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SPEEDS OF CONVERGENCE AND ASYMPTOTIC EXPANSIONS IN THE CENTRAL LIMIT THEOREM: A TREATMENT BY OPERATORS.

By T. J. Sweeting

University of Surrey

Speeds of convergence to normality and asymptotic expansions for sums of independent random vectors in $\mathbb{R}^k$, $k > 1$ are investigated using the method of operators. Existing results are improved and some new results obtained. In particular, asymptotic expansions for smooth functions are derived.

1. Introduction. Let $X_1, X_2, \cdots, X_n$ be independent random vectors in $\mathbb{R}^k$ with distributions $F_1, F_2, \cdots, F_n$. Assume that $EX_i = 0$, $n^{-1}\sum_{i=1}^n \operatorname{Cov}(X_i) = I_k$, the identity matrix of order k. Let Q_n be the distribution of $S_n = n^{-\frac{1}{2}}\sum_{i=1}^n X_i$ and N be the standard normal distribution in $\mathbb{R}^k$. In Sweeting (1977) the error in approximating Q_n by N was investigated in the case of a common summand distribution using the method of operators. The main theorem improved and extended a result in Bhattacharya (1975) by removing a logarithmic term in n in the error bound, and by treating more general moment conditions. In this paper, a series of results in Bhattacharya and Rao (1976) and elsewhere are derived by the above method and improved in a similar manner.

After some preliminary definitions and results in Section 2, an expansion of the operator $\tilde{\mathfrak{Q}}_n$ associated with the normalized sum of the suitably truncated variables is obtained in Section 3. This is used to obtain the main lemma (Lemma 7) regarding the smoothed distribution of S_n. In Section 4 a general result is obtained (Theorem 1) applicable for summand distributions and functions subject only to certain moment and boundedness conditions respectively. A sharper form of this result (Theorem 2) is obtained in Section 5 when Cramér's condition holds, leading to known asymptotic expansions for a large class of functions. Under certain smoothness conditions on the functions, a sharper form of Theorem 1 is obtained in Section 6 (Theorem 3) not requiring Cramér's condition; this leads to asymptotic expansions, which were also obtained independently by Götze and Hipp (1978). Finally, the bounds occurring in Theorems 1–3 are investigated further; in particular conditions are given under which $o(\cdot)$ rates may be deduced.

For a general review of the work on rates of convergence and asymptotic expansions in the central limit theorem see Bhattacharya and Rao (1976).

2. Definitions and preliminaries. If $x = (x_1, \cdots, x_k) \in \mathbb{R}^k$, $j = (j_1, \cdots, j_k) \in (\mathbb{Z}^+)^k$ write $x^j = x_1^{j_1} \cdots x_k^{j_k}$, $\|x\| = (x_1^2 + \cdots + x_k^2)^{\frac{1}{2}}$. If $|x^j|$ is integrable with

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respect to F_i, the jth moment $\mu_{j,i}$ of X_i is $\mu_{j,i} = EX_i^j$ and the jth cumulant is $\chi_{j,i}$ (defined in Section 3); the rth absolute moment of X_i is $\beta_{r,i} = E\|X_i\|^r$ $(r \geq 0)$. Note that $|\mu_{j,i}| \leq \beta_{|j|,i}$ where we write $|j| = \Sigma_{i=1}^k j_i$. Define $\chi_j = n^{-1}\Sigma_{i=1}^n \chi_{j,i}$, $\beta_r = n^{-1}\Sigma_{i=1}^n \beta_{r,i}$.

LEMMA 1. *If* $\beta_r < \infty$ *and* $2 \leq j \leq r$ *then*

$$(\beta_j/k)^{r-2} \leq (\beta_r/k)^{j-2}.$$

PROOF. The function $j \to (\beta_{j,i}/\beta_{2,i}^{\frac{1}{2}j})^{1/(j-2)}$ is nondecreasing on $(2, r]$ for each i (Lemma 6.2 in [2], or on application of Hölder's inequality). Thus for $j > 2$

$$\begin{aligned}\beta_j/k &= (nk)^{-1}\Sigma_{i=1}^n \beta_{j,i}\\ &\leq \Sigma_{i=1}^n(\beta_{r,i}/nk)^{(j-2)/(r-2)}(\beta_{2,i}/nk)^{(r-j)/(r-2)}\\ &\leq (\beta_r/k)^{(j-2)/(r-2)}\end{aligned}$$

using Hölder's inequality and $\beta_2 = \operatorname{tr}(I_k) = k$. The case $j = 2$ is trivial.

Let M^k be the space of finite signed measures on the Borel σ-field of $\mathbb{R}^k$, B^k the class of real Borel measurable functions on $\mathbb{R}^k$ and B_1^k the subclass of bounded functions. As in [6], define the scaled measure $\bar{P}$ by $d\bar{P}(x) = dP(n^{\frac{1}{2}}x)$ where $P \in M^k$. Define the operator $\mathcal{P}$ on B_1^k associated with P to be the function whose values at $v \in B_1^k$ are given by

$$\mathcal{P}v(x) = \int v(x - y)\, dP(y).$$

Let C_m^k be the class of functions in B^k having continuous partial derivatives of order $m \geq 1$ and let D_i be the partial differential operator with respect to the ith variable. If $D = (D_1, \cdots, D_k)$ and $j \in (\mathbb{Z}^+)^k$ write $D^j = D_1^{j_1} \cdots D_k^{j_k}$. If $v \in C_m^k$, $x, h \in \mathbb{R}^k$, Taylor's theorem in $\mathbb{R}^k$ asserts that

$$(1) \qquad v(x + h) = \Sigma_{|j|=0}^{m-1}(j!)^{-1}h^j D^j v(x) + \Sigma_{|j|=m}(j!)^{-1}h^j D^j v(x + \theta h)$$

where $j! = j_1! \cdots j_k!$, $0 < \theta < 1$. Let $\mathcal{F}_i$, $\mathcal{N}$ be the operators associated with the distributions F_i, N given in Section 1; then $\mathcal{Q}_n \equiv \bar{\mathcal{F}}_1 \cdots \bar{\mathcal{F}}_n$ is the operator associated with the distribution Q_n of S_n.

Define $X_{i,n} = X_i I\{\|X_i\| \leq n^{\frac{1}{2}}\}$ and let $\mu^{(i)} = EX_{i,n}$, $\mu = n^{-1}\Sigma_{i=1}^n \mu^{(i)}$, $\Sigma = n^{-1}\Sigma_{i=1}^n \operatorname{Cov}(X_{i,n})$. Define

$$\Lambda_{r,i} = \int_{\|x\| > n^{\frac{1}{2}}} \|x\|^r\, dF_i(x), \ \Lambda_r = n^{-1}\Sigma_{i=1}^n \Lambda_{r,i}.$$

It is not hard to show that

$$(2) \qquad n^{\frac{1}{2}}|\mu^{(i)}| \leq \Lambda_{2,i}$$

$$(3) \qquad 0 \leq x'(I_k - \Sigma)x \leq (k + 1)\Lambda_2$$

where $\|x\| = 1$. Assume $\Lambda_2 \leq \rho < (k + 1)^{-1}$; then Σ is nonsingular and one can define $\tilde{X}_i = T(X_{i,n} - \mu^{(i)})$ where T is a positive definite matrix satisfying $T^2 = \Sigma^{-1}$. Thus $E\tilde{X}_i = 0$, $n^{-1}\Sigma_{i=1}^n \mathcal{C}_i = I_k$ where $\mathcal{C}_i = \operatorname{Cov}(\tilde{X}_i)$. Let $\tilde{\mu}_{j,i}$, $\tilde{\chi}_{j,i}$ be the jth moment and cumulant ($j \in (\mathbb{Z}^+)^k$) and, for $r \geqslant 0$, $\tilde{\beta}_{r,i}$ the rth absolute moment of $\tilde{X}_i$; $\tilde{\chi}_j = n^{-1}\Sigma_{i=1}^n \tilde{\chi}_{j,i}$, $\tilde{\beta}_r = n^{-1}\Sigma_{i=1}^n \tilde{\beta}_{r,i}$. As in [6] we have $\|T\| \leqslant (1 - (k+1)\rho)^{-\frac{1}{2}}$ where $\|\cdot\|$ is the norm of the linear operator T, and so

$$\|\tilde{X}_i\| \leqslant c_1 n^{\frac{1}{2}} \tag{4}$$

$$\tilde{\beta}_{r,i} \leqslant c_2(r) E\|X_{i,n}\|^r. \tag{5}$$

Also one may show (extending the proof of Lemma 14.1 in [2] for example) that for $|j| \geqslant 2$

$$|\tilde{\mu}_{j,i} - \mu_{j,i}| \leqslant c_3 \Lambda_{|j|,i}. \tag{6}$$

Let $F_{i,n}$, $\tilde{F}_i$ be the distributions of $X_{i,n}$, $\tilde{X}_i$ respectively with associated operators $\mathcal{F}_{i,n}$, $\tilde{\mathcal{F}}_i$ and let $\mathcal{Q}_{n,n} = \overline{\mathcal{F}}_{1,n} \cdots \overline{\mathcal{F}}_{n,n}$, $\tilde{\mathcal{Q}}_n = \bar{\tilde{\mathcal{F}}}_1 \cdots \bar{\tilde{\mathcal{F}}}_n$; thus $\tilde{\mathcal{Q}}_n$ is the operator associated with $\tilde{Q}_n$, the distribution of $\tilde{S}_n = n^{-\frac{1}{2}}\Sigma_{i=1}^n \tilde{X}_i$. From the identity (1) in [6] we have

$$(\mathcal{Q}_n - \mathcal{Q}_{n,n})v(x) = \Sigma_{i=1}^n \int_{\|y\|>1}\left[\overline{\mathcal{P}}_i v(x-y) - \overline{\mathcal{P}}_i v(x)\right] d\overline{F}_i(y) \tag{7}$$

where $\mathcal{P}_i = \mathcal{F}_0 \cdots \mathcal{F}_{i-1}\mathcal{F}_{i+1,n} \cdots \mathcal{F}_{n+1,n}$ and $\mathcal{F}_0 \equiv \mathcal{F}_{n+1,n} \equiv \mathcal{I}$, the identity operator.

3. Expansion of $\mathcal{Q}_n$ and main lemma. Let $Y_1, \cdots, Y_n$ be independent random vectors with distributions $G_1, \cdots, G_n$ where G_i is the normal distribution in $\mathbb{R}^k$ with zero mean and covariance matrix $\mathcal{C}_i$; thus $\mathfrak{N} = \overline{\mathcal{G}}_1 \cdots \overline{\mathcal{G}}_n$. If $j \in (\mathbb{Z}^+)^k$, $r \geqslant 0$, then $\nu_{j,i}$, $\alpha_{r,i}$ denote the jth moment and rth absolute moment respectively of Y_i. Write $\mathcal{C}_i = U_i^2$; then $\|U_i\|^2$ is the largest eigenvalue of $\mathcal{C}_i$ and so for some unit vector x, $\|U_i\|^2 = x'\mathcal{C}_i x = \operatorname{Var}(x'\tilde{X}_i) \leqslant \tilde{\beta}_{2,i}$. Thus if Z is standard normal we have $\alpha_{r,i} = E\|U_i Z\|^r \leqslant \|U_i\|^r E\|Z\|^r \leqslant c_1(r)\tilde{\beta}_{2,i}^{r/2}$ and so from the standard moment inequality

$$\alpha_{r,i} \leqslant c_1(r)\tilde{\beta}_{r,i}. \tag{8}$$

We first give expansions for the operators $\bar{\tilde{\mathcal{F}}}_i$ and $\overline{\mathcal{G}}_i$.

LEMMA 2. *Let $p \in \mathbb{Z}^+$ and $\phi \in C_{p+1}^k$. Suppose*

$$\max_{|j|=p+1}(1 + \|x\|^s)|D^j\phi(x)| \leqslant \Delta_{p+1,s} < \infty \tag{9}$$

for some $s \geqslant 0$. Then

$$(1 + \|x\|^s)\left|\left(\bar{\tilde{\mathcal{F}}}_i - \Sigma_{|j|=0}^p (j!)^{-1}\tilde{\mu}_{j,i} n^{-\frac{1}{2}|j|}(-D)^j\right)\phi(x)\right| \leqslant c_2 \tilde{\beta}_{p+1,i} n^{-\frac{1}{2}(p+1)}\Delta_{p+1,s}. \tag{10}$$

Furthermore the above inequality holds on replacing $\bar{\tilde{\mathcal{F}}}_i$ by $\overline{\mathcal{G}}_i$ and $\tilde{\mu}_{j,i}$ by $\nu_{j,i}$.

PROOF. From the Taylor expansion (1), $(1 + \|x\|^s)^{-1}$ times the left-hand side in (10) equals

$$\Sigma_{|j|=p+1}(j!)^{-1}\int(-y)^j D^j\phi(x - \theta y)\, d\bar{\tilde{F}}_i(y).$$

Using the bound (4) we have $n^{-\frac{1}{2}}\|\tilde{X}_i\| \leq \frac{1}{2}\|x\|$ if $\|x\| > c_3$ for sufficiently large c_3, and for this range of x the result follows from (9); for $\|x\| < c_3$ the bound follows immediately from (9). The proof of the second statement is similar except that we split the integration over the regions $\|y\| \leq \frac{1}{2}\|x\|$, $\|y\| > \frac{1}{2}\|x\|$. This gives the bound in the lemma with $\tilde{\beta}_{p+1,i}$ replaced by $\alpha_{p+1,i}$ and the assertion follows from (8).

A real continuous even function v on $\mathbb{R}^k$ is *positive definite* (p.d.) if it is the Fourier-Stieltjes transform of a finite measure A on $\mathbb{R}^k$; that is

$$v(x) = \int e^{ix'y}\, dA(y).$$

A p.d. function is *h-smooth* if its associated measure A vanishes outside the closed sphere of radius $h > 0$ centered at zero (see [6]). Let V be a distribution on $\mathbb{R}^k$ having moments of all orders and possessing a p.d. 1-smooth density v, and $V_h(x) = V(hx)$ where $h = \kappa\tilde{\beta}_3^{-1}$ (κ to be determined). Lemma 3 below may be proved via operators, but it is simpler to use characteristic functions here; in addition, this approach will be required in Section 5. Define $M_n = J_1 * \cdots * J_n$ with operator $\mathfrak{M}_n = \mathfrak{J}_1 \cdots \mathfrak{J}_n$ where q of the $\mathfrak{J}_i$ are identity operators ($0 \leq q \leq r - 1$) and the remaining $\mathfrak{J}_i$ equal to either $\tilde{\mathfrak{F}}_i$ or $\bar{\mathfrak{G}}_i$; let ψ be the characteristic function (c.f.) of M_n. Let A be the measure with Fourier-Stieltjes transform v; for all $m \in (\mathbb{Z}^+)^k$ we have

$$\|D^m \mathfrak{M}_n \bar{v}_h\| \leq \int |\zeta^m| |\psi(\zeta)|\, dA(\kappa^{-1}\varepsilon_n \zeta) \tag{11}$$

where $\varepsilon_n = \tilde{\beta}_3 n^{-\frac{1}{2}}$.

LEMMA 3. *There exist constants $\kappa < 1$, c_4, c_5 such that if $\varepsilon_n \leq c_4(k, q)$ then*

$$\|D^m \mathfrak{M}_n \bar{v}_h\| \leq c_5(k, |m|, q).$$

PROOF. Suppose without loss of generality that $\mathfrak{J}_1, \cdots, \mathfrak{J}_q$ are the identities and let ψ_i be the c.f. corresponding to J_i ($q + 1 \leq i \leq n$). By expanding the c.f. $|\psi_i|^2$ (see [2], Theorem 8.9)

$$|\psi_i(\zeta)| \leq \exp\left(-\tfrac{1}{2}\zeta'\mathcal{C}_i\zeta + c_6\|\zeta\|^3\tilde{\beta}_{3,i}\right)$$

and so

$$|\psi(\zeta)| = \Pi_{i=q+1}^{n}|\psi_i(n^{-\frac{1}{2}}\zeta)| \leq \exp\left(-\tfrac{1}{2}\|\zeta\|^2 + c_6\|\zeta\|^3\varepsilon_n + R\|\zeta\|^2\right)$$

where $R = c_7 n^{-1}\Sigma_{i=1}^{q}\tilde{\beta}_{2,i}$. By splitting the integral over $\|y\| \leq (\tilde{\beta}_3 n)^{\frac{1}{3}}$, $\|y\| > (\tilde{\beta}_3 n)^{\frac{1}{3}}$ it may be shown that $\tilde{\beta}_{2,i}n^{-1} \leq 2\varepsilon_n^{\frac{2}{3}}$ for all i. Thus by appropriate choice of c_4 we have $R \leq 1/8$; thus, for suitable κ and c_4, if $\|\zeta\| \leq \kappa\varepsilon_n^{-1}$ one has

$$|\psi(\zeta)| \leq \exp\left(-\tfrac{1}{4}\|\zeta\|^2\{1 - 4R\}\right) \leq \exp\left(-\tfrac{1}{8}\|\zeta\|^2\right).$$

The assertion now follows from (11).

If $v \in B^k$ and $\|x\|^s v(x)$ is bounded ($s \geq 0$) define the function $v^{[s]} \in B_1^k$ by $v^{[s]}(x) = (c'x)^s v(x)$ where c is some fixed unit vector. Similarly, if $P \in M^k$ and $\int \|x\|^s d|P|(x) < \infty$, define $dP^{[s]}(x) = (c'x)^s\, dP(x)$. Some properties of these functions are given in [6]. Let $U = V_h^{*(s+1)}$ and u be the density of U; the following result is a generalized version of Lemma 2 in [6].

LEMMA 4. *Under the condition of Lemma* 3, *there exists a constant* c_8 *such that for any* $s \in \mathbb{Z}^+$ *and* $m \in (\mathbb{Z}^+)^k$

$$(1 + \|x\|^s)|D^m \mathfrak{M}_n \bar{u}(x)| \leqslant c_8(k, |m|, r).$$

PROOF. In view of Lemma 3, it suffices to prove that

$$\|(D^m \mathfrak{M}_n \bar{u})^{[s]}\| \leqslant c_9 \tag{12}$$

where c_9 is independent of the vector c since, taking $c = \|x\|^{-1}x$, (12) implies that $\|x\|^s|D^m \mathfrak{M}_n \bar{u}(x)| \leqslant c_9$. We indicate the slight modifications required to the proof of Lemma 2 in [6]; the moment condition given in the statement of that lemma is satisfied from (5). With notation as in [6] we find by an identical argument that

$$\|D^b(\mathfrak{M}_n \bar{u})^{[a]}\| \leqslant c_{10}\Sigma_{l=0}^a n^l Q(a, b, l) \tag{13}$$

where $Q(a, b, l) = \sup_{\alpha \in \Omega_a(l)}\|\mathcal{G}_{q+1}^{[a_{q+1}]} \cdots \mathcal{G}_n^{[a_n]}\bar{\mathcal{Q}}_\alpha(D^b \bar{v}_h)\|$ and $\mathcal{G}_1, \cdots, \mathcal{G}_q$ are taken to be the identities. We show that

$$Q(a, b, l) \leqslant c_{11}n^{-l}. \tag{14}$$

If $f \in C_1^k$ then for all $j \geqslant 0$, $\|\bar{\mathcal{V}}_h^{[j]}f\| \leqslant c_{12}\|f\|$ and

$$\|\bar{\bar{\mathcal{F}}}_i^{[j]}f\| \leqslant \tilde{\beta}_{j,i}n^{-\frac{1}{2}j}\|f\| \leqslant c_{13}n^{-1}\|f\|, \qquad j > 1$$

$$\leqslant c_{14}n^{-1}\sup_{|m|=1}\|D^m f\|, \qquad j = 1.$$

Furthermore, from (8) these bounds hold with $\bar{\bar{\mathcal{F}}}_i$ replaced by $\bar{\mathcal{G}}_i$. Suppose $\alpha \in \Omega_a(l)$ and let l' be the number of a_i equal to 1 ($0 \leqslant l' \leqslant l$). Iterating the above bounds we have

$$\|\mathcal{G}_{q+1}^{[a_{q+1}]} \cdots \mathcal{G}_n^{[a_n]}\bar{\mathcal{Q}}_\alpha(D^b \bar{v}_h)\| \leqslant c_{15}n^{-l}\sup_{|m|=b+l'}\|D^m f\|$$

where f is $\mathfrak{M}_n \bar{v}_h$ omitting the l terms for which $a_i > 0$. Thus from Lemma 3 (14) holds and (12) follows from (13) and an application of an equivalent form of the identity (19) in [6] with D notation.

Define the formal power series $f(t)$ in $t = (t_1, \cdots, t_k)$ by $f(t) = \Sigma_{|j|>0}(j!)^{-1}\mu_j t^j = E\exp(t'X)$ where $\{\mu_i\}$ is the moment sequence of a random vector X; call f the moment series. If f, g are the moment series of independent random vectors X and Y then fg is the moment series of $X + Y$. The coefficient of $t^j/j!$ in the formal expansion of $\log f(t)$ is the jth *cumulant* χ_j; that is

$$\Sigma_{|j|\geqslant 1}(j!)^{-1}\chi_j t^j = \Sigma_{l=1}^{\infty}(-)^{l+1}l^{-1}\left[\Sigma_{|j|\geqslant 1}(j!)^{-1}\mu_j t^j\right]^l. \tag{15}$$

It is easily seen that χ_j is a function of μ_i for $1 \leqslant |i| \leqslant |j|$; the familiar additive property of cumulants is easily deduced. In the next lemma $\chi_j = n^{-1}\Sigma_{i=1}^n \chi_{j,i}$, $\tilde{\chi}_j = n^{-1}\Sigma_{i=1}^n \tilde{\chi}_{j,i}$; the proof of this lemma will be omitted as it follows from Lemmas 6.3 and 14.5 in [2] on taking $m = 1$, $s = |\nu|$ in the latter result. Alternatively it may be deduced from (15) and the estimates (5), (6) given here.

LEMMA 5. *Suppose* $|j| \geqslant 3$; *then*

(i)

$$|\chi_j| \leqslant c_{16}\beta_{|j|},\ |\tilde{\chi}_j| \leqslant c_{16}\tilde{\beta}_{|j|}.$$

(ii) *If*

then
$$n^{-\frac{1}{2}(|j|-2)}\beta_{|j|} \leqslant c_{17}$$
$$|\chi_j - \tilde{\chi}_j| \leqslant c_{18}\Lambda_{|j|}.$$

Following Bhattacharya and Rao (1976) define the formal polynomials $P_i(t, \{\chi_j\})$ in $t = (t_1, \cdots, t_k)$ by the following identity between two formal power series in u:

$$\Sigma_{i=0}^{\infty} P_i(t, \{\chi_j\}) u^i = \Sigma_{m=0}^{\infty}(m!)^{-1}\left[\Sigma_{|j|\geqslant 3}(j!)^{-1}\chi_j t^j u^{|j|-2}\right]^m. \tag{16}$$

The following facts are immediate: the degree of $P_i(t, \{\chi_j\})$ is $3i$ and each term in P_i involves a product $\chi_{j_1} \cdots \chi_{j_l}$ where $l \leqslant i$ and $|j_1 + \cdots + j_l| = i + 2l$. It follows from (i) of Lemma 5 that the coefficients of all terms in $P_i(t, \{\chi_j\})$ are $\leqslant c_{19}\beta_{|j_1|} \cdots \beta_{|j_l|} \leqslant c_{20}\beta_{i+2}$ in absolute value (using Lemma 1), with a similar result for $P_i(t, \{\tilde{\chi}_j\})$.

Consider the cumulant series of $\Sigma_{i=1}^n \tilde{X}_i$; for $|j| \geqslant 3$ the jth cumulant is $\Sigma_{i=1}^n \tilde{\chi}_{j,i} = n\tilde{\chi}_j$ so that if $\tilde{q}_n(t)$ is the moment series of $\tilde{S}_n$ we have

$$\log \tilde{q}_n(t) = \Sigma_{|j|\geqslant 3}(j!)^{-1}\tilde{\chi}_j t^j n^{-\frac{1}{2}(|j|-2)} + \tfrac{1}{2}\|t\|^2$$

since $\mathrm{Cov}(\tilde{S}_n) = I_k$. From the definition of the polynomials P_i we therefore have the formal identity

$$\tilde{q}_n(t) = \Sigma_{i=0}^{\infty} n^{-\frac{1}{2}i} P_i(t, \{\tilde{\chi}_j\}) z(t) \tag{17}$$

where $z(t) = \exp(\frac{1}{2}\|t\|^2)$ is the moment series of N. In Lemma 6 we estimate the error involved when the first few terms in (17) evaluated at $t = -D$ are used to approximate $\tilde{\mathfrak{Q}}_n$ when operating on the function $\bar{u}$.

Let $E = \{0, 1\}$. Suppose a_i, b_i $(i = 1, \cdots, n)$ are elements of some commutative ring and write $a = (a_1, \cdots, a_n)$, $b = (b_1, \cdots, b_n)$, $a^l = a_1^{l_1} \cdots a_n^{l_n}$, $l \in (\mathbb{Z}^+)^n$. The following binomial expansion may be established by induction on n. Let $l \in E^n$, $t \in \mathbb{Z}^+$; then

$$(a + b)^l = \Sigma_{i\leqslant l|i|<t} a^i b^{l-i} + \Sigma_{i\leqslant l|i|=t} a^i (a + b)^d b^{l-i-d} \tag{18}$$

where $d_j = l_j$ if $j < m(i)$, $d_j = 0$ otherwise, and $m(i) = \min(j : i_j = 1)$ (taken to be $n + 1$ if $i = 0$). We shall also use the following identity, which follows on repeated application of (18) to $(1 - a + a)^{i_j}$, assuming there is an identity element. Let $l \in E^n$, $0 \leqslant i \leqslant l$ and t, p be positive integers; then

$$a^{l-i} = \Sigma_{|m_p|<t}(1 - a)^{m_p} a^{l-i_p} + \Sigma_{|m_p|=t}(1 - a)^{m_\rho} a^{l-i_\rho-d_\rho} \tag{19}$$

where the summations are over $0 \leqslant i_p \leqslant \cdots \leqslant i_1 \leqslant i$, $m_p = i_1 + \cdots + i_p$ and $\rho = \max(j : i_j > 0)$.

Write $\tilde{T}_{p,n} = \Sigma_{i=0}^p n^{-\frac{1}{2}i} P_i(-D, \{\tilde{\chi}_j\})$ and $\gamma_n = \tilde{\beta}_{r+1} n^{-\frac{1}{2}(r-1)}$.

LEMMA 6. *Under the conditions of Lemma* 3,

$$(1 + \|x\|^s)|(\tilde{\mathfrak{Q}}_n - \tilde{T}_{r-2,n}\mathfrak{N})\bar{u}(x)| \leqslant c_{21}(k, r, s)\gamma_n.$$

PROOF. Let $e = (1, \cdots, 1) \in E^n$. Write $\bar{\tilde{\mathcal{F}}} = (\bar{\tilde{\mathcal{F}}}_1, \cdots, \bar{\tilde{\mathcal{F}}}_n)$, $\bar{\mathcal{G}} = (\bar{\mathcal{G}}_1, \cdots, \bar{\mathcal{G}}_n)$, $\mathfrak{D} = \bar{\tilde{\mathcal{F}}} - \bar{\mathcal{G}}$. Then from (18) we have

$$\tilde{\mathcal{Q}}_n = \bar{\tilde{\mathcal{F}}}^e = (\mathfrak{D} + \bar{\mathcal{G}})^e = \Sigma_{|i|<r-1}\mathfrak{D}^i\bar{\mathcal{G}}^{e-i} + \Sigma_{|i|=r-1}\mathfrak{D}^i\bar{\tilde{\mathcal{F}}}^d\bar{\mathcal{G}}^{e-i-d} \tag{20}$$

where the summations are over $i \in E^n$. Suppose $r \geqslant 3$; applying (19) with $a = \bar{\mathcal{G}}$, $l = e, p = t \geqslant 2$ and noting that $i_t = 0$ necessarily in the first sum we have, writing $I = (\mathcal{I}, \cdots, \mathcal{I})$,

$$\bar{\mathcal{G}}^{e-i} = \Sigma^*(I - \bar{\mathcal{G}})^{m_{t-1}}\mathfrak{N} + \Sigma^{**}(I - \bar{\mathcal{G}})^{m_t}\bar{\mathcal{G}}^{e-i_p-d_p}$$

where the summation is over $0 \leqslant i_{t-1} \leqslant \cdots \leqslant i_1 \leqslant i$, $|m_{t-1}| < t$ in Σ^* and over $0 \leqslant i_t \leqslant \cdots \leqslant i_1 \leqslant i$, $|m_t| = t$ in Σ^{**}. We therefore write

$$\tilde{\mathcal{Q}}_n = \mathcal{A}_n\mathfrak{N} + \mathcal{B}_n + \mathcal{C}_n \tag{21}$$

where

$$\mathcal{A}_n = \Sigma_{|i|<r-1}\Sigma^*\mathfrak{D}^i(I - \bar{\mathcal{G}})^{m_{t-1}}$$

$$\mathcal{B}_n = \Sigma_{0<|i|<r-1}\Sigma^{**}\mathfrak{D}^i(I - \bar{\mathcal{G}})^{m_t}\bar{\mathcal{G}}^{e-i_p-d_p}$$

$$\mathcal{C}_n = \Sigma_{|i|=r-1}\mathfrak{D}^i\bar{\tilde{\mathcal{F}}}^d\bar{\mathcal{G}}^{e-i-d}$$

and $t = [3(r-1)/2]$. Consider $\mathcal{C}_n\bar{u}$. Applying Lemma 2 with $p = 2$ to each $\mathfrak{D}_j = \tilde{\mathcal{F}}_j - \bar{\mathcal{G}}_j$ occurring in the sum and noting that $\tilde{\mu}_{j,i} = \nu_{j,i}$ for $|j| \leqslant 2$ it follows on iteration from Lemma 4 that

$$(1 + \|x\|^s)|\mathcal{C}_n\bar{u}(x)| \leqslant c_{22}n^{-3(r-1)/2}\Sigma\,\tilde{\beta}_{3,i_1}\cdots\tilde{\beta}_{3,i_{r-1}}$$

where the summation is over all choices of $i_1 < \cdots < i_{r-1}$ from $\{1, \cdots, n\}$. Thus

$$(1 + \|x\|^s)|\mathcal{C}_n\bar{u}(x)| \leqslant c_{22}n^{-3(r-1)/2}\left(\Sigma_{i=1}^n\tilde{\beta}_{3,i}\right)^{r-1} \leqslant c_{22}k^{r-1}\gamma_n \tag{22}$$

from Lemma 1. Consider next $\mathcal{B}_n\bar{u}$. Applying Lemma 2 with $p = 1$ to each $\mathfrak{D}_j$ and $(1 - \bar{\mathcal{G}}_j)$ occurring in the sum it follows on iteration from Lemma 4 that

$$(1 + \|x\|^s)|\mathcal{B}_n\bar{u}(x)| \leqslant c_{23}\Sigma_{l=1}^{r-2}\Sigma'\Sigma_{|s|=t}n^{-(l+t)}\tilde{\beta}_{2,i_1}^{1+s_1}\cdots\tilde{\beta}_{2,i_l}^{1+s_l}$$

where the summation in Σ' is over all choices of $i_1 < \cdots < i_l$, and $s = (s_1, \cdots, s_l)$. Thus

$$\begin{aligned}(1 + \|x\|^s)|\mathcal{B}_n\bar{u}(x)| &\leqslant c_{23}\Sigma_{l=1}^{r-2}n^{-(l+t)}\Sigma_{|s|=t}\Pi_{j=1}^l\left[\Sigma_{i=1}^n\tilde{\beta}_{2,i}^{1+s_j}\right]\\ &\leqslant c_{23}\Sigma_{l=1}^{r-2}n^{-(l+t)}\Sigma_{|s|=t}\Pi_{j=1}^l\left[\Sigma_{i=1}^n\tilde{\beta}_{2+2s_j,i}\right]\\ &= c_{23}\Sigma_{l=1}^{r-2}n^{-t}\Sigma_{|s|=t}\Pi_{j=1}^l\tilde{\beta}_{2+2s_j}\\ &\leqslant c_{24}n^{-t}\tilde{\beta}_{r+1}^{2t/(r-1)}\\ &= c_{24}\gamma_n^{2t/(r-1)} \leqslant c_{25}\gamma_n\end{aligned} \tag{23}$$

since $2t/(r-1) > 1$ and $\gamma_n \leqslant c_{26}$. Let $\mathcal{S}_n$ denote the expression obtained by replacing each $\tilde{\mathcal{F}}_j$ and $\bar{\mathcal{G}}_j$ occurring in $\mathcal{Q}_n$ by the expansion given in Lemma 2 with $p = 3(r-2)$. By repeated application of Lemma 2 along with Lemma 4 and $\tilde{\beta}_{j,i} n^{-\frac{1}{2}j} \leqslant c_{27}(j)$ it may be shown that

$$(1 + \|x\|^s)|(\mathcal{Q}_n - \mathcal{S}_n)\mathfrak{N}\bar{u}(x)| \leqslant c_{28}\tilde{\beta}_{3(r-2)+1} n^{-\frac{1}{2}[3(r-2)-1]} \leqslant c_{29}\gamma_n. \tag{24}$$

In view of (22), (23) and (24) it remains to prove that

$$(1 + \|x\|^s)|(\mathcal{S}_n - \tilde{T}_{r-2,n})\mathfrak{N}\bar{u}(x)| \leqslant c_{30}\gamma_n. \tag{25}$$

Let $\tilde{f}_i$, g_i be the moment series of $\tilde{X}_i$, Y_i. Write $\tilde{f} = (\tilde{f}_1, \cdots, \tilde{f}_n)$, $g = (g_1, \cdots, g_n)$, $d = \bar{\tilde{f}} - \bar{g}$ where $\bar{\tilde{f}}_i(t) = \tilde{f}(n^{-\frac{1}{2}}t)$ etc. The identity (21) remains valid when the operators are replaced by their corresponding moment series; that is

$$\tilde{q}_n = a_n z + b_n + c_n$$

where a_n, b_n, c_n are the expressions $\mathcal{Q}_n$, $\mathcal{B}_n$, $\mathcal{C}_n$ with $\tilde{f}, g, d$ replacing $\tilde{\mathcal{F}}, \mathcal{G}, \mathfrak{D}$ respectively. Let s_n be the expression obtained by replacing each $\bar{\tilde{f}}_j$, $\bar{g}_j$ in a_n by the terms of order not greater than $3(r-2)$. Noting that $\bar{g}_i^{-1} = \exp(-\frac{1}{2}n^{-1}t'\mathcal{C}_i t)$ it is seen that all terms in $\tilde{q}_n/z$ of order not greater than $3(r-2)$ are contained in s_n. Since $\mathcal{S}_n$ is the expression s_n evaluated at $t = -D$ it follows from (17) and the fact that the degree of P_i is $3i$ that one has

$$\mathcal{S}_n = \tilde{T}_{r-2,n} + R_n$$

where R_n consists of a finite number of terms of order $> 3(r-2)$ in $\tilde{q}_n/z$, evaluated at $t = -D$. But for $i > r-2$ the coefficients of all terms in $P_i(t, \{\tilde{\chi}_j\})n^{-\frac{1}{2}i}$ are $\leqslant \tilde{\beta}_{i+2}n^{-\frac{1}{2}i} \leqslant c_{31}\gamma_n$. Operating R_n on $\mathfrak{N}\bar{u}$, (25) follows and the result follows for $r \geqslant 3$. If $r = 2$, the result follows immediately from (22) since the first sum in (20) is simply $\mathfrak{N}$.

The alternative approach adopted by Bhattacharya and Rao (1976) involves expansions of derivatives of characteristic functions; see Chapter 2, Section 9 in [2]. Lemma 6 follows from Fourier inversion and the estimates (15.37)–(15.47) in [2], along with the estimate for $\|T\|$ given here in Section 2.

Define the class Γ of functions on $[0, \infty)$ as in [6] and let $\beta_{r,g} = n^{-1}\sum_{i=1}^n E\|X_i\|^r g(\|X_i\|)$ where $g \in \Gamma$. Assume $\beta_{r,g} < \infty$ and define

$$\eta_n = n^{-\frac{1}{2}r}\sum_{i=1}^n\Big[n^{-\frac{1}{2}}\int_{\|x\| \leqslant n^{\frac{1}{2}}}\|x\|^{r+1}dF_i(x) + \int_{\|x\| > n^{\frac{1}{2}}}\|x\|^r g\big(n^{-\frac{1}{2}}\|x\|\big)\, dF_i(x)\Big].$$

When $k = 1$ the bound

$$E\|S_n\|^r g(\|S_n\|) \leqslant c_{32}(k, r)(1 + \eta_n) \tag{26}$$

is a consequence of the inequality proved in Sazonov (1974); that (26) is true for all $k \geqslant 1$ may be deduced as in [6] for the i.i.d. case. Note that

$$\gamma_n \equiv \tilde{\beta}_{r+1} n^{-\frac{1}{2}(r-1)} \leqslant c_2(r+1)\eta_n \tag{27}$$

from (5) and that for $j \leqslant r$

(28) $$\beta_j n^{-\frac{1}{2}(j-2)} \leqslant \beta_2 + \eta_n = k + \eta_n.$$

Write $h(t) = 1 + t^r g(t)$, $t \geqslant 0$.

LEMMA 7. *Suppose* $\beta_{r,g} < \infty$. *Under the conditions of Lemma* 3, *for all* $n \geqslant 1$

$$\int h(\|x\|)|(\mathcal{Q}_n - \tilde{T}_{r-2,n}\mathfrak{N})\bar{u}(x)|\, dx \leqslant c_{33}(k, r)\eta_n.$$

PROOF. Taking $s = k + r + 2$ in Lemma 6 we find (cf. Lemma 4 in [6])

(29) $$\int h(\|x\|)|(\tilde{\mathcal{Q}}_n - \tilde{T}_{r-2,n}\mathfrak{N})\bar{u}(x)|dx \leqslant c_{34}\eta_n.$$

Again, as in [6], it follows from (7) and (26) that

(30) $$\int h(\|x\|)|(\mathcal{Q}_n - \mathcal{Q}_{n,n})\bar{u}(x)|dx \leqslant c_{35}\eta_n.$$

We use

(31) $$|(\mathcal{Q}_{n,n} - \tilde{T}_{r-2,n}\mathfrak{N})\bar{u}(x)|$$
$$\leqslant |(\tilde{\mathcal{Q}}_n - \tilde{T}_{r-2,n}\mathfrak{N})\bar{u}(a(x))| + |\bar{\mathfrak{U}}\tilde{T}_{r-2,n}\mathfrak{n}(a(x)) - \bar{\mathfrak{U}}\tilde{T}_{r-2,n}\mathfrak{n}(x)|$$

where $a(x) = T(x - n^{\frac{1}{2}}\mu)$ and $\mathfrak{n}$ is the density of N. Applying the change of variable $y = a(x)$ and using the fact that the determinant of a symmetric matrix is the product of its eigenvalues we find that

$$\int h(\|x\|)|(\tilde{\mathcal{Q}}_n - \tilde{T}_{r-2,n}\mathfrak{N})\bar{u}(a(x))|dx \leqslant c_{36}\eta_n.$$

Using $|D^m\mathfrak{n}(x)| \leqslant c_{37}(m)(1 + \|x\|^{|m|})\mathfrak{n}(x)$ one can show that the second term on the right-hand side in (31) is less than $c_{38}(1 + \|x\|^{k+r+2})^{-1}\eta_n$ (cf. [6], Lemma 4). The result follows from (30), (31).

4. Speeds of convergence for general summand distributions. In this section speeds of convergence are derived for expectations of all N-continuous functions, F_i-integrable under prescribed moment conditions. Theorem 1 extends and improves some results in Bhattacharya and Rao (1976) (Section 3.15). Define $\omega_\phi^\varepsilon(x) = \sup\{|\phi(y) - \phi(z)| : y, z \in S(x, \varepsilon)\}$ where $\phi \in B^k$ and $S(x, \varepsilon)$ is the open sphere centered at x radius ε and, following Bhattacharya and Rao (1976),

$$\bar{\omega}_\phi(\varepsilon : \mu) = \int \omega_\phi^\varepsilon(x)\, d\mu(x),$$

the average modulus of oscillation of ϕ with respect to the measure μ. Define

$$M_{r,g}(\phi) = \sup_x[h(\|x\|)]^{-1}|\phi(x)|$$

where $h(t) = 1 + t^r g(t)$, $g \in \Gamma$. We need a slight generalization of Lemma 5 in [6] to cover the case where Q is a finite signed measure on $\mathbb{R}^k$. With the notation in [6] we have

$$\mathcal{Q}\phi^{s,a\varepsilon}(x) - \mathcal{Q}\phi(x),\ \mathcal{Q}\phi(x) - \mathcal{Q}\phi^{i,a\varepsilon}(x) \leqslant \mathcal{Q}^+\omega_\phi^{2a\varepsilon}(x)$$

and an identical proof yields the result of Lemma 5 in [6] with $\mathcal{Q}$ replaced by $\mathcal{Q}^+$, and the constants A_1, A_2 depending on the integrals of $h(\|x\|)$ with respect to P and $|Q|$. Write $\tilde{\psi} = \tilde{T}_{r-2,n}N$.

LEMMA 8. *Let* $h < 1$, $\|x\| \leqslant c_1 \varepsilon_n^{1/p}$ $(p \geqslant 1)$. *There exists a constant* c_2 *such that if* $\varepsilon_n \leqslant c_2$ *one has*

$$\int \omega_\phi^h(x-y)\, d\tilde{\psi}^+(y) \leqslant c_3\left[\mathfrak{N}\omega_\phi^h(x) + M_{r,g}(\phi)\eta_n\right].$$

PROOF. Let q be the density of $\tilde{\psi}$; then $q(y) = (1 + S_n(y))\mathrm{n}(y)$ where

$$|S_n(y)| \leqslant c_4 \Sigma_{s=1}^{r-2}(1 + \|y\|^{3s})\tilde{\beta}_{s+2} n^{-\frac{1}{2}s}.$$

Suppose $\|y\|^{3(r-1)} \leqslant c_5 \gamma_n^{-1}$; then by suitable choice of c_2 and c_5 we have $|S_n(y)| \leqslant \frac{1}{2}$, since $\tilde{\beta}_{s+2} n^{-\frac{1}{2}s} \leqslant c_6 \varepsilon_n$ and $\|y\|^{3s} \leqslant c_5^{s/(r-1)}[\tilde{\beta}_{s+2} n^{-\frac{1}{2}s}]^{-1}$. Let $A_n = \{c_5 \gamma_n^{-1}\}^{1/3(r-1)}$; then

$$\begin{aligned}\int \omega_\phi^h(x-y)\, d\tilde{\psi}^+(y) &\leqslant \tfrac{3}{2}\int_{\|y\| \leqslant A_n} \omega_\phi^h(x-y)\mathrm{n}(y)\, dy \\ &\quad + c_7 M_{r,g}(\phi)\int_{\|y\|>A_n}\left[1 + \|x-y\|\right]^{r+1}\left[1 + \|y\|^{3(r-2)}\right]\mathrm{n}(y)\, dy \\ &\leqslant \tfrac{3}{2}\mathfrak{N}\omega_\phi^h(x) + c_8 M_{r,g}(\phi)\eta_n\end{aligned}$$

and the result follows.

Applying (35) in [6] with $\|x\| \leqslant c_1\varepsilon_n^{1/p}$ and using the moment inequality $\varepsilon_n^{r-1} \leqslant \gamma_n \leqslant c_9 \eta_n$ we have

$$\mathfrak{N}\omega_\phi^h(x) \leqslant c_{10}\left[\mathfrak{N}\omega_\phi^h(0) + M_{r,g}(\phi)\eta_n\right]. \tag{32}$$

Write

$$\psi = \Sigma_{i=0}^{r-2} n^{-\frac{1}{2}i} P_i(-D, \{\chi_j\})N.$$

We now give the main result of this section.

THEOREM 1. *Suppose* $\beta_{r,g} < \infty$ *where* $g \in \Gamma$ *and* $r \geqslant 2$ *is an integer. Then for all* $\phi \in B^k$ *with* $M_{r,g}(\phi) < \infty$ *and all* $n \geqslant 1$

$$\left|\int \phi\, d(Q_n - \psi)\right| \leqslant C(k,r)\left[M_{r,g}(\phi)\eta_n + \bar{\omega}_\phi(C'\varepsilon_n : N)\right]$$

where $\varepsilon_n = \tilde{\beta}_3 n^{-\frac{1}{2}} \leqslant c_2(3) n^{-3/2}\Sigma_{i=1}^n \int_{\|x\| \leqslant n^{\frac{1}{2}}} \|x\|^3\, dF_i(x)$.

PROOF. Note that $\varepsilon_n \leqslant \gamma_n^{1/(r-1)} \leqslant c_2(r+1)\eta_n^{1/(r-1)}$ from (27); choose c_{11} such that if $\eta_n \leqslant c_{11}$ then ε_n satisfies the requirements of Lemma 3 and 8. If $\eta_n > c_{11}$ we have

$$\left|\int \phi\, d(Q_n - \psi)\right| \leqslant c_{12}(k,r) M_{r,g}(\phi)\eta_n$$

using (28). We may therefore assume that $\eta_n \leqslant c_{11}$. In the modified version of Lemma 5 of [6] take $P = Q_n$, $Q = \tilde{\psi}$, $K_\varepsilon = \bar{U}$ where $a^2\varepsilon = \varepsilon_n$, $\varepsilon' = \varepsilon^{\frac{1}{2}}$, $t = [(a\varepsilon')^{-\frac{1}{2}}]$. Applying Lemma 8 with $p = 4$ and (32) for the estimation of $\tau(t)$, it follows from Lemma 7 that

$$\left|\int \phi\, d(Q_n - \tilde{\psi})\right| \leqslant c_{13}\left[M_{r,g}(\phi)\eta_n + \bar{\omega}_\phi(C'\varepsilon_n : N)\right].$$

Finally, each coefficient in $[P_i(t, \{\chi_j\}) - P_i(t, \{\tilde{\chi}_j\})]n^{-\frac{1}{2}i}$ is less in absolute value than a constant times a sum of terms

$$n^{-\frac{1}{2}i}|\chi_{j_1} - \tilde{\chi}_{j_1}|\beta_{|j_2|} \cdots \beta_{|j_l|} \leq c_{14}\eta_n \Pi_{m=2}^{l} n^{-\frac{1}{2}(|j_m|-2)}\beta_{|j_m|}$$
$$\leq c_{15}\eta_n$$

since $|j_1 + \cdots + j_l| = i + 2l$, using Lemma 1 and $\eta_n \leq c_{11}$. Thus

$$|\int h(\|y\|)d(\psi - \tilde{\psi})| \leq c_{16}\eta_n \tag{33}$$

and the result follows.

Suppose $\beta_r < \infty$ for some integer $r \geq 3$; then $\varepsilon_n \leq \beta_3 n^{-\frac{1}{2}}$. Taking $g(x) = 1$, Theorem 15.1 in [2] follows from Theorem 2 on application of Lemma 11 in Section 7, with the removal of a moment condition. Theorem 15.1 in [2] applies when the translates of ϕ form an N-uniformity class; our result applies for arbitrary ϕ and hence improves Theorem 15.4 in [2] by removing a logarithmic term from the error bound. Götze and Hipp (1978) prove a similar result (Theorem 3.13) to Theorem 1 with $g(x) = x^t$ (again, assuming the translates of ϕ form an N-uniformity class). For various consequences and applications of these results see [2]; evidently some of these results may now be improved.

In the case $r = 2$, we have $\psi = N$ and from (27) $\varepsilon_n \leq c_2(3)\eta_n$. By applying the result to $g(x) = x^t$, $0 \leq t \leq 1$, we have an extension of Theorem 18.1 in [2] since in that theorem (a) for $t < 1$, ϕ must satisfy $|\phi(x)| \leq c_{18}(1 + \|x\|^2)$, and (b) the translates of ϕ must form an N-uniformity class.

For the form of error bound in Theorem 1 for arbitrary $g \in \Gamma$, see Section 7.

5. Speeds of convergence and asymptotic expansions under Cramér's condition. One is able to considerably improve the speed of convergence in Theorem 1 when the summand distributions satisfy a Cramér condition. Under certain circumstances the result gives rise to asymptotic expansions. The main result (Theorem 2) improves results in [2]. Define

$$\rho_i = \sup_{\|\zeta\| > \kappa\tilde{\beta}_3^{-1}}|\theta_i(\zeta)|,\ \rho = n^{-1}\Sigma_{i=1}^{n}\rho_i$$

where θ_i is the characteristic function of X_i and κ is the constant in Lemma 3. Assume that $\rho < 1$; in the i.i.d. case this is equivalent to assuming that *Cramér's condition* is satisfied (see [2], Chapter 4, for example). For all ζ we have $|\theta_i(\zeta)| \geq 1 - \frac{1}{2}\zeta' \operatorname{Cov}(X_i)\zeta$; taking $\|\zeta\| = 1$ (noting that $\kappa\tilde{\beta}_3^{-1} < 1$) it follows that $\rho \geq 1 - \frac{1}{2}\|\zeta\|^2 = \frac{1}{2}$. Let $T = \rho_1^{-n}$ where $\rho_1 = \rho^{1/(r-1)}$; we use the smoothing distribution V_T where V is the distribution defined in Section 3. Let $|m| \leq r - 1$; we have

$$\|D^m\tilde{\mathfrak{Q}}_n' v_T\| \leq \int|\zeta^m||\tilde{\theta}(\zeta)|dA(T^{-1}\zeta) \tag{34}$$

where $\tilde{\mathfrak{Q}}_n'$ is $\tilde{\mathfrak{Q}}_n$ omitting an arbitrary $q \leq s$ $\tilde{\mathfrak{F}}_i$'s, and $\tilde{\theta}$ is the ch.f. of $\tilde{\mathfrak{Q}}_n'$. Assume that $\kappa\rho_1^n \leq \varepsilon_n$. By the argument in Lemma 3 the integral in (34) over the region $\|\zeta\| \leq \kappa\varepsilon_n^{-1}$ is less than $c_1(k, |m|, q)$. Consider the integral over $\|\zeta\| > \kappa\varepsilon_n^{-1}$. If $\tilde{\theta}_i$ is the ch.f. of $\tilde{X}_i$ then

$$|\tilde{\theta}_i(\zeta) - \theta_i(\zeta)| = \int_{\|x\| > n^{\frac{1}{2}}}(1 - e^{i\zeta' x})\, dF_i(x) \leq 2\beta_{2,i}n^{-1};$$

thus $\tilde{\rho}_i \equiv \sup_{\|\zeta\| > \kappa\tilde{\beta}_3^{-1}} |\tilde{\theta}_i(\zeta)| \leqslant \rho_i + 2\beta_{2,i} n^{-1}$. Assuming without loss of generality that the first q $\tilde{\mathcal{F}}_i$'s are omitted one has

$$\int_{\|\zeta\| > \kappa\varepsilon_n^{-1}} |\zeta^m| |\tilde{\theta}(\zeta)| \, dA(T^{-1}\zeta) \leqslant \left[\Pi_{i=q+1}^n \tilde{\rho}_i\right] T^{|m|} \int \|\zeta\|^{|m|} \, dA(\zeta)$$
$$\leqslant c_1 (n/n-q)^n \left[n^{-1} \Sigma_{i=1}^n \tilde{\rho}_i\right]^n T^{|m|}$$
$$\leqslant c_2 \rho^n e^{k/\rho} T^{|m|} \leqslant c_3$$

by definition of T and using $\rho \geqslant \frac{1}{2}$. It follows that if $\kappa\rho_1^n \leqslant \varepsilon_n$ then

$$\|D^m \tilde{\mathfrak{Q}}_n' v_T\| \leqslant c_4. \tag{35}$$

LEMMA 9. *If $\kappa\rho_1^n \leqslant \varepsilon_n$ then Lemma 6 holds with $\bar{u}$ replaced by u_T.*

PROOF. We use the identity

$$\mathcal{I} = \Sigma_{j=0}^{t-1} (\mathcal{I} - \overline{\mathcal{U}})^j \overline{\mathcal{U}} + (\mathcal{I} - \overline{\mathcal{U}})^t = \mathcal{Q}_t \overline{\mathcal{U}} + \mathcal{B}_t. \tag{36}$$

Suppose that $\max_{|m|=l} (1 + \|x\|^s) |D^m f(x)| \leqslant A$, where $l = 0$ or $1, f \in B^k$. By splitting the range of integration over $\|y\| \leqslant \frac{1}{2}B$, $\|y\| > \frac{1}{2}B$ where $B = \max(\|x\|, 1)$ it is easily seen that

$$(1 + \|x\|^s) |(\mathcal{I} - \overline{\mathcal{U}}) f(x)| \leqslant c_5 A \varepsilon_n^l. \tag{37}$$

We operate (36) with $t = r - 1$ on $\tilde{\Delta} u_T$ where $\tilde{\Delta} = \tilde{\mathfrak{Q}}_n - \tilde{T}_{r-2,n} \mathfrak{N}$. We have $\overline{\mathcal{U}} \tilde{\Delta} u_T = \tilde{\Delta} \bar{u}_1$ where $U_1 = V_1^{*(s+1)}$ and V_1 is p.d. 1-smooth having moments of all orders. Thus, from Lemma 6, $(1 + \|x\|^s) |\tilde{\Delta} \bar{u}_1(x)| \leqslant c_6 \gamma_n$ and it follows from (37) with $l = 0$ that

$$(1 + \|x\|^s) |\mathcal{Q}_{r-1} \overline{\mathcal{U}} \tilde{\Delta} u_T(x)| \leqslant c_7 \gamma_n. \tag{38}$$

In view of (35) it is easily verified that with $\mathfrak{M}_n = \tilde{\mathfrak{Q}}_n$ the proof of Lemma 4 applies without change on replacing $\bar{u}$ by u_T, and so $(1 + \|x\|^s) |D^m \tilde{\mathfrak{Q}}_n u_T(x)| \leqslant c_8$; thus on repeated application of (37) with $l = 1$ one has

$$(1 + \|x\|^s) |\mathcal{B}_{r-1} \tilde{\Delta} u_T(x)| \leqslant c_9 \varepsilon_n^{r-1} \leqslant c_{10} \gamma_n \tag{39}$$

from Lemma 1 and using the fact that the coefficients of $P_i(t, \{\tilde{\chi}_j\}) n^{-\frac{1}{2}i}$ are less than $\tilde{\beta}_{i+2} n^{-\frac{1}{2}i} \leqslant c_{11}(i)$. The result follows from (36), (38) and (39).

By using Lemma 9, the proof of Lemma 7 is easily seen to apply on replacing $\bar{u}$ by u_T, and with the notation of Section 4 we have the following result.

THEOREM 2. *Suppose $\beta_{r,g} < \infty$ where $g \in \Gamma$, $r \geqslant 2$ is an integer, and suppose that $\rho < 1$. Then for all $\phi \in B^k$ with $M_{r,g}(\phi) < \infty$ and all $n \geqslant 1$*

$$\left|\int \phi d(Q_n - \psi)\right| \leqslant C(k, r) \left[M_{r,g}(\phi) \eta_n + \bar{\omega}_\phi (C' \rho_1^n : N)\right].$$

PROOF. Assume $\kappa\rho_1^n \leqslant \varepsilon_n$. Replacing ε_n by ρ_1^n in the proof of Theorem 1 and using Lemma 7 with $\bar{u}$ replaced by u_T, the proof applies without change. If $\kappa\rho_1^n > \varepsilon_n$ the result follows directly from Theorem 1.

The above result is a more general version of Theorem 20.6 in [2] (when the latter is specialized to the case $n^{-1} \Sigma_{i=1}^n \operatorname{Cov}(X_i) = I_k$). Under certain conditions, Theorem 2 yields asymptotic expansions for a large class of functions. Suppose $r \geqslant 3$ and

C1: (β_r) is uniformly bounded in n;

C2: For all $\varepsilon > 0$, $n^{-1}\Sigma_{i=1}^n \int_{\|x\|>\varepsilon n^{\frac{1}{2}}} \|x\|^r \, dF_i(x) \to 0$ as $n \to \infty$;

C3: $\limsup_{i\to\infty} \sup_{\|\zeta\|>b} |\theta_i(\zeta)| < 1$ for all $b > 0$.

Let $\mathcal{C}$ be a class of $\phi \in B^k$ satisfying $\sup_{\phi\in\mathcal{C}} M_r(\phi) < \infty$ where $M_r(\phi) = \sup_x (1 + \|x\|^r)^{-1}|\phi(x)|$, and $\sup_{\phi\in\mathcal{C}} \bar{\omega}_\phi(h : N) = o([-\log h]^{-\frac{1}{2}(r-2)})$. Corollary 1 below is proved in [2] (Theorem 20.6 along with (20.45)).

COROLLARY 1. *Suppose* C1-C3 *hold. Then*

$$\sup_{\phi\in\mathcal{C}} \left|\int \phi \, d(Q_n - \psi)\right| = o\left(n^{-\frac{1}{2}(r-2)}\right).$$

PROOF. Taking $g(x) = 1$ conditions C1 and C2 ensure that $\eta_n = o(n^{-\frac{1}{2}(r-2)})$ (see Lemma 11 and the remark following Lemma 10 in Section 7). We have $\tilde{\beta}_3 \leqslant c_{12}\beta_r^{1/(r-2)} \leqslant c_{13}$ for all n from C1, and it follows from C3 that $\limsup_{n\to\infty}\rho < 1$. Therefore, from the definition of the class $\mathcal{C}$, $\sup_{\phi\in\mathcal{C}} \bar{\omega}_\phi(\rho_1^n : N) = o(n^{-\frac{1}{2}(r-2)})$ and the result follows from Theorem 2.

6. Speeds of convergence and asymptotic expansions for smooth functions. We now restrict our attention to smooth functions and obtain bounds applicable for arbitrary summand distributions; in particular, asymptotic expansions not requiring Cramér's condition will be deduced. Theorem 3 below includes known results in this area in [2], [4] and more recently [3]. The proof uses the main lemma in Section 3 (Lemma 7) and Theorem 1 applied to $D^i\phi$ for $|i| = s$.

THEOREM 3. *Suppose $\beta_{r,g} < \infty$ where $g \in \Gamma$ and $r \geqslant 3$ is an integer. Then for all $\phi \in C_s^k$, $1 \leqslant s \leqslant r - 2$, with $M_{r,g}(\phi) < \infty$ and $\max_{|i|=s} M_{r,g}(D^i\phi) < \infty$ and all $n \geqslant 1$*

$$\left|\int \phi \, d(Q_n - \psi)\right| \leqslant C(k, r)\left[A_{r,g}^s(\phi)\eta_n + \varepsilon_n^s \max_{|i|=s} \bar{\omega}_{D^i\phi}(C'\varepsilon_n : N)\right]$$

where $A_{r,g}^s(\phi) = M_{r,g}(\phi) + \max_{|i|=s} M_{r,g}(D^i\phi)$.

PROOF. As in the proof of Theorem 1 we may assume that $\eta_n \leqslant c_1$ and hence $\varepsilon_n \leqslant c_2$. We operate (36) with $t = s$ on $\Delta\phi$ where $\Delta = \mathcal{Q}_n - T_{r-2,n}\mathfrak{N}$, so that

$$\Delta\phi = \mathcal{C}_s(\bar{\mathcal{U}}\Delta\phi) + \mathcal{B}_s(\Delta\phi). \tag{40}$$

Note that $M_{r,g}(\phi_x) \leqslant c_3(1 + \|x\|^{r+1})M_{r,g}(\phi)$ where $\phi_x(y) = \phi(x - y)$. We have

$$\begin{aligned} |\bar{\mathcal{U}}\Delta\phi(x)| &\leqslant M_{r,g}(\phi_x)\int h(\|y\|)|\Delta\bar{u}(y)| \, dy \\ &\leqslant c_4(1 + \|x\|^{r+1})M_{r,g}(\phi)\eta_n \end{aligned} \tag{41}$$

on application of Lemma 7 along with (33). If $|f(x)| \leqslant A(1 + \|x\|^{r+1})$, $f \in B^k$, it is easily verified that $|(\mathcal{I} - \bar{\mathcal{U}})f(x)| \leqslant c_5 A(1 + \|x\|^{r+1})$. It follows from (41) that

$$|\mathcal{C}_s\bar{\mathcal{U}}\Delta\phi(0)| \leqslant c_6 M_{r,g}(\phi)\eta_n. \tag{42}$$

Suppose that there exists a function f' such that

$$\max_{|m|=1}|D^m f(x)| \leqslant A\left[(1 + \|x\|^{r+1})M_{r,g}(f')\eta_n + \mathfrak{N}\omega_{f'}^h(x)\right] = H(x)$$

say. Then

$$|(\mathcal{G} - \overline{\mathfrak{U}})f(x)| \leqslant k^{\frac{1}{2}}\int\|y\| \max_{|m|=1}|D^m f(x - \theta y)|\, d\overline{U}(y)$$
$$\leqslant c_7 A\left[\varepsilon_n(1 + \|x\|^{r+1})M_{r,g}(f')\eta_n + \int\|y\|\mathfrak{N}\omega_{f_x'}^h(-\theta y)\, d\overline{U}(y)\right].$$

Let I, J be the integrals over the regions $\|y\| \leqslant \varepsilon_n^{\frac{1}{2}}$, $\|y\| > \varepsilon_n^{\frac{1}{2}}$. Using (32) with $p = 2$ gives

$$I \leqslant c_8\varepsilon_n\left[\mathfrak{N}\omega_{f_x'}^h(0) + M_{r,g}(f_x')\eta_n\right]$$
$$\leqslant c_9\varepsilon_n\left[\mathfrak{N}\omega_{f'}^h(x) + (1 + \|x\|^{r+1})M_{r,g}(f')\eta_n\right]$$

and, omitting details,

$$J \leqslant c_{10}M_{r,g}(f_x')\int_{\|y\|>\varepsilon_n^{\frac{1}{2}}}\|y\|(1 + \|y\|^{r+1})\, d\overline{U}(y)$$
$$\leqslant c_{11}(1 + \|x\|^{r+1})M_{r,g}(f')\varepsilon_n^r$$
$$\leqslant c_{12}\varepsilon_n(1 + \|x\|^{r+1})M_{r,g}(f')\eta_n$$

from Lemma 1, $\varepsilon_n \leqslant c_2$ and (27). We therefore have

$$|(\mathcal{G} - \overline{\mathfrak{U}})f(x)| \leqslant c_{13}\varepsilon_n H(x). \tag{43}$$

But applying Theorem 1 to $D^i\phi$ for $|i| = s$ gives

$$|D^i\Delta\phi(x)| \leqslant c_{14}(1 + \|x\|^{r+1})M_{r,g}(D^i\phi)\eta_n + \mathfrak{N}\omega_{D^i\phi}^{C'\varepsilon_n}(x)$$

and it follows on s applications of (43) that

$$|\mathfrak{B}_s\Delta\phi(0)| \leqslant c_{15}\max_{|i|=s}\left[M_{r,g}(D^i\phi)\eta_n + \varepsilon_n^s\overline{\omega}_{D^i\phi}(C'\varepsilon_n : N)\right] \tag{44}$$

using $\varepsilon_n \leqslant c_2$. The result follows from (42), (44) and (40).

Note in particular that if $\max_{|i|=r-2}\overline{\omega}_{D^i\phi}(h : N) \leqslant B_r(\phi)h$ then

$$|\int\phi d(Q_n - \psi)| \leqslant C(k, r)\left[A_{r,g}^{r-2}(\phi) + B_r(\phi)\right]\eta_n$$

since $\varepsilon_n^{r-1} \leqslant c_2(r + 1)\eta_n$ from (27). Let $\mathcal{Q}_r$ be a class of functions $\phi \in C_{r-2}^k$ satisfying $\sup_{\phi\in\mathcal{Q}_r} A_{r,g}^{r-2}(\phi) < \infty$ with $g(t) \equiv 1$ and $\sup_{\phi\in\mathcal{Q}_r}\overline{\omega}_{D^i\phi}(h : N) \to 0$ as $h \to 0$ for $|i| = r - 2$. Conditions C1 and C2 in Section 5 ensure that $\eta_n = o(n^{-\frac{1}{2}(r-2)})$ and we can deduce the following result.

COROLLARY 2. *Suppose* C1 *and* C2 *are satisfied. Then*

$$\sup_{\phi\in\mathcal{Q}_r}|\int\phi d(Q_n - \psi)| = o\left(n^{-\frac{1}{2}(r-2)}\right).$$

Bhattacharya and Rao (1976) derive asymptotic expansions in the i.i.d. case for bounded functions f which are Fourier-Stieltjes transforms of finite signed measures μ satisfying $\int\|x\|^{r-2}d|\mu|(x) < \infty$. Since this condition implies that D^if

exists for $|i| = r - 2$ we have $\lim_{h\to 0} \bar{\omega}_{D^i\phi}(h : N) = 0$ for $|i| = r - 2$. Thus Theorem 20.7 in [2] follows from Corollary 2. In the i.i.d. case when $k = 1$, Hipp (1977) obtained the above asymptotic expansion with $r = 4$ for functions possessing a bounded uniformly continuous derivative of order 4. Very recently Götze and Hipp (1978) have obtained asymptotic expansions for smooth functions. Their result (Theorem 3.6) follows from our Theorem 3 with $g(x) = x^t$ $(0 \leqslant t < 1)$. In particular, they prove Corollary 2 with C1, C2 replaced by $n^{(r-2)/2}\eta_n \to 0$.

The functions $\phi_t(x) = x^t$ $(t \in (\mathbb{Z}^+)^k)$ are infinitely differentiable; in fact $D^j\phi_t(x) = c_{j,t}x^{t-j}$ if $j \leqslant t$ and $D^j\phi_t \equiv 0$ otherwise. We may therefore apply Corollary 2 with $r = |t|$. In fact the expansions here are *exact* as may be shown directly from the arguments preceding Lemma 6 applied to the original variables. In the case $k = 1$ this result is due to Von Bahr (1965). Note that for $s \in \mathbb{Z}^+$, $\|x\|^{2s} = \Sigma_{|i|=s}(s!/i!)x^{2i}$ and so the even-order moments of $\|S_n\|$ also have an exact expansion (in powers of n^{-1}). The functions $x \to \|x\|^i$ for $i \in (r-2, r]$ are $r - 2$ times differentiable and form an appropriate class $\mathcal{C}_r$ for application of Corollary 2. In the case $k = 1$ Von Bahr (1967) proved that for this class of functions the error in Corollary 2 is $O(n^{-\frac{1}{2}(r-2)})$.

7. On the form of the error bound. In this section the bounds in the preceding theorems are investigated further. Let

$$T^i_{r,g}(u) = \int_{\|x\|>u}\|x\|^r g(\|x\|)\, dF_i(x)$$

$$T_{r,g}(u) = n^{-1}\Sigma^n_{i=1}T^i_{r,g}(u)$$

and define

$$\beta_{r,g}(n) = \inf_{0<\varepsilon\leqslant 1}\left[\frac{\varepsilon g(n^{\frac{1}{2}})}{g(\varepsilon n^{\frac{1}{2}})}n^{-1}\Sigma^n_{i=1}\int_{\|x\|\leqslant \varepsilon n^{\frac{1}{2}}}\|x\|^r g(\|x\|)\, dF_i(x) + T_{r,g}(\varepsilon n^{\frac{1}{2}})\right].$$

The rates of convergence in the preceding theorems may be formulated in terms of $\beta_{r,g}(n)$; it is then possible to deduce error bounds or $o(\cdot)$ rates, as may be seen from the following lemma.

LEMMA 10. (i) $\beta_{r,g}(n) \leqslant \beta_{r,g}$ *for all* $n \geqslant 1$. (ii) *Suppose that* $u/g(u) \to \infty$ *and* $\sup_i T^i_{r,g}(u) \to 0$ *as* $u \to \infty$. *Then* $\beta_{r,g}(n) \to 0$ *as* $n \to \infty$.

PROOF. (i) Take $\varepsilon = 1$. (ii) The uniform integrability condition ensures that (a) $\beta_{r,g}$ is uniformly bounded for all n, and (b) $T_{r,g}(u_n) \to 0$ as $n \to \infty$ for every sequence $u_n \to \infty$. Suppose first that $g(u) \not\to \infty$ so that $g(u)\uparrow l$, $1 \leqslant l < \infty$. The result follows in this case on taking $\varepsilon = n^{-1/4}$. Suppose then that $g(u) \to \infty$. Take $\varepsilon = [g(n^{\frac{1}{2}})]^{-1}$; since $[g(n^{\frac{1}{2}})]^{-1}n^{\frac{1}{2}} \to \infty$ as $n \to \infty$ the result follows.

REMARK. If $g(x) = x^t$ $(0 \leqslant t < 1)$ then (ii) holds if we assume $\beta_{r,g} \leqslant K$ for all n and the Lindeberg-type condition $T_{r,g}(\varepsilon n^{\frac{1}{2}}) \to 0$ for every $\varepsilon > 0$. For then we have $\limsup_{n\to\infty} \beta_{r,g}(n) \leqslant K\varepsilon^{1-t}$ for every ε, $0 < \varepsilon \leqslant 1$.

Define $g'(x) = \inf_{t \geqslant 1} \dfrac{g(tx)}{g(t)}$ if $x \geqslant 1$, $g'(x) = x$, $x < 1$. Then $g' \in \Gamma$ and $g' \leqslant g$. Let

$$\delta_n = \left\{ n^{\frac{1}{2}(r-2)} g\left(n^{\frac{1}{2}}\right) \right\}^{-1} \beta_{r,g}(n)$$

$$\delta_n' = \delta_n + \left\{ n^{\frac{1}{2}(r-2)} g'\left(n^{\frac{1}{2}}\right) \right\}^{-1} T_{r,g}\left(n^{\frac{1}{2}}\right).$$

Let η_n be defined as in Section 3 and η_n' be η_n with g replaced by g'.

LEMMA 11. $\eta_n \leqslant \delta_n'$, $\eta_n' \leqslant \delta_n$.

PROOF. We have

$$n^{\frac{1}{2}(r-2)}\eta_n = n^{-1}\Sigma_{i=1}^n \Big\{ n^{-\frac{1}{2}} \int_{\|x\| \leqslant \varepsilon n^{\frac{1}{2}}} \|x\|^{r+1}\, dF_i(x) + n^{-\frac{1}{2}} \int_{\varepsilon n^{\frac{1}{2}} < \|x\| < n^{\frac{1}{2}}} \|x\|^{r+1}\, dF_i(x)$$

$$+ \int_{\|x\| > n^{\frac{1}{2}}} \|x\|^r g\left(n^{-\frac{1}{2}}\|x\|\right) dF_i(x).$$

Using properties of $g \in \Gamma$ and the fact that $g(n^{-\frac{1}{2}}\|x\|) \leqslant [g'(n^{\frac{1}{2}})]^{-1} g(\|x\|)$ when $\|x\| > n^{\frac{1}{2}}$ establishes the first inequality. The second inequality follows in a similar way, using properties of $g \in \Gamma$ and $g'(n^{-\frac{1}{2}}\|x\|) \leqslant [g(n^{\frac{1}{2}})]^{-1} g(\|x\|)$ when $\|x\| > n^{\frac{1}{2}}$.

REMARK. Since $\delta_n < \delta_n'$ a bound involving δ_n will be preferred whenever Theorems 1–3 may be applied with $g' \in \Gamma$; this will be the case whenever $M_{r,g'}(\phi) < \infty$ in Theorems 1 and 2 and in addition $\max_{|i| = r-2} M_{r,g'}(D^i\phi) < \infty$ in Theorem 3. Note that if $g(x) = x^t$ $(0 \leqslant t \leqslant 1)$ then $g' = g$. In particular Theorem 15.1 in [2] follows from Theorem 1 on taking $t = 0$.

EXAMPLE. If $g(x) = x^t \log(1 + x)/\log 2$, $0 \leqslant t < 1$, then $g'(x) = x^t$ $(x \geqslant 1)$. Suppose $E\|X_i\|^2 \log(1 + \|X_i\|) < \infty$, $1 \leqslant i \leqslant n$. If $\phi \in B^k$ satisfies $|\phi(x)| \leqslant A(1 + \|x\|^2)$ and $\bar{\omega}_\phi(\varepsilon : N) \leqslant c_1(k)\varepsilon$, then applying Lemmas 10 and 11 to Theorem 1 gives the rate of convergence

$$|E\phi(S_n) - E\phi(Z)| \leqslant c_2(k) A\left[(n \log n)^{-1} \Sigma_{i=1}^n E\|X_i\|^2 \log(1 + \|X_i\|)\right].$$

However, if $\phi(x) = \|x\|^2 \log(1 + \|x\|)$ for example, to the quantity in square brackets must be added the term

$$n^{-1}\Sigma_{i=1}^n \int_{\|x\| > n^{\frac{1}{2}}} \|x\|^2 \log(1 + \|x\|)\, dF_i(x).$$

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF SURREY GU2 5XH
ENGLAND