove a general rate of escape result (Theorem 1) for processes with stationary independent increments.

We now state Theorem 2a, the analogue in infinite dimensions of the Erdös-Dvoretzky test (1.1). Let us call a function h continuous on $(0, \infty)$ admissible if

$$(1.2a)\qquad 0 < h(t)\downarrow 0 \quad \text{and} \quad t^{\frac{1}{2}}h(t)\uparrow\infty,\ t\uparrow\infty,$$

and

$$(1.2b)\qquad h \text{ varies slowly at } \infty;\ h(tx)/h(t) \to 1,\ t\uparrow\infty, \qquad \text{for all } x > 0.$$

THEOREM 2a. *Let X be a genuinely infinite dimensional Brownian motion on V with $X(0) = 0$. Let $|\cdot|$ be a continuous seminorm on V of rank at least 3 with respect to X (see Section 3) and let h be admissible. If for some $\theta < 1$ we have*

$$(1.3)\qquad \Sigma_k h^{-2}(2^k)P\big[|X(1)| \leqslant \theta h(2^k)\big] = \infty \quad \textit{then}$$

$$(1.4)\qquad P\big[|X(t)| \leqslant t^{\frac{1}{2}}h(t) \text{ i.o. } as\ t\uparrow\infty\big] = 1.$$

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However, if the series in (1.3) *converges for some* $\theta > 1$, *then the probability in* (1.4) *is* 0.

(For an apparently stronger version, see Theorem 2 in Section 3. Appearances notwithstanding, the two versions are equivalent.)

REMARK 1. From (1.2a) and (1.2b) we see that for each fixed $\theta > 0$ the function $t \mapsto t^{-1}h^{-2}(t)P[|X(1)| \leqslant \theta h(t)]$ is nonincreasing on $(0, \infty)$. Consequently (1.3) holds if and only if

$$(1.5) \qquad \int_1^\infty t^{-1}h^{-2}(t)P[|X(1)| \leqslant \theta h(t)]\,dt = \infty.$$

(Express the integral as a sum of integrals over intervals $[2^k, 2^{k+1})$, $k = 0, 1, \cdots$). From this observation one sees that if (1.3), (1.4), (1.5) hold for the function h, then they must also hold for the function $t \mapsto h(t^\beta)$ for any fixed $\beta > 0$ (provided $h(t^\beta)$ is admissible).

REMARK 2. Since $t \mapsto tX(1/t)$ is also a Brownian motion on V, Theorem 2a has an analogue for $t \downarrow 0$, e.g., if (1.3) holds for some admissible h and $\theta < 1$, then $P[|X(t)| \leqslant t^{\frac{1}{2}}h(1/t)$ i.o. as $t \downarrow 0] = 1$, etc. A similar remark applies to Theorem 2 in Section 3.

REMARK 3. If for some closed subspace V_0, we have

$$(1.6) \qquad P[X(t) - X(0) \in V_0 \quad \text{for all} \quad t] = 1,$$

but for every proper subspace W of V_0 we have

$$(1.7) \qquad P[X(t) - X(0) \in W] < 1 \qquad \text{for all} \quad t > 0,$$

then we say that the process X is genuinely d-dimensional, $d = \dim(V_0)$, and that V_0 is the support of X. It follows from the 0-1 law of Kallianpur (1970) that every Brownian motion with values in a separable Banach space has a support space V_0 and that the probability in (1.7) is in fact 0. See Fernique (1974). Compare also Kuelbs (1970).

2. In this section we give a criterion for $|X(t)| \leqslant \gamma(t)$ infinitely often as $t \to \infty$ where $X = \{X(t);\, t \geqslant 0\}$ is a process with stationary independent increments and γ is a regularly varying function. This criterion is similar to a criterion in Kesten (1970) page 1176, Theorem 2, for a point in R' to be an accumulation point of a random sequence $\{S_n/Y_n\}$ where $\{S_n\}$ is a one dimensional random walk. Kesten attributes his criterion, in a special case, to K. G. Binmore and M. Katz.

Let V be as in Section 1 and let $X = \{X(t),\, t \geqslant 0\}$ be a strong Markov process on V which has stationary independent increments, $X(0) = 0$, and, if the time parameter t is continuous, which has right continuous sample paths. The most important property of such a process that we use is this: if T is any stopping time (more specifically, if T is any hitting time for X) relative to $\{\sigma(X(s);\, s \leqslant t);\, t \geqslant 0\}$, then, conditional on $T < \infty$, the process $\{X(T + t) - X(T); t \geqslant 0\}$ is independent of $\sigma(X(s); s \leqslant T +)$ and is a probabilistic replica of X. Let γ be a positive, nondecreasing function on $(0, \infty)$ which is continuous and varies regularly at ∞ with positive exponent; for some $\beta > 0$ (the exponent)

$$\gamma(tx)/\gamma(t) \to x^{\beta}, \qquad t\uparrow\infty, x > 0. \tag{2.1}$$

THEOREM 1. *Let* $|\cdot|$ *be any continuous seminorm on* V *not identically* 0, *let* X *and* γ *be as above and let* $b > 1$. *Then* (2.2) *implies* (2.3) *implies* (2.4) *where*

$$\liminf_{t\uparrow\infty}|X(t)|/\gamma(t) < 1 \text{ w.p. 1 } (\textit{with probability } 1), \tag{2.2}$$

$$\Sigma_k P\big[\,|X(t)| \leqslant \gamma(t) \textit{ for some } t \in \big[\,b^k, b^{k+1}\big)\big] \textit{ diverges}, \tag{2.3}$$

$$\liminf_{t\uparrow\infty}|X(t)|/\gamma(t) \leqslant 1 \text{ w.p. } 1. \tag{2.4}$$

NOTE. The continuity of the seminorm $|\cdot|$ can be replaced by the considerably weaker condition that the seminorm process $t \mapsto |X(t)|$ have continuous sample paths. This remark applies to Theorem 2 as well.

PROOF. We partially follow Kesten (1970), page 1176. Clearly (2.2) implies (2.3) by the Borel-Cantelli lemma, so let us suppose that (2.3) holds. Let $\varepsilon > 0$ be fixed but arbitrary and choose $q > 2$ so large that

$$\gamma(s_2)/\gamma(s_1) \leqslant 1 + \tfrac{1}{2}\varepsilon \qquad \text{when} \quad q \leqslant s_1 \leqslant s_2 \leqslant s_1 q/(q-1), \tag{2.5}$$

$$\gamma(s_1)/\gamma(s_2) \leqslant \tfrac{1}{2}\varepsilon \qquad \text{when} \quad q \leqslant s_1 \leqslant s_2/(q-1). \tag{2.6}$$

To prove (2.5) and (2.6) use Bojanic (1971) after noting that $\gamma(x)/x^{\beta}$ is slowly varying. Keep in mind that x^{β} is continuous and monotone and $\beta > 0$ and that $\gamma(x)$ is monotone. Since for each $y \geqslant 1$ we have

$$t_{k+y}/t_{k+s} \to 0,\ s\uparrow\infty, \qquad \text{uniformly in} \quad k \geqslant 0, \tag{2.7}$$

where $t_k = b^k$, we may choose $s > 1$ so large that $t_s \geqslant 2$ and

$$t_{k+1}/t_{k+s} = 1/b^{s-1} \leqslant 1/q \qquad \text{for} \quad k \geqslant 1. \tag{2.8}$$

Fix $k \geqslant k_0$ where $t_{k_0} \geqslant q$. Define the stopping times

$$\lambda = \min\{t\colon t \geqslant t_k, |X(t)| \leqslant \gamma(t)\}.$$

Note that $P[\lambda < \infty] > 0$ since for any $j > k$, $P[\lambda < \infty] \geqslant P[|X(t)| \leqslant \gamma(t)$ for some $t \in [t_j, t_{j+1})]$ and the latter probability must be nonzero for infinitely many j on account of (2.3).

On $[\lambda < \infty]$ we have $|X(\lambda)| \leqslant \gamma(\lambda)$, by continuity, so

$$\begin{aligned} D_k &\equiv \big[\,|X(\alpha)| > \gamma(\alpha) \text{ for all } \alpha \geqslant t_{k+s}, \lambda \in \big[\,t_k, t_{k+1}\big)\big] \\ &\supset \big[\,|X(\lambda + t) - X(\lambda)| \leqslant \gamma(\lambda + t) + \gamma(\lambda) \qquad \text{for all} \\ &\qquad \times t \geqslant t_{k+s} - \lambda, \lambda \in \big[\,t_k, t_{k+1}\big)\big]. \end{aligned}$$

But for $t_k \leqslant \lambda < t_{k+1}$ and $t + \lambda \geqslant t_{k+s}$, we have $\lambda \geqslant q$, $t > t_{k+s} - t_{k+1} \geqslant q$, and, more importantly, $t + \lambda \leqslant tq/(q-1)$, $\lambda \leqslant t/(q-1)$ by (2.8). Applying (2.5) and (2.6) we get

$$D_k \supset \left[|X(\lambda + t) - X(\lambda)| \geqslant (1 + \varepsilon)\gamma(t) \text{ for all } t \geqslant q, \lambda \in \left[t_k, t_{k+1} \right) \right].$$

Now $\{X(\lambda + t) - X(\lambda);\ t \geqslant q\}$ and λ are independent on $[\lambda < \infty]$ and Law $(\{X(\lambda + t) - X(\lambda);\ t \geqslant q\} | \lambda < \infty) =$ Law $\{X(t);\ t \geqslant q\}$, consequently

$$P(D_k) \geqslant P\left[|X(t)| \geqslant (1 + \varepsilon)\gamma(t) \text{ for all } t \geqslant q \right] P(A_k) \tag{2.9}$$

where $A_k = [\lambda \in [t_k, t_{k+1})] = [|X(t)| \leqslant \gamma(t) \text{ some } t \in [b^k, b^{k+1})]$. Clearly at most s of the events D_k can occur at one time, so $E[\# D_k \text{ which occur}] \equiv \Sigma_k P(D_k) \leqslant s < \infty$. Hence, by (2.3) and (2.9) we must have

$$P\left[|X(t)| \geqslant (1 + \varepsilon)\gamma(t) \text{ for all } t \geqslant q \right] = 0$$

and this holds for all large q (and hence all $q \geqslant 1$). In other words $P[|X(t)| < (1 + \varepsilon)\gamma(t) \text{ i.o. } t\uparrow\infty] = 1$. But this is the same thing as (2.4) since $\varepsilon > 0$ is arbitrary.

COROLLARY. *$P[\liminf |X(t)|/\gamma(t) > 1] = 1 \Rightarrow$ series in* (2.3) *converges* $\Rightarrow P[\liminf |X(t)|/\gamma(t) \geqslant 1] = 1$.

REMARK 4. The only property of the sequence $t_k = b^k$ needed for the proof is given by (2.7) and, of course, $t_{k-1} < t_k \to \infty$. So we can replace $\{b^k\}$ by any such sequence, e.g., $t_k = \exp(k \log k)$, $t_k = \exp(k^2)$. This may be useful in showing convergence in (2.3) rather than divergence. It would be of interest to determine whether or not sequences $\{t_k\}$ slower than exponential could be used in Theorem 1. See Feller (1968), pages 210–211, problem 7, where the sequence $t_k = \exp(k/\log k)$ is put to good use in a similar situation.

3. **Proof of Theorem 2.** Let $X = \{X(t);\ t \geqslant 0\}$ be a Brownian motion on $(V, \|\cdot\|)$ starting at 0, that is, (i) X has stationary independent increments, (ii) X is strong Markov, (iii) X has continuous sample paths, and (iv) X is centered Gaussian; for any continuous linear functional f and for any $t \geqslant 0$, $\langle X(t), f\rangle \equiv f(X(t))$ has a Gaussian distribution with mean $E\langle X(t), f\rangle = 0$ and variance $E\langle X(t), f\rangle^2 = tE\langle X(1), f\rangle^2$. (We added (iv) mostly for emphasis; it may be easily seen that (i) and (iii) with $E\langle X(t), f\rangle = 0$ implies (iv).) The essential implication of (iv) (for our proof) is the Brownian scaling property; for each fixed t, $X(t)$ and $t^{\frac{1}{2}}X(1)$ are identical in law. For the essential implication of (i) and (ii) see Section 2. Note that the law of a Brownian motion on V is completely determined by the covariance function given by $(f, g) \to Ef(X(1))g(X(1))$, $f, g \in V^*$.

Let us say that a continuous seminorm $|\cdot|$ on V is of rank at least d with respect to X if there is a continuous linear transformation $L: V \to V$ of rank d such that the process LX is genuinely d-dimensional and

$$\|Lv\| \leqslant |v| \text{ for all } v \in V. \tag{3.1}$$

Observe that LX is also a Brownian motion and its support is $\overline{L(V_0)}$ where V_0 is the support of X, see Remark 3, Section 1. The bar denotes closure in the norm topology. We need only require $|\cdot|$ be continuous on V_0.

THEOREM 2. *Let X (with $\dim(V_0) = \infty$) be as above and let $|\cdot|$ be a seminorm of rank $\geqslant 3$ with respect to X. Fix $b > 1$ and let h be admissible, see* (1.2). *Then*

$$\liminf |X(t)|/t^{\frac{1}{2}}h(t) \geqslant 1 \text{ w.p. } 1, \qquad \textit{or} \quad \leqslant 1 \text{ w.p. } 1,$$

according as

$$\Sigma_k h^{-2}(b^k)P\big[|X(1)| \leqslant h(b^k)\big] \tag{3.2}$$

converges or diverges. Convergence (divergence) of (3.2) *is equivalent to convergence or (divergence) of the integral in* (1.5), *Remark* 1, *with $\theta = 1$.*

PROOF. Put

$$\gamma(t) = t^{\frac{1}{2}}h(t)$$

where h satisfies (1.2). Then γ satisfies the hypothesis of Theorem 1 ($\beta = \frac{1}{2}$ in (2.1)). Therefore, the conclusion follows from Theorem 1 and Lemmas 1 and 2 below.

Let us write

$$p(\gamma, t_1, t_2) = P\big[|X(s)| \leqslant \gamma(s) \quad \text{for some} \quad s \in [t_1, t_2)\big].$$

LEMMA 1. *For any $b > 1$ and $k = 1, 2, \cdots$*

$$p(\gamma, b^{k-1}, b^k) \geqslant a_1 h^{-2}(b^k)P\big[|X(1)| \leqslant h(b^k)\big] \tag{3.3}$$

where

$$a_1 = (1 - b^{-1})\Big(\textstyle\int_0^\infty P\big[|X(s)| \leqslant 2\big]\, ds\Big)^{-1} > 0.$$

LEMMA 2. *For any $b > 1$, $\varepsilon > 0$, $k = 0, 1, \cdots$,*

$$p(\gamma, b^k, b^{k+1}) \leqslant a_2 h^{-2}(b^k)P\big[|X(1)| \leqslant (1 + \varepsilon)h(b^k)\big] \tag{3.4}$$

where

$$a_2 = \max\{4\varepsilon^{-2}(b^2 - 1)E|X(1)|^2, 2(1 + b^{-1})h^2(b)\} < \infty.$$

In what follows, if Z is any random variable and A any set in the range of Z, we write $I[Z \in A]$ for the indicator of the event $\{w: Z(w) \in A\}$, also, P^x, E^x denote probability, expectation when the process starts at x, $P^0 = P$, $E^0 = E$.

PROOF OF LEMMA 1. Let $T = \min\{t: t \geqslant b^{k-1}, |X(t)| \leqslant \gamma(t)\}$. Then T is a stopping time, $p(\gamma, b^{k-1}, b^k) = P[T < b^k]$ and we have

$$E\textstyle\int_{b^{k-1}}^{b^k} I\big[|X(s)| \leqslant \gamma(s)\big]\, ds = E\big\{\int_T^{b^k} I\big[|X(s)| \leqslant \gamma(s)\big]\, ds;\ T \leqslant b^k\big\}$$

$$= E\Big[\big\{E^y \textstyle\int_0^{b^k - t} I\big[|X(u)| \leqslant \gamma(u + t)\big]\, du\big\}\big|_{y = X(T),\, t = T};\quad T \leqslant b^k\Big] \tag{3.5}$$

by the strong Markov property. But for $y = X(T)$ we have $|y| \leqslant \gamma(b^k)$ on $T \leqslant b^k$, so

$$E^y \int_0^{b^k - t} I[|X(u)| \leqslant \gamma(u + t)]\, du \leqslant E\int_0^{b^k} I[|X(u) + y| \leqslant \gamma(b^k)]\, du$$
$$\leqslant E\int_0^{b^k} I[|X(u)| \leqslant 2\gamma(b^k)]\, du = \gamma^2(b^k) E\int_0^{b^k/\gamma^2(b^k)} I[|X(s)| \leqslant 2]\, ds$$
$$\leqslant b^k h^2(b^k) E\int_0^\infty I[|X(s)| \leqslant 2]\, ds = b^k h^2(b^k)\int_0^\infty P[|X(s)| \leqslant 2]\, ds.$$

The first equality in the preceding calculation follows from the Brownian scaling; Law $(X(at))$ = Law $(a^{\frac{1}{2}}X(t))$, see (iv) above. The last equality is by Fubini's theorem. Returning to (3.5) we get

$$E\int_{b^{k-1}}^{b^k} I[|X(s)| \leqslant \gamma(s)]\, ds \leqslant b^k h^2(b^k)\int_0^\infty P[|X(s)| \leqslant 2]\, ds\, p(\gamma, b^{k-1}, b^k).$$

On the other hand, by monotonicity of h,

$$E\int_{b^{k-1}}^{b^k} I[X(s)| \leqslant \gamma(s)]\, ds = \int_{b^{k-1}}^{b^k} P[|X(1)| \leqslant h(s)]\, ds$$
$$\geqslant (b^k - b^{k-1}) P[|X(1)| \leqslant h(b^k)].$$

The last two inequalities give (3.3).

It remains to show that $a_1 > 0$, i.e., that

$$\int_0^\infty P[|X(t)| \leqslant a]\, dt < \infty, \qquad a > 0. \tag{3.6}$$

Let V_0 be the support of X and let L be as in (3.1), L continuous. Put $N = \overline{L(V_0)}$, then N is the support of the process $LX = Z$ and, by hypothesis, we can suppose that N is 3-dimensional, i.e., that Z is genuinely 3-dimensional. Let v_1, v_2, v_3 be a basis for N and let us write

$$LX(t) = Z(t) = Z_1(t)v_1 + Z_2(t)v_2 + Z_3(t)v_3.$$

Then the covariance matrix $D = (EZ_i(1)Z_j(1))$ $i, j = 1, 2, 3$ must be nonsingular, hence, positive definite, and we have

$$P[|Z(t)|_2 \leqslant b] = (2\pi t)^{-\frac{3}{2}}\delta^{-\frac{1}{2}}\int_{z_1^2+z_2^2+z_3^2 \leqslant b^2} \exp\left(-\tfrac{1}{2}t^{-1}z'D^{-1}z\right) dz'$$

where $|Z(t)|_2 = (Z_1^2(t) + Z_2^2(t) + Z_3^2(t))^{\frac{1}{2}}$, $\delta = \det(D)$, $z' =$ transpose $(z) = (z_1, z_2, z_3)$. It follows that for $b < \infty$

$$\int_0^\infty P[|Z(t)|_2 \leqslant b]\, dt = \int_0^1 + \int_1^\infty < 1 + (18\pi\delta)^{-\frac{1}{2}} b^3 \int_1^\infty t^{-\frac{3}{2}}\, dt < \infty. \tag{3.7}$$

All norms on the finite dimensional space N are equivalent, so there exists a constant C, $0 < C < \infty$, such that

$$|Z(t)|_2 \leqslant C\|Z(t)\|.$$

But from (3.1) we have

$$\|Z(t)\| = \|LX(t)\| \leqslant |X(t)|.$$

Hence

$$P[|X(t)| \leqslant a] \leqslant P[Z(t)|_2 \leqslant aC],$$

and (3.6) follows from this and (3.7).

PROOF OF LEMMA 2. Let $\lambda = \min\{t: |X(t)| \leqslant \gamma(t), t \geqslant b^k\}$. Then

$$
\begin{aligned}
(3.8)\qquad & E\int_{b^k}^{b^{k+2}} I[|X(s)| \leqslant (1+\varepsilon)\gamma(s)]\,ds \\
&\geqslant E\left[\int_{\lambda}^{b^{k+2}} I[|X(s)| \leqslant (1+\varepsilon)\gamma(s)]\,ds;\ \lambda < b^{k+1}\right] \\
&= E\Big[E\Big(\int_0^{b^{k+2}-t} I[|X(u)+y| \\
&\qquad \leqslant (1+\varepsilon)\gamma(u+t)]\,du\Big)\Big|_{y=X(\lambda),\,t=\lambda};\ \lambda < b^{k+1}\Big]
\end{aligned}
$$

as in the proof of Lemma 1. When $y = X(\lambda)$ and $b^k \leqslant t = \lambda < b^{k+1}$ we have $|X(u) + y| \leqslant |X(u)| + \gamma(t)$, so,

$$
\begin{aligned}
\int_0^{b^{k+2}-t} I[|X(u)+y| \leqslant (1+\varepsilon)\gamma(u+t)]\,du &\geqslant \int_0^{b^{k+2}-t} I[|X(u)| \leqslant \varepsilon\gamma(t)]\,du \\
&\geqslant \int_0^{b^{k+2}-b^{k+1}} I[|X(u)| \leqslant \varepsilon\gamma(b^k)]\,du,
\end{aligned}
$$

by monotonicity of γ. Using this and Fubini's theorem in (3.8) we get

(3.9)

$$\int_{b^k}^{b^{k+2}} P[|X(s)| \leqslant (1+\varepsilon)\gamma(s)]\,ds \geqslant p(\gamma, b^k, b^{k+1})\int_0^{(b-1)b^{k+1}} P[|X(u)| \leqslant \varepsilon\gamma(b^k)]\,du.$$

Choose

$$a^2 = \max\{2E|X(1)|^2, \varepsilon^2 h^2(b)/(b^2 - b)\}, \qquad a > 0,$$

then $a < \infty$ (the finiteness of $E|X(1)|^2$ is immediate from Fernique (1974), page 11), and we have

$$a^{-2}\varepsilon^2\gamma^2(b^k) = a^{-2}\varepsilon^2 h^2(b^k) b^k \leqslant (b-1)b^{k+1}$$

for all $k \geqslant 0$ (recall that h is nonincreasing). For any $0 < u \leqslant a^{-2}\varepsilon^2\gamma^2(b^k)$, we get

$$
\begin{aligned}
P[|X(u)| \leqslant \varepsilon\gamma(b^k)] &= P[|X(1)| \leqslant u^{-\frac{1}{2}}\varepsilon\gamma(b^k)] \\
&\geqslant P[|X(1)| \leqslant a] \geqslant \tfrac{1}{2},
\end{aligned}
$$

by Chebyshev's inequality and Brownian scaling, and thus,

$$(3.10)\qquad \int_0^{(b-1)b^{k+1}} P[|X(u)| \leqslant \varepsilon\gamma(b^k)]\,du \geqslant \tfrac{1}{2}a^{-2}\varepsilon^2 b^k h^2(b^k).$$

Clearly

(3.11)

$$
\begin{aligned}
\int_{b^k}^{b^{k+2}} P[|X(s)| \leqslant (1+\varepsilon)\gamma(s)]\,ds &= \int_{b^k}^{b^{k+2}} P[|X(1)| \\
&\leqslant (1+\varepsilon)h(s)]\,ds \leqslant (b^{k+2} - b^k)P[|X(1)| \leqslant (1+\varepsilon)h(b^k)].
\end{aligned}
$$

Going back to (3.9) with the bounds (3.10) and (3.11) we get (3.4).

4. Examples. In this section we shall construct some Brownian motions with support in l^2 or l^∞ and determine their rates of escape. These examples illustrate the difference between the finite and infinite dimensional cases and suggest some interesting conjectures and problems (discussed in the last section).

Throughout this section let $\{\{B_k(t), t \geq 0\}\}_{k=1}^{\infty}$ be a sequence of mutually independent one dimensional standard ($EB(t) = 0$), $EB^2(t) = t$ all t) Brownian motions all defined on the same probability space. Let $\{\sigma_k\}$ be a sequence of constants such that

(4.1a) $\qquad \sigma_k > 0$ for all k sufficiently large,

(4.1b) $\qquad \{\sigma_k\}$ ultimately decreasing and $\sigma_k \to 0$, $k \to \infty$.

Our examples will be of the form

$$X(t) = (\sigma_1 B_1(t), \sigma_2 B_2(t), \cdots) = \Sigma_{k=1}^{\infty} \sigma_k B_k(t) e_k, \qquad t \geq 0, \tag{4.2}$$

for various choices of σ_k. Here e_k = kth unit coordinate vector = $(\underbrace{0, 0, \cdots, 0}_{k-1}, 1, 0, \ldots)$. Note that (4.1a) and mutual independence of the B_k guarantees that X is genuinely infinite dimensional. We also note the following lemma whose proof will be left to the reader.

LEMMA 3. *Let* $1 \leq p < \infty$. $P[X(t) \in l^p$ *for all* $t] = 1$ *if and only if*

$$\Sigma_{k=1}^{\infty} \sigma_k^p < \infty. \tag{4.3}$$

Moreover, under (4.3), X is a Brownian motion on l^p.

If $X = (X_1, X_2, \cdots)$, we write $|X|_p$ for the l^p norm of X: $|X|_p = (\Sigma |X_k|^p)^{1/p}$, $p < \infty$, $|X|_\infty = LUB\{|X_k| : k \geq 1\}$. For our examples we want to find admissible h so that

$$0 < \liminf |X(t)|_p / t^{\frac{1}{2}} h(t) < \infty \tag{4.4}$$

w.p. 1. (Note that (4.4) is impossible in the finite dimensional case; the lim inf is either 0 w.p. 1 or ∞ w.p. 1.) From Theorem 2 it is clear that to determine such h we must get very sharp asymptotic estimates for the probabilities

$$P\big[|X(1)|_p \leq \varepsilon\big] \text{ as } \varepsilon \downarrow 0.$$

Unfortunately this seems to be a very delicate problem even in the case $p = \infty$. (See Section 5.) However, the recent work of Dudley, Hoffman-Jørgensen and Shepp (1979) provides estimates in the cases $p = 2$ and $p = \infty$ which are sufficient for a wide variety of sequences $\{\sigma_k\}$.

EXAMPLE 1. *Let* $\sigma_k = 1/k^\beta$ $(\beta > \frac{1}{2})$ *in* (4.2). *For this process we have* (4.4) *with* $p = 2$ *and* $h(t) = (\lg \lg t)^{-(\beta - \frac{1}{2})}$ (lg *stands for the natural logarithm*). *More precisely*

$$(1/2p)^{1/2p} \leq \liminf \frac{(\lg \lg t)^{\beta - \frac{1}{2}}}{t^{\frac{1}{2}}} |X(t)|_2 \leq (2p)^{\frac{1}{2}} \beta^\beta, \tag{4.5}$$

where $p = (2\beta - 1)^{-1}$.

PROOF. From Example (4.5) in [2], we have

$$F(h) \equiv P[|X(1)|_2 \leq h] \geq Bh^{p(3-\beta)}\exp(-\beta(1+p)h^{-2p}),$$
$$F(h) \leq Ah^{p(1-\beta)}\exp\left(-\beta - \tfrac{1}{2}\right)h^{-2p}$$

where A and B are finite positive and independent of h. Replacing h by $h_\theta(e^n) = \theta h(e^n) = \theta(\lg n)^{-1/2p}$ we get $A_1(\lg n)^{c_1} n^{-\gamma_1} \leq h_\theta^{-2}(e^n)F(h_\theta(e^n)) \leq A_2(\lg n)^{c_2} n^{-\gamma_2}$ where $\gamma_1 = \beta(1+p)^p\theta^{-2p}$, $\gamma_2 = (\beta - \frac{1}{2})\theta^{-2p}$, A_1, A_2, C_1, C_2 are independent of n. An application of Theorem 2 easily yields (4.5).

REMARK 5. Occasionally, the probabilities $P[|X(1)|_2 \leq h]$ are quite obliging. In Example 1, if $\beta = 1$ we find, see Theorem 5.3 of [2], that this probability is bounded above by a constant times $h^{-1}\exp(-(\pi^2/8)h^{-2})$ and is bounded below by a constant times $h\exp(-(\pi^2/8)h^{-2})$. From this we get

$$\liminf(\lg\lg t/t)^{\frac{1}{2}}\left(\Sigma_{k=1}^{\infty}(1/k^2)B_k^2(t)\right)^{\frac{1}{2}} = \pi/8^{\frac{1}{2}},$$

w.p. 1. (Take $h = \theta(\lg\lg t)^{-\frac{1}{2}}$ in Theorem 2 with θ slightly smaller and then slightly larger than $\pi/8^{\frac{1}{2}}$, keep in mind that the lim inf must be a constant w.p. 1.)

REMARK 6. For any process of the form (4.2) let us write

$$X_d(t) = \Sigma_{k=1}^{d}\sigma_k B_k(t)e_k, \qquad X^d(t) = X(t) - X_d(t).$$

In Example 1, we find that for every $d \geq 1$,

$$\liminf s(t)(\lg\lg t)^{\frac{1}{2}}|X_d(t)|_2/t^{\frac{1}{2}} = \infty$$
$$\liminf s(t)(\lg\lg t)^{\frac{1}{2}}|X_d(t)|_2/t^{\frac{1}{2}} = 0$$

w.p. 1 whenever $s(t)\uparrow\infty$ as $t\uparrow\infty$ and $\Sigma[s(2^k)(\lg\lg 2^k)^{\frac{1}{2}}]^{-(d-2)} = \infty$. (This follows from (1.1) for X_d and (4.4) for X^d.) Thus, as one might expect, with respect to $|\ \ |_2$, X of Example 1 escapes to ∞ slightly faster than any of its finite dimensional projections.

EXAMPLE 2. Let $\sigma_k = k^{-\frac{1}{2}}e^{-\lambda k}(\lambda > 0$, const.). Then (4.4) holds with $p = 2$ and

$$h(t) = \exp\left[-(2\lambda\lg\lg t)^{\frac{1}{2}}\right].$$

PROOF. Arguing as in Example 4.7 of [2] we obtain

$$F(t) \equiv P[|X(1)|_2 \leq t] \leq At^{p_1}|\lg t|^{-\frac{1}{4}}\exp\left[-(1/2\lambda)\lg^2 t\right],$$
$$F(t) \geq Bt^{p_2}|\lg t|^{-\frac{1}{4}}\exp\left[-(1/2\lambda)\lg^2 t\right]$$

where $p_1 = -\frac{3}{2}$ and $p_2 = (1/2\lambda)(1 + \lambda + \lg 4)$. Setting $t = h_\theta(e^n) \equiv \theta\exp[-(2\lambda\lg n)^{\frac{1}{2}}]$, we find that, apart from a constant multiple, the series (3.2) (with $b = e$, $h = h_\theta$) is dominated by and in turn dominates a series of the form

$$\Sigma(\lg n)^{-\frac{1}{8}}\exp\left[C(\theta)(2\lambda\lg n)^{\frac{1}{2}} - \lg n\right]$$

where $C(\theta) = 2 - p_1 + \lambda^{-1} \lg \theta$ in the dominating series and $C(\theta) = 2 - p_2 + \lambda^{-1} \lg \theta$ in the dominated series. It follows that (3.2) converges for any $\theta < \exp[\lambda(p_1 - 2)]$ and diverges for any $\theta > \exp[\lambda(p_2 - 2)]$ (as a by product we have obtained fair upper and lower bounds for the lim inf in (4.4). For example, if $\lambda = \frac{1}{2}$, the lim inf lies between .17 and 1.56.

REMARK 7. The $k^{-\frac{1}{2}}$ in $\sigma_k = k^{-\frac{1}{2}}e^{-\lambda k}$ of the last example is merely a "convenience factor"; it greatly simplifies the computations, required by the results of [2], leading to the bounds on $P[|X(1)|_2 \leq t]$. However, one is naturally more curious about the behavior of (4.2) when $\sigma_k = e^{-\lambda k}$. With a little patience one can show that in this case (4.4) holds with $p = 2$ and $h(t) = (\lg \lg t)^{\frac{1}{4}} \exp[-(2\lambda \lg \lg t)^{\frac{1}{2}}]$.

EXAMPLE 3. When the sequence $\{\sigma_k\}$ does not satisfy (4.3), the process (4.2) no longer has support in l^p and indeed, $|X(t)|_p = \infty$ w.p. 1. The next example shows that such processes may have a very rapid growth rate in the l^∞ norm. The example is

$$X(t) = \Sigma_{k=2}^{\infty} (\lg k)^{-\frac{1}{2}} B_k(t) e_k.$$

For this process we have $|X(t)|_\infty < \infty$ w.p. 1, but

$$\liminf |X(t)|_\infty / t^{\frac{1}{2}} = 2^{\frac{1}{2}} \text{ w.p. } 1. \tag{4.6}$$

In other words, $\gamma(t) = t^{\frac{1}{2}}$ is the natural rate of escape for this process.

PROOF. The finiteness of $|X(t)|_\infty$ follows from Example 3.2 and Theorem 3.1 of [2]. It also follows from Example (3.2)

$$P\big[|X(1)|_\infty \leq s\big] = 0 \text{ for } 0 \leq s \leq 2^{\frac{1}{2}}, \qquad > 0 \text{ for } s > 2^{\frac{1}{2}}.$$

Let Q denote the positive rationals then

$$\begin{aligned} p_n &\equiv P\Big[|X(t)|_\infty \leq t^{\frac{1}{2}} h(t) \quad \text{for some} \quad t \in \big[2^n, 2^{n+1}\big)\Big] \\ &= P\Big[|X(t)|_\infty \leq t^{\frac{1}{2}} h(t) \quad \text{for some} \quad t \in \big[2^n, 2^{n+1}\big) \cap Q\Big] \\ &\leq \Sigma_{t \in [2^n, 2^{n+1}) \cap Q} P\Big[|X(t)|_\infty \leq t^{\frac{1}{2}} h(t)\Big]. \end{aligned}$$

But for $t \in [2^n, 2^{n+1})$, $P[|X(t)|_\infty \leq t^{\frac{1}{2}}h(t)] = P[|X(1)|_\infty \leq h(t)] \leq P[|X(1)|_\infty \leq h(2^n)] = 0$ as soon as $h(2^n) \leq 2^{\frac{1}{2}}$. Hence, on taking $h(t) \equiv 2^{\frac{1}{2}}$, we get $\Sigma p_n = 0 < \infty$ which entails $P[|X(t)|_\infty \leq t^{\frac{1}{2}}$ i.o. as $t \uparrow \infty] = 0$ by the Borel-Cantelli lemma. From this we get (4.6) with $\geq$ instead of $=$. To get $\leq$, note that $q \equiv P[|X(t)|_\infty \leq st^{\frac{1}{2}}$ i.o. as $t \uparrow \infty] \geq \lim_n P[|X(t)|_\infty \leq st^{\frac{1}{2}}$ for some $t \geq n] \geq \lim P[|X(n)|_\infty \leq sn^{\frac{1}{2}}] = P[|X(1)|_\infty \leq s] > 0$ for any $s > 2^{\frac{1}{2}}$. But q is either 0 or 1, hence $q = 1$ for every $s > 2^{\frac{1}{2}}$. We are done.

REMARK 8. The (stochastic) separability of the process $\{|X(t)|_\infty, t \geq 0\}$ needed in the preceeding proof follows from the sample path continuity, with respect to $|\cdot|_\infty$, of the process X. This fact is not exactly trivial to verify but we omit the proof.

5. In this section we study the rate of escape of Brownian motion with respect to various sup-norms. These results complement those of Section 4, and throughout we let

$$X(t) = \Sigma_{k=1}^{\infty} \frac{B_k(t)e_k}{a_k} \tag{5.1}$$

where $\{\{B_k(t)\}: k \geqslant 1, t \geqslant 0\}$ and $\{e_k\}$ are as in Section 4, and $\{a_k\}$ is a strictly positive increasing sequence. We may view X as a Brownian motion (with continuous sample paths) in the Hilbert space of real sequences

$$V = \{x = \{x_k\} \in R^\infty : \Sigma_k \lambda_k x_k^2 < \infty\}$$

where $\{\lambda_k\}$ is a fixed sequence of strictly positive numbers such that $\Sigma_k \lambda_k < \infty$ (which entails $\Sigma_k \lambda_k / a_k^2 < \infty$ since $\{a_k\}$ is increasing). The inner product on V is given by

$$(x, y) = \Sigma_k \lambda_k x_k y_k.$$

THEOREM 3. *For $\{x_k\}$ in R^∞ define*

$$|\{x_k\}|_\infty = \sup_{k \geqslant 1} |x_k|, \tag{5.2}$$

and let $\{X(t): t \geqslant 0\}$ denote the sample continuous Brownian motion defined in (5.1). *Then the following hold.*

(i) *If $P(|X(t)|_\infty < \infty) > 0$ for any $t > 0$, then*

$$(\log k)^{\frac{1}{2}} = O(a_k) \quad as \quad k \to \infty. \tag{5.3}$$

(ii) *If $P(x(t) \in c_0) >$ for any $t > 0$, then*

$$(\log k)^{\frac{1}{2}} = o(a_k) \quad as \quad k \to \infty. \tag{5.4}$$

Here c_0 is the subspace of $\mathbb{R}^\infty$ consisting of sequences which converge to zero.

(iii) *If $a_k = k^p$ for $0 < p < \infty$, then*

$$0 < \liminf_{t\to\infty} (\log \log t)^p \frac{|X(t)|_\infty}{t^{\frac{1}{2}}} < \infty \text{ w.p. } 1. \tag{5.5}$$

REMARK 9. Using simple comparison arguments with (iii) of Theorem 3, we get many different results. For example, if $a_k \leqslant Mk^p (a_k \geqslant Mk^p)$ for all k sufficiently large, then

$$\liminf_{t\to\infty} (\log \log t)^p \frac{|X(t)|_\infty}{t^{\frac{1}{2}}} > 0 (< \infty) \text{ w.p. } 1.$$

PROOF. If $P(|X(t)|_\infty < \infty) > 0$, then $P(|X(1)|_\infty < \infty) > 0$ and hence $P(|X(1)|_\infty < \infty) = 1$ by the Kolmogorov zero-one law. Letting $F(t) = P(|X(1)|_\infty \leqslant t)$ we have

$$F(t) = \Pi_{k=1}^{\infty} P(|B_k(1)| \leqslant ta_k)$$
$$= \Pi_{k=1}^{\infty}(1 - R(ta_k))$$

where $R(t) = (2/\pi)^{\frac{1}{2}} \int_t^{\infty} \exp(-x^2/2)\, dx$. Hence $F(t) > 0$ iff $\Sigma_k R(ta_k) < \infty$, and, using the inequality

$$C_1 \exp(-t^2/2)/(1+t) < R(t) < C_2 \exp(-t^2/2)/(1+t) \qquad t > 0$$

for some infinite positive C_1, C_2, we see, as in Dudley, Hoffman-Jørgensen, and Shepp (1977), that $F(t) > 0$ iff

(5.6) $$\Sigma_{k=1}^{\infty} \exp(-t^2 a_k^2/2)/(1 + ta_k) < \infty.$$

Now (5.6) converges for all $t > \delta$ iff

$$\Sigma_{k=1}^{\infty} \exp(-t^2 a_k^2/2) < \infty \text{ for every } t > \delta.$$

Hence $P(|X(1)| < \infty) = 1$ implies there exists a $t_0 < \infty$ such that

(5.7) $$\Sigma_{k=1}^{\infty} \exp(-t_0^2 a_k^2/2) < \infty.$$

Now $\{a_k\}$ increasing implies $\{\exp(-t_0^2 a_k^2/2)\}$ is decreasing and thus (5.7) implies

$$\exp(-t_0^2 a_k^2/2) = c_k/k, \qquad k \geqslant 1,$$

where $c_k \to 0$ as $k \to \infty$, $c_k > 0$. Taking logarithms and noting that $-\log c_k > 0$ eventually, we see that

(5.8) $$\limsup_k (\log k)^{\frac{1}{2}}/a_k \leqslant t_0/2^{\frac{1}{2}}.$$

Thus (i) holds.

If $P(X(t) \in c_0) > 0$, then by the above argument we also have (5.8) holding. Furthermore, it is easy to see that the number $t_0 > 0$ in (5.8) can be taken arbitrarily small. Hence the lim sup in (5.8) is 0 and (iii) holds.

Part (iii) of Theorem 3 is established using Theorem 2 with $b = e$. That is, one shows that the series

(5.9) $$\Sigma_{n=1}^{\infty} P\big[|X(1)|_{\infty} \leqslant A(\log n)^{-p}\big](\log n)^{2p}$$

converges for A sufficiently small, $A > 0$, and diverges for A sufficiently large, $A < \infty$. These facts are established on noting first the fact

(5.10) $$P\big[|X(1)|_{\infty} \leqslant t\big] = \Pi_{k=1}^{\infty} P\big[|V| \leqslant tk^p\big]$$

where $t = A(\log n)^{-p}$, and V is a Gaussian random variable with mean 0 and variance 1.

To get convergence in (5.9) note that

$$1 + \Sigma_{k=1}^{\infty} P\big[|Z| \geqslant k^p\big] \geqslant E|Z|^{1/p}, \qquad Z = V(\log n)^p A^{-1},$$

hence

$$\Pi_{k=1}^{\infty} P\big[|V| \leqslant Ak^p(\log n)^{-p}\big] \leqslant \exp\big(-\Sigma_{k=1}^{\infty} P\big[|V| \geqslant Ak^p(\log n)^{-p}\big]\big)$$
$$\leqslant 3\exp(-b\log n) = 3n^{-b},$$

where $b = A^{-1/p}E|V|^{1/p}$. Consequently for $0 < A < (E|V|^{1/p})^p$, (5.9) does converge.

The proof that (5.9) diverges for $A < \infty$ sufficiently large is somewhat more involved. We break the product in (5.10) into two parts: $\Pi_{k \leqslant r(n)}$ and $\Pi_{k > r(n)}$ where $r(n)$ is chosen to satisfy $-1 < r(n) - A^{-1/p} \log n \leqslant 0$ for all $n \geqslant 1$. The second product dominates n^{-m_1} where $m_1 = 2A^{-1/p}E|V|^{1/p}$. Using Stirling's approximation, we see that the first product dominates $(\log n)^{p/2} n^{m_2}$ where $m_2 = A^{-1/p}(\frac{1}{2}\log(2/\pi) - p - \frac{1}{4}(1+p)^{-1})$. Consequently, the series (5.9) dominates the series $\Sigma_n (\log n)^{5p/2} n^{(m_2 - m_1)}$ and by choosing A sufficiently large, we can make $m_2 - m_1 \geqslant -1$ forcing divergence. The details of these assertions are straightforward but lengthy and are therefore omitted.

6. Let X be a Brownian motion on the Banach space $(V, \|\cdot\|)$ with support V_0. Let $\gamma(t) = t^{\frac{1}{2}}h(t)$ where h is any numerical function satisfying (1.2) or only (1.2a). Put

$$C(X, \gamma, \|\cdot\|) = \liminf_{t\to\infty} \gamma(t)^{-1}\|X(t)\|.$$

As noted before, $C(X, \gamma, \|\cdot\|)$ is a constant w.p. 1. The Erdös-Dvoretzky test (1.1) shows that 0 and ∞ are the only possible values of $C(X, \gamma, \|\cdot\|)$ when X is finite dimensional. (Replacing h by εh does not affect convergence or divergence of $\Sigma h^{d-2}(2^k)$.) On the other hand, for some infinite dimensional Brownian motions, there is a natural rate of escape; a function $\gamma = t^{\frac{1}{2}}h$ such that

$$0 < C(X, \gamma, \|\cdot\|) < \infty. \tag{6.1}$$

See examples in Section 4. Note that in Example 3 $h \equiv 1$ which does not go to 0.

Conjecture 1. If X is genuinely infinite dimensional and if

$$P[\|X(t)\| \leqslant \varepsilon] > 0 \qquad \text{for all} \quad \varepsilon > 0, \quad t > 0, \tag{6.2}$$

then (6.1) holds for some γ. As one may easily show, (6.2) holds for any process (4.2), $\|\cdot\| = |\cdot|_p$, with $\{\sigma_k\}$ satisfying (4.3).

Conjecture 2. If (6.1) holds, then γ can be taken to the $\gamma(t) = t^{\frac{1}{2}}h(t)$ where h is the (asymptotic as $t \to \infty$) solution to

$$P[\|X(t)\| \leqslant h(t)] = 1/\lg t \qquad t > e.$$

Of course even if Conjecture 1 is false, the problem remains: find necessary and sufficient conditions on $(X, V_0, \|\cdot\|)$ so that (6.1) holds for some γ. Particular processes of the form (4.2) are of independent interest.

Conjecture 3. If Z_β is the process

$$Z_\beta(t) = \Sigma_{k=1}^{\infty} k^{-\beta} B_k(t) e_k, \qquad t \geqslant 0,$$

notation as in Example 1, Section 4, then a natural rate of escape for Z_β with respect to the l^p norm $|\cdot|_p$ is given by

$$\gamma_{p,\beta}(t) = t^{\frac{1}{2}}/(\lg \lg t)^{\beta - 1/p}$$

provided $\beta p > 1$.

Note that Conjecture 3 is true in the case $p = 2$ and $p = \infty$; see Example 1, Section 4, and Theorem 3 part (iii) in Section 5. A related problem is to compute the exact value of $C(Z_\beta, \gamma_{p,\beta}, |\cdot|_p)$. One value is $C(Z_1, \gamma_{2,1}, |\cdot|_2) = \pi/8^{\frac{1}{2}}$ found in Remark 5. In [9] Mogulsky states a result (no proof given) which seems to imply $C(Z_\beta, \gamma_{2,\beta}, |\cdot|_2) = A^{\beta - \frac{1}{2}}$, $\beta > \frac{1}{2}$ and $A = (\beta - \frac{1}{2})[\pi/(2\beta \sin(\pi/2\beta))]^{\beta/(\beta - \frac{1}{2})}$.

7. While writing up the results of this paper, Professor Kuelbs brought to my attention item [9]. In [9], Mogulsky has stated a rate of escape type result for Brownian motions satisfying the very strong assumption

$$\log P[\|X(1)\| \leqslant t] \sim -t^{-a}L(t), \qquad \text{as} \quad t \downarrow 0,$$

for some $a > 0$ and some slowly varying function L. No proofs of his results have appeared. Part (e) of his Theorem 1 and part (i) of his Theorem 2 seem to be wrong. Parts (d) and (g) of his Theorems 1 and 2, respectively, follow easily, in the case of Brownian motion, from our Theorem 2. I omit the details.

Added in proof. Dennis D. Cox has recently (1-80) established that Conjecture 1 is true in the l^p case, $1 \leqslant p \leqslant \infty$! Also, Conjecture 3 is true.

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