The Annals of Probability
1980, Vol. 8, No. 3, 584–599

DOMAINS OF PARTIAL ATTRACTION AND TIGHTNESS CONDITIONS[1]

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Let $X_1, X_2, \cdots$ be a sequence of independent, identically distributed, random variables with a common distribution function F. S_n denotes $X_1 + \cdots + X_n$. An increasing sequence of positive integers (n_i) is defined to belong to $\mathfrak{N}(F)$ if there exist normalizing sequences (b_k) and (a_k), with $a_k \to \infty$, so that every subsequence of $(a_{n_i}^{-1}S_{n_i} - b_{n_i})$ has a further subsequence converging in distribution to a nondegenerate limit. The main concern here is a description of $\mathfrak{N}(F)$ in terms of F. This includes also conditions for $\mathfrak{N}(F)$ to be void, as well as for $(1, 2, \cdots) \in \mathfrak{N}(F)$, thus improving on some classical results of Doeblin. It is also shown that if there exists a unique type of laws so that F is in the domain of partial attraction of a probability law if and only if the law belongs to that type, then in fact F is in the domain of attraction of these laws.

O. Introduction. Let $X_1, X_2, \cdots$ be independent, real-valued, random variables with a common distribution function F. Let

$$S_n = \sum_{i=1}^n X_i.$$

Throughout this paper (a_k) will denote a sequence of positive real numbers tending to infinity and (b_k) will denote a sequence of real numbers. (n_i) will denote a strictly increasing sequence of positive integers. Our concern is related to the following classical topic: the convergence in distribution of

$$T_i = \left(\frac{S_{n_i}}{a_{n_i}} - b_{n_i}\right).$$

Before proceeding we must fix some terminology. For any random variable Y we denote by $\mathcal{L}(Y)$ its probability distribution function (pdf). If $\mu, \mu_1, \mu_2, \cdots$ are positive measures on R^1 (the real line), each with finite total mass, we write $\mu_n \to_c \mu$ (read "μ_n converges completely to μ") if $\int g\, d\mu_n \to \int g\, d\mu$ for every bounded continuous function g on R^1. For random variables "Y_n converges to Y in distribution" means $\mathcal{L}(Y_n) \to_c \mathcal{L}(Y)$, where complete convergence of pdf's is, of course, identified with the complete convergence of the corresponding measures. A pdf having only one point of increase will be called *degenerate*. Our pdf's will be normalized to be right-continuous. A collection of pdf's is *tight* if every sequence of pdf's taken from the collection has a subsequence which converges completely.

Received September 19, 1978; revised February 1979.

[1]This work was partially supported by NSF Grant No. MCS 78-01168.

AMS 1970 *subject classification.* Primary 60G50.

Key words and phrases. Sums of independent random variables, domain of partial attraction, domain of attraction, tight sequence, set of uniform decrease.

In our work the case where F has a finite variance will be of minor interest, but when that is the case it will be convenient, and entails no loss, to assume that the mean is zero. So henceforth we assume

$$\text{(0.1)} \qquad \text{either } \int x^2\, dF = \infty \text{ or } \int x\, dF = 0.$$

Given sequences (X_i), (a_k) and (n_i), as above, we say that (a_k) is *admissible* for (n_i) if (b_k) can be found so that the corresponding sequence $\mathcal{L}(T_i)$ is tight and no subsequence converges to a degenerate distribution. Let $\mathfrak{N}(F)$ denote the class of (n_i) for which an admissible sequence exists. If F is fixed we will simply write $\mathfrak{N}$ rather than $\mathfrak{N}(F)$. Our main concern is the structure of $\mathfrak{N}$. We write (n) for $(1, 2, \cdots)$.

In the traditional terminology, if (a_k), (b_k) and (n_i) exist so that $\mathcal{L}(T_i) \to_c G$, and G is nondegenerate, F is said to be in the *domain of partial attraction* of G. For a brief summary of some classical results see [3], and for streamlined proofs consult Feller [2]. One of the great classical works on the subject is Doeblin [1]; the classical era of limit theorems is the 1930s. We will have more to say about [1] below.

Our main results appear in Section 2. These include necessary and sufficient conditions for $\mathfrak{N}(F) = \varnothing$, and for $(n) \in \mathfrak{N}$, in terms of the tails of F; the first situation is somewhat of an anomaly, the second, on the other hand, is usually encountered. We introduce a certain subclass $\mathfrak{N}_0$ of $\mathfrak{N}$ and give necessary and sufficient conditions for $(n_i) \in \mathfrak{N}_0$. We have no good necessary and sufficient conditions for $(n_i) \in \mathfrak{N}$, but it is shown that $(n_i) \in \mathfrak{N}$ if and only if there exists a positive integer k such that $(kn_i) \in \mathfrak{N}_0$. For further results the reader may consult the body of the paper. One further result, though, should be explained.

Evidently, if F is in the domain of partial attraction of G, it is also in the domain of partial attraction of any distribution of the same type as G, i.e., any distribution of the form $G(a^{-1}x + b)$, $a > 0$, b real. Also, as pointed out by Doeblin [1], G must be infinitely divisible, and if the characteristic function of G is $\exp\{\psi(t)\}$, then for every positive constant u, $\exp\{u\psi(t)\}$ is again the characteristic function of a distribution function to whose domain of partial attraction F belongs; see Theorem II of [1]. It follows at once that if all distribution functions, to whose domain of partial attraction F belongs, belong to the same type, then this must be a stable type. The question arises whether under these circumstances F must actually be in the domain of attraction of this type. Under the assumption that $(n) \in \mathfrak{N}$, it is a very simple exercise to provide an affirmative answer. That the answer is always affirmative is proved in Theorem 2.12.

At the time we started working on these problems, we knew Doeblin's famous paper [1] only second hand. When we finally turned to the original we were surprised to learn that the main part of our Proposition 1.11 was already in [1]. However, all the theorems established in Section 2 go well beyond [1], and to the best of our knowledge are new. Actually, our partial ignorance of the contents of

[1] was fortuitous because without it we might well have been discouraged from attempting further progress.

1. Preliminaries. As usual, a collection of random variables (X_{ik}), $k = 1, 2, \cdots, n_i$, $i = 1, 2, \cdots$ will be called a *triangular array* provided the random variables in each row are independent, where i is the row index. If for every $\varepsilon > 0$

$$\lim_{i\to\infty} P[|X_{ik}| > \varepsilon] = 0, \text{ uniformly in } k,$$

then the array is said to be *infinitesimal*. Let $F_{ik} = \mathcal{L}(X_{ik})$. Let $c > 0$ be fixed and define

$$\beta_{ik} = \int_{|x|\leqslant c} x \, dF_{ik}.$$

Following Feller [2], we call the array *centered* if for some $c > 0$

$$\lim_{i\to\infty} \frac{\sum_{k=1}^{n_i} \beta_{ik}^2}{\sum_{k=1}^{n_i} \int_{|x|\leqslant c} x^2 \, dF_{ik}} = 0. \tag{1.1}$$

For an infinitesimal array if relation (1.1) holds for c then if it holds for $c' > c$. As pointed out in [2], considerable simplifications can be achieved by working with centered arrays. If (X_{ik}) is any infinitesimal triangular array, we can always find constants c_{ik} so that $(X_{ik} - c_{ik})$ is a centered infinitesimal array.

Our first proposition summarizes some classical results in a convenient form.

PROPOSITION 1.1. *Suppose the infinitesimal triangular array* (X_{ik}) *is centered. There exist constants* (b_i) *such that the sequence*

$$(\textstyle\sum_{k=1}^{n_i} X_{ik} - b_i), \; i = 1, 2, \cdots \tag{1.2}$$

is tight, if and only if for every $s > 0$

$$\sup_i \textstyle\sum_{k=1}^{n_i} \left(\int_{|x|\leqslant s} x^2 \, dF_{ik} + s^2 \int_{|x|>s} dF_{ik} \right) < \infty \tag{1.3}$$

and

$$\lim_{x\to\infty} \textstyle\sum_{k=1}^{n_i} [1 - F_{ik}(x) + F_{ik}(-x)] = 0, \textit{ uniformly in } i. \tag{1.4}$$

When (1.3) *and* (1.4) *hold,* (1.2) *will be tight if and only if*

$$b_i = \textstyle\sum_{k=1}^{n_i} \beta_{ik} + 0(1).$$

It is possible to choose (b_i) *so that* (1.2) *converges in distribution if and only if*

$$\Psi_i(dx) = \sum_{k=1}^{n_i} \frac{x^2}{1 + x^2} dF_{ik} \to_c \Psi(dx), \tag{1.5}$$

where Ψ *is a nonnegative measure on* R^1 *with finite total mass.* ($\Psi \equiv 0$ *is possible*). *Then the limiting distribution will be infinitely divisible, and* Ψ *will be the Lévy-Khinchine spectral measure for the corresponding characteristic function. In particular,* $\Psi \equiv 0$ *if and only if the limit is degenerate.*

PROOF. The first part of this proposition is essentially Lemma 2 in Feller [2], IX. 7; see also the beginning of IX.9. In [2] a condition involving truncated variances appears in place of (1.3), but under the assumption that the array is centered and (1.4) the conditions are equivalent. The second part of the proposition is obtained from the well-known convergence criterion for triangular arrays. For a convenient reference see [3], Section 25, Theorem 4. Notice that the usual conditions reduce to (1.5) because of the assumption that the array is centered.

We now return to the notations and assumptions of Section 0. Let

$$X_{ik} = \frac{X_k}{a_{n_i}}, \quad k = 1, 2, \cdots, n_i; \ i = 1, 2, \cdots. \tag{1.6}$$

Since $a_k \to \infty$, (X_{ik}) is an infinitesimal triangular array, and, as remarked in [2], IX. 8, condition (0.1) is easily seen to imply that the array is centered. Clearly $F_{ik}(x) = F(a_{n_i}x)$.

For given F, we define the following functions on $(0, \infty)$:

$$L(x) = 1 - F(x) + F(-x-)$$

$$K(x) = \frac{1}{x^2}\int_{|y|\leq x} y^2 \, dF(y)$$

$$Q(x) = L(x) + K(x).$$

Observe that

$$K(x) = -L(x) + \frac{1}{x^2}\int_0^x 2yL(y)\, dy$$

so that

$$Q(x) = \frac{2}{x^2}\int_0^x yL(y)\, dy = \frac{\int_0^x yL(y)\, dy}{\int_0^x y\, dy}. \tag{1.7}$$

It follows that Q is continuous, nonincreasing, and strictly decreasing for $x > x_0 = \sup\{t : L(t) = 1\}$, and $Q(x) \to 0$ as $x \to \infty$. Note also that L is right-continuous; furthermore, we will assume throughout that

$$L(x) > 0, \qquad x > 0.$$

If this is not the case, then F must be in the domain of attraction of a Gaussian law, a situation not interesting in the present context.

PROPOSITION 1.2. *There exist constants* (b_k) *such that the sequence*

$$T_i = \left(\frac{S_{n_i}}{a_{n_i}} - b_{n_i}\right)$$

is tight if and only if the following conditions hold:

$$\sup_i n_i Q(a_{n_i}) < \infty; \tag{1.8}$$

$$\lim_{\lambda\to\infty} n_i L(\lambda a_{n_i}) = 0, \text{ uniformly in } i. \tag{1.9}$$

If (1.8) *and* (1.9) *hold, then*

$$b_k = b(a_k) \equiv \frac{1}{a_k} \int_{|x|<a_k} x \, dF \tag{1.10}$$

makes (T_i) *tight.*

If $\mathcal{L}(T_i) \to_c G$, *G will be nondegenerate if and only if*

$$\inf_i n_i Q(a_{n_i}) > 0. \tag{1.11}$$

PROOF. The first part of the proposition follows from Proposition 1.1. Note that (1.3) will be true for all $s > 0$ provided it holds for some $s > 0$, and (1.8) corresponds to (1.3) with $s = 1$.

For the final assertion of the proposition observe that $\mathcal{L}(T_i) \to_c G$ implies

$$\Psi_i(dx) \equiv n_i \frac{x^2}{1 + x^2} dF_{i1}(x) \to_c \Psi(dx)$$

with Ψ the Lévy-Khinchine measure of the characteristic function of G, and so G will be degenerate if and only if

$$\lim_{i\to\infty} n_i \int_{-\infty}^{\infty} \frac{x^2}{1 + x^2} dF_{i1}(x) = 0. \tag{1.12}$$

From the definition of Q one obtains, for any $s > 0$,

$$\frac{s^2}{1 + s^2} Q(s) \leqslant \int_{-\infty}^{\infty} \frac{x^2}{1 + x^2} dF(x) \leqslant (1 + s^2) Q(s).$$

Defining Q_{i1} in terms of F_{i1} as Q is defined in terms of F, one has $Q_{i1}(x) = Q(a_{n_i} x)$. Hence the term in (1.12) can be bounded both above and below by positive multiples of $n_i Q(a_{n_i})$, and (1.11) is seen to hold if and only if (1.12) fails.

COROLLARY 1.3. *For* (a_k) *to be admissible for* (n_i) *it is necessary and sufficient that* (1.9) *hold and that for some* $c > 1$

$$c^{-1} \leqslant n_i Q(a_{n_i}) \leqslant c. \tag{1.13}$$

REMARK 1.4. It is clear now that if (a_k) is admissible for (n_i), then the corresponding normalizing (b_k) must satisfy

$$b_{n_i} = b(a_{n_i}) + 0(1)$$

with $b(a_{n_k})$ as defined in (1.10).

We say that two sequences of positive reals (α_k) and (α'_k) are *equivalent* if there exists a positive constant $c > 1$ such that

$$c^{-1} \leqslant \alpha'_k / \alpha_k \leqslant c.$$

It is evident that if (a_k) is admissible for (n_i), a second sequence (a'_k) will also be admissible if and only if (a_{n_i}) and (a'_{n_i}) are equivalent sequences.

In what follows, the behavior of the functions L and Q will play a central role. To obtain an intuitive idea about the relation between these functions, note that $Q \geqslant L$, but if L is constant over a long interval (or varies sufficiently slowly) then

Q/L will eventually be near 1. If then L starts to drop rapidly, Q will also drop rapidly, but more slowly, so that the ratio Q/L may increase.

DEFINITION 1.5. A subset A of $[0, \infty)$ will be called a *set of uniform decrease for* L if it is bounded and

$$\lim_{\lambda\to\infty}\frac{L(\lambda x)}{L(x)} = 0, \text{ uniformly for } x \in A.$$

Substituting Q for L gives the definition of a set of uniform decrease for Q.

The definitions of L and Q imply that for $\mu \geqslant 1$, $x > 0$

$$(1.14)\qquad Q(\mu x) = \frac{1}{\mu^2}Q(x) + 2\frac{\int_x^{\mu x} yL(y)\,dy}{(\mu x)^2} \leqslant \frac{Q(x)}{\mu^2} + \left(1 - \frac{1}{\mu^2}\right)L(x).$$

It follows that if A is a set of uniform decrease for L, it is a set of uniform decrease for Q.

PROPOSITION 1.6. *If there exists an increasing, unbounded, sequence of positive reals* (x_i) *which satisfies either*

$$(1.15)\qquad \{x_i\} \text{ is a set of uniform decrease for } L,$$

or

$$(1.16)\qquad \lim_{i\to\infty}\frac{Q(x_i)}{L(x_i)} = \lim_{x\to\infty}\left(\frac{K(x_i)}{L(x_i)} + 1\right) = \infty,$$

then for any increasing sequence of positive integers (n_i) *such that*

$$(1.17)\qquad n_i \sim \frac{1}{Q(x_i)},$$

any sequence (a_k) *with* $a_{n_i} = x_i$ *is admissible.*

REMARK 1.7. The notation in (1.17) means $n_i Q(x_i) \to 1$.

PROOF. We apply Corollary 1.3 and Proposition 1.2. Only (1.9) needs verifying.

If (1.15) holds, we write

$$(1.18)\qquad n_i L(\lambda a_{n_i}) = n_i L(x_i)\frac{L(\lambda x_i)}{L(x_i)},$$

since $L \leqslant Q$, the first factor on the right is bounded in i, and then (1.15) implies (1.9).

If (1.16) holds, then by (1.17) for $\lambda \geqslant 1$

$$n_i L(\lambda a_{n_i}) \sim \frac{L(\lambda a_{n_i})}{Q(a_{n_i})} \leqslant \frac{L(x_i)}{Q(x_i)}$$

and this shows that (1.9) holds.

COROLLARY 1.8. *If there exists a set of uniform decrease for* L, *then* $\mathfrak{N} \neq \varnothing$.

COROLLARY 1.9. *If*

$$\limsup_{x\to\infty} \frac{K(x)}{L(x)} = \infty \tag{1.19}$$

then $\mathfrak{N} \neq \varnothing$.

These corollaries are not new. See the remark after Proposition 1.11. Also Lévy [4] proved that (1.19) is necessary and sufficient for F to be in the domain of partial attraction of a Gaussian distribution.

PROPOSITION 1.10. *If* $\mathfrak{N} \neq \varnothing$, *there exists an increasing, unbounded, sequence* (x_i) *such that* (1.15) *or* (1.16) *holds. Furthermore, the* x_i *can be chosen so that the* $Q(x_i)^{-1}$ *are integers.*

PROOF. Let $(n_i) \in \mathfrak{N}$, and let (a_k) be admissible for (n_i). Let $x_i = a_{n_i}$. Proceeding to a subsequence, if necessary, we may assume that

$$0 \leqslant \lim_{i\to\infty} n_i L(x_i) \equiv \lambda_o \leqslant \infty$$

exists. Since (1.13) must hold, and $L \leqslant Q$, it follows that $\lambda_o < \infty$. If $\lambda_o = 0$, (1.13) implies the condition (1.16). If $\lambda_o > 0$, the identity (1.18) together with (1.9) imply (1.15).

For the last assertion of the proposition, we replace the original a_k by

$$a'_k = \min\{t : t \geqslant a_k,\ Q(t)^{-1} \text{ is an integer}\}.$$

We must show that (a'_k) is admissible for (n_i), that is, (a_{n_i}) and (a'_{n_i}) are equivalent. As noted above, if (1.15) holds then (x_i), $x_i = a_{n_i}$, is also a set of uniform decrease for Q and the equivalence of (a_k) and (a'_k) follows at once. Also if $Q(a_{n_i})/L(a_{n_i}) \to \infty$, which is certainly the case if (1.16) holds, then writing $\mu_i = a'_{n_i}/a_{n_i} \geqslant 1$ and using (1.14) we have

$$\frac{Q(a'_{n_i})}{Q(a_{n_i})} \leqslant \frac{1}{\mu_i^2} + \left(1 - \frac{1}{\mu_i^2}\right)\frac{L(a_{n_i})}{Q(a_{n_i})}$$

which shows that (μ_i) must remain bounded because the left side approaches 1 by the definition of the a'_k. Thus (a_{n_i}) and (a'_{n_i}) are equivalent.

PROPOSITION 1.11. $\mathfrak{N} \neq \varnothing$ *if and only if there exists a sequence* (x_i) *of positive reals satisfying* (1.15) *or* (1.16). *In that case there exists* $(n_i) \in \mathfrak{N}$ *for which the sequence* $a(k)$ *defined by*

$$Q(a(k)) = \frac{1}{k}, \qquad k = 1, 2, \cdots \tag{1.20}$$

is admissible.

PROOF. Propositions 1.6 and 1.11 immediately give the first assertion. If $\mathfrak{N} \neq \varnothing$, by Proposition 1.10 there exist x_i satisfying (1.15) or (1.16) such that $Q(x_i)^{-1}$ are integers. Let $n_i = Q(x_i)^{-1}$, and $a(n_i) = x_i$. It follows from Proposition 1.6 that $(a(k))$, with $Q(a(k)) = k^{-1}$, is admissible for (n_i).

The first assertion of Proposition 1.11 was already established by Doeblin [1], Theorem VII′.

PROPOSITION 1.12. *Let* (x_n) *and* (y_n) *be two sequences of positive reals, with* $x_n \to \infty$, $x_n < y_n$, *and let* $I_n = [x_n, y_n)$. *Let*

$$R(x) = \frac{Q(x)}{L(x)},\ M_n = \inf_{x \in I_n} R(x),\ N_n = \sup_{x \in I_n} R(x).$$

The set $\{x_n\}$ *will be a set of uniform decrease for L provided one of the following two conditions holds:*

$$\sup_n \frac{y_n}{x_n} < \infty \text{ and } \lim_{n\to\infty} \frac{N_n}{R(x_n)} = \infty. \tag{1.21}$$

$$\lim_{n\to\infty} \frac{y_n}{x_n} = \infty,\ \liminf_n M_n = M_o > 1,\ \sup_n \frac{R(x_n)}{M_n} = r < \infty. \tag{1.22}$$

PROOF. Assume (1.21). By the first condition in (1.21) there exists $\mu > 0$ such that $y_n \leqslant \mu x_n$ for all n. For any $\varepsilon > 0$ there exists $z_n \in I_n$ such that

$$(N_n - \varepsilon) \leqslant \frac{Q(z_n)}{L(z_n)} \leqslant \frac{Q(x_n)}{L(z_n)}$$

since Q is decreasing. Hence for $y \geqslant \mu x_n$, $n \geqslant 1$

$$L(y) \leqslant \frac{Q(x_n)}{N_n - \varepsilon} = \frac{R(x_n)}{N_n - \varepsilon} \cdot L(x_n)$$

and by the second assumption in (1.21) it follows that $\{x_n\}$ is a set of uniform decrease for L.

We now consider (1.22). Dividing (1.14) through by $L(\mu x)$ gives

$$\frac{Q(\mu x)}{L(\mu x)} \leqslant \frac{Q(x)}{L(x)} \frac{L(x)}{L(\mu x)} \frac{1}{\mu^2} + \left(1 - \frac{1}{\mu^2}\right) \frac{L(x)}{L(\mu x)} \tag{1.23}$$

and hence, for $\mu \geqslant 1$, $x \in I_n$, $\mu x \in I_n$

$$\frac{L(x)}{L(\mu x)} \geqslant \frac{\mu^2 (R(\mu x)/R(x))}{1 + \mu^2/R(x)} \geqslant \frac{\mu^2}{1 + \mu^2/M_n} \frac{R(\mu x)}{R(x)}.$$

If $\mu^m x_n \in I_n$, where m is a positive integer, then by interation we get

$$\frac{L(x_n)}{L(\mu^m x_n)} \geqslant \left(\frac{\mu^2}{1 + \mu^2/M_n}\right)^m \frac{R(\mu^m x_n)}{R(x_n)}.$$

If (1.22) holds, then there exists $M_o' > 1$ such that for $n \geqslant n_o$

$$\frac{L(x_n)}{L(\mu^m x_n)} \geqslant \left(\frac{\mu^2}{1 + \mu^2/M_o'}\right)^m \frac{1}{r}.$$

We pick μ large so that $\mu^2(1 + \mu^2/M_o')^{-1} > 1$. This shows that $\{x_m\}$ is a set of uniform decrease for L.

2. The main results. Throughout, $(a(k))$ is the sequence defined by (1.20). Let $\mathcal{N}_o$ be the class of all sequences (n_i) for which $(a(k))$ is admissible. Further let $R(x) = Q(x)/L(x)$. Note that R is a right-continuous function on $(0, \infty)$ and has left-limits.

THEOREM 2.1. *$\mathcal{N} \neq \varnothing$ if and only if there exists a set of uniform decrease for L.*

PROOF. Applying Proposition 1.11 it suffices to check that if $R(x)$ is unbounded then there is a set of uniform decrease for L. We define

$$y_n' = \inf\{t > 0 : R(t) \geqslant n!\}$$

and

$$x_n' = \sup\{t < y_n : R(t) \leqslant (n-1)!\}.$$

Since R is right-continuous and has left-limits we can find x_n (equal to x_n' or slightly smaller) and y_n (equal to y_n' or slightly larger) such that if $I_n = [x_n, y_n)$, N_n and M_n are defined as in Proposition 1.12, then

$$N_n \geqslant n!, R(x_n) \leqslant 2M_n, \text{ and } R(x_n) \leqslant (n-1)!.$$

We now apply Proposition 1.12. Either for a subsequence n' the conditions in (1.21) hold, in which case $\{x_{n'}\}$ is a set of uniform decrease for L, or, all the conditions of (1.22) with the possible exception of $M_o > 1$ hold. If $M_o \leqslant 1$, then along a subsequence n' we have $M_{n'} \leqslant 2$, therefore $R(x_{n'}) \leqslant 4$. Since $R(x_{n'}') \geqslant (n-1)!$ and R is decreasing, we have

$$\frac{(n-1)!}{4} \leqslant \frac{R(x_{n'}')}{R(x_{n'})} \leqslant \frac{L(x_{n'})}{L(x_{n'}')}$$

and it follows that $\{x_{n'}\}$ is a set of uniform decrease for L. If $M_o > 1$, then, of course, (1.22) applies.

COROLLARY 2.2. *$\mathcal{N} = \varnothing$ if*

$$\lim_{\lambda\to\infty} \liminf_{x\to\infty} \frac{L(\lambda x)}{L(x)} > 0. \tag{2.1}$$

It was shown by Doeblin [1], Theorem VII, that (2.1) implies $\mathcal{N} = \varnothing$.

COROLLARY 2.3. *$\mathcal{N} = \varnothing$ if*

$$\lim_{x\to\infty} R(x) = 1. \tag{2.2}$$

PROOF. The equality in (1.14) gives the estimate

$$Q(\mu x) \geqslant \frac{Q(x)}{\mu^2} + \left(1 - \frac{1}{\mu^2}\right) L(\mu x)$$

and on dividing by $L(\mu x)$ we get

$$R(\mu x) \geqslant R(x) \frac{L(x)}{L(\mu x)} \frac{1}{\mu^2} + \left(1 - \frac{1}{\mu^2}\right). \tag{2.3}$$

If a set A of uniform decrease for L exists, then picking μ so that for $x \in A$

$$L(x) \geqslant 3L(\mu x)$$

and using (2.2) we obtain from (2.3)

$$R(\mu x) \geqslant 1 + \frac{1}{\mu^2}$$

for all $x \in A$ sufficiently large, which contradicts (2.2).

REMARK. It has been pointed out to us by William Pruitt that (2.2) is equivalent to the condition that L is slowly varying. So Corollary 2.3 follows from Corollary 2.2.

The following example shows that (2.1) may fail even when no set of uniform decrease for L exists. We hope it provides some intuition for the conditions.

EXAMPLE 2.4. Let r_n, $n \geqslant 1$, be a sequence of real numbers exceeding 1 and increasing to ∞. Let $a_{n,k}$, $k = 1, 2, \cdots, n$; $n = 1, 2, \cdots$, be positive real numbers such that $a_{n,k} < a_{m,j}$ if $n < m$ or $n = m$ and $k < j$, and

$$a_{n,(k+1)} \geqslant a_{n,k} r_k^k r_n, \qquad k = 1, 2, \cdots, n-1$$

$$a_{(n+1),1} \geqslant a_{n,n} r_n^n r_{n+1}.$$

The function L is defined to be constant except for jumps at the points

$$a_{n,k} r_k^i,$$

$$i = 1, 2, \cdots, k;\ k = 1, 2, \cdots, n;\ n = 1, 2, \cdots.$$

Each jump is such that the limit from the right equals one half the limit from the left. Since

$$L(\lambda a_{n,k}) \leqslant 2^{-k} L(a_{n,k}),\ k \leqslant 1;\ \lambda \geqslant r_k^k;\ n \geqslant k,$$

(2.1) fails.

We will now show that no set of uniform decrease for L may exist. Let $\{x_i\}$, $x_i \nearrow \infty$, be such a set. If for a fixed i

$$a_{n,n} \leqslant x_i < a_{(n+1),1}$$

let $n(i) = n + 1$, $k(i) = 1$; if

$$a_{n,(k-1)} \leqslant x_i < a_{n,k}, \text{ for some } k \geqslant 2,$$

let $n(i) = n$, $k(i) = k$. Observe that

$$L(\lambda x_i) \leqslant L(x_i)/4 \text{ implies } \lambda > r_{k(i)}$$

and so, if $\{x_i\}$ is a set of uniform decrease for L, there must exist a k such that $k(i) \leqslant k$ for all i. In that case, however,

$$L(\lambda x_i) < 2^{-k-1} L(x_i) \text{ implies } \lambda > r_{n(i)},$$

and since $n(i) \to \infty$, $\{x_i\}$ cannot be a set of uniform decrease for L.

Let (n) denote the sequence $1, 2 \cdots$. Our next theorem gives necessary and sufficient conditions for $(n) \in \mathfrak{N}$. Doeblin also gives necessary and sufficient conditions for $(n) \in \mathfrak{N}$ in Theorem VIII of [1], but those conditions are less transparent than (2.4) below.

THEOREM 2.5. *$(n) \in \mathfrak{N}$ if and only if*

$$\lim_{\lambda \to \infty} nL(\lambda a(n)) = 0, \text{ uniformly in } n, \tag{2.4}$$

and then $(n) \in \mathfrak{N}_o$.

PROOF. From the definition of $a(n)$, given in (1.20), and Corollary 1.3 it follows that (2.4) implies $a(n)$ is admissible for (n), i.e., $(n) \in \mathfrak{N}_o$.

Suppose, on the other hand, that (a_n) is some sequence admissible for (n). By (1.13) we may write

$$Q(a_n) = \gamma_n / n, \qquad c^{-1} < \gamma_n < c, n \geq 1.$$

Let n' be the least m such that $a_m \geq a(n)$. Then

$$\frac{\gamma_{n'-1}}{n'-1} = Q(a_{n'-1}) \geq Q(a(n)) = \frac{1}{n} \geq Q(a_{n'}) = \frac{\gamma_{n'}}{n'}$$

and so

$$2c \geq \frac{n'}{n} \geq c^{-1}.$$

Since (a_n) must satisfy (1.9)

$$0 = \lim_{\lambda \to \infty} \sup_{n'} (n'-1) L(\lambda a_{n'-1}) \geq \lim_{\lambda \to \infty} \sup_n \left(\frac{n}{c} - 1\right) L(\lambda a(n))$$

$$= \frac{1}{c} \lim_{\lambda \to \infty} \sup_n nL(\lambda a(n)).$$

This proves the theorem.

If the behavior of L is erratic, $\mathfrak{N}$ can have strange properties. This is illustrated by Example 2.6. The peculiarities are summarized in Theorem 2.7.

EXAMPLE 2.6. Let $1 = z_1' < z_2' < \cdots$; $(z_n' - z_{n-1}') \to \infty$. Let

$$L(1) = \tfrac{1}{4};\ L(x) = L(z_k'),\ z_k' \leq x < z_{k+1}';$$

$$L(z_{k+1}') = \frac{1}{k+2} L(z_k') = \frac{1}{(k+2)!2}.$$

Let $z_k = z_k' - \frac{1}{2}$, $k \geq 1$. The differences $z_{k+1} - z_k$ are chosen so big that the point z_k^* defined by $Q(z_k^*) = 2L(z_k^*)$, $z_k' < z_k^* < z_{k+1}$, satisfies $z_{k+1}' = (k^2 - 2)^{\frac{1}{2}} z_k^*$. By using our remark preceding Definition 1.5 we can pick z_k^* to satisfy the first requirement, then z_{k+1}' is defined to be $(k^2 - 2)^{\frac{1}{2}} z_k^*$. Then we have with n_k defined to be $(k-2)!2k(k^2-2)$

$$z_k^* = a((k+1)!), \qquad z_{k+1}' = a(n_k).$$

THEOREM 2.7. *If F is chosen so that L and Q satisfy the conditions of Example* 2.6, *then all the following assertions hold*:

(2.5) $(n_i) \in \mathfrak{N}$ *provided* $R(a(n_i))$ *is bounded in* i.

(2.6) $(n_i) \in \mathfrak{N}_o$ *provided* $R(a(n_i)) \to \infty$.

(2.7) *Every sequence* (n_i) *has a subsequence* $(n_i') \in \mathfrak{N}$.

(2.8) $(n) \notin \mathfrak{N}$.

(2.9) *There exists no sequence* (a_n) *admissible for every* $(n_i) \in \mathfrak{N}$.

(2.10) $\mathfrak{N}_o \neq \mathfrak{N}$.

PROOF. Observe that $\{z_k\}$ is a set of uniform decrease for L. For n given, let $a_n = z_{k+1}$ if $z_k' \leqslant a(n) < z_{k+1}'$. Assume now that $R(a(n_i)) \leqslant M$ for all i. If $a_{n_i} = z_{k+1}$, then

$$Q(a_{n_i}) = Q(z_{k+1}) \leqslant 2Q(a(n_i)) = \frac{2}{n_i}.$$

Also,

$$Q(a_{n_i}) \geqslant L(a_{n_i}) = L(a(n_i)) \geqslant \frac{1}{M} Q(a(n_i)) = \frac{1}{Mn_i}.$$

Therefore the condition (1.13) of Corollary 1.3 is satisfied. We now check (1.9).

$$n_i L(\lambda a_{n_i}) = \frac{L(\lambda a_{n_i})}{Q(a(n_i))} = \frac{L(\lambda a_{n_i})}{L(a_{n_i})} \cdot \frac{1}{R(a_{n_i})} \leqslant \frac{L(\lambda a_{n_i})}{L(a_{n_i})} \tag{2.11}$$

and $\{a_{n_i}\}$, being a subset of $\{z_k\}$, is a set of uniform decrease for L. Thus (a_n) is admissible for (n_i) by Corollary 1.3 and (2.5) holds. In our example $n_k = k!$, and $a(n_k) = z_{k-1}^*$ correspond to this situation.

In case $R(a(n_i)) \to \infty (n_k = (k-2)!2k(k^2-2), a(n_k) = z_{k+1}'$ correspond to this situation in the example), then $n_i Q(a(n_i)) = 1$ and for $\lambda > 1$

$$n_i L(\lambda a(n_i)) = \frac{L(\lambda a(n_i))}{Q(a(n_i))} \leqslant \frac{1}{R(a(n_i))}, \tag{2.12}$$

which gives (2.6). The assertion (2.7) is just a consequence of (2.5) and (2.6). Since (2.4) is clearly violated (look at the choice of n_k and $a(n_k)$ in (2.5)) (2.8) holds.

To prove (2.9), suppose (a_n) exists which is admissible for all $(n_i) \in \mathfrak{N}$. Since $(n) \notin \mathfrak{N}$, by Proposition 1.2 one of the conditions (1.8), (1.9) and (1.11) must fail for (n) and (a_n), and then along a subsequence (n_i) either $R(a(n_i))$ is bounded or tends to ∞ and one of the conditions (1.8), (1.9) and (1.11) is still violated, thus (a_n) is not admissible for some (n_i) satisfying (2.5) or (2.6), which is a contradiction. Since $a(n))$ is not admissible for the sequence $(k!)$, and $(k!) \in \mathfrak{N}$, it follows that $\mathfrak{N}_o \neq \mathfrak{N}$.

We now consider necessary and sufficient conditions for $(n_i) \in \mathfrak{N}_o$.

THEOREM 2.8. *The following conditions are equivalent.*

(2.13) $$(n_i) \in \mathfrak{N}_o.$$

(2.14) $$\lim_{\lambda\to\infty} n_i L(\lambda a(n_i)) = 0, \text{ uniformly in } i.$$

(2.15) *Either* $R(a(n_i)) \to \infty$, *or there exists* $\alpha > 0$ *such that for each* $M \geqslant \alpha$ *the set* $\{a(n_i) : R(a(n_i)) \leqslant M\}$ *is a set of uniform decrease for* L

PROOF. The equivalence of (2.13) and (2.14) follows from Corollary 1.3. Assume (2.14). If $R(a(n_i)) \nrightarrow \infty$, then the second equality in (2.11) shows that the second alternative in (2.15) holds. Hence (2.14) implies (2.15). Now assume (2.15). If $R(a(n_i)) \to \infty$ then (2.12) again shows that (2.14) holds: otherwise, for all M sufficiently large the set $\{a(n_i) : R(a(n_i)) \leqslant M\}$ is a set of uniform decrease for L. If $\{a(n_i)\}$ is a set of uniform decrease for L then (2.11) shows that (2.14) holds. For the general case, let (n_i^M) be the subsequence obtained from (n_i) by deleting those n_i for which $R(a(n_i)) > M$. Under the assumption we have $(n_i^M) \in \mathfrak{N}_o$ for all M sufficiently large. If (n_i') is a subsequence of (n_i) such that $R(a(n_i')) \to \infty$ then, by (2.12), Corollary 1.3 shows that $(n_i') \in \mathfrak{N}_o$. The assumption that $(n_i') \notin \mathfrak{N}_o$ now leads to a contradiction by the same reasoning used in the proof of (2.9).

COROLLARY 2.9. *If* $[0, \infty)$ *is a set of uniform decrease for* L, *then* $(n) \in \mathfrak{N}_o$.

PROOF. This follows from (2.11) with n in place of n_i.

COROLLARY 2.10. $(n) \in \mathfrak{N}_o$ *if*

(2.16) $$\liminf_{x\to\infty} R(x) > 1.$$

PROOF. If $R(a(n)) \to \infty$, then $(n) \in \mathfrak{N}_o$ by Theorem 2.8. Also, if

$$\limsup_n R(a(n)) < \infty$$

then Proposition 1.12 (applying (1.22)) shows that $(n) \in \mathfrak{N}_o$. Hence we may assume that for some $\delta > 0$

$$1 + \delta = \liminf_n R(a(n)) < \limsup_n R(a(n)) = \infty.$$

By Theorem 2.8 we only need to check that $\{a(n) : R(a(n)) \leqslant M\}$ is a set of uniform decrease for L, for all M sufficiently large. This is evident from (1.22) of Proposition 1.12.

THEOREM 2.11. $(n_i) \in \mathfrak{N}$ *if and only if there exists a positive integer* k *such that* $(kn_i) \in \mathfrak{N}_o$.

PROOF. Evidently if (kn_i) belongs to $\mathfrak{N}_o$ (or even just to $\mathfrak{N}$) then $(n_i) \in \mathfrak{N}$. Suppose now $(n_i) \in \mathfrak{N}$, so that there is a sequence (a_k) admissible for (n_i). Arguing as in Proposition 1.10 we can choose $a_k' \geqslant a_k$ so that $Q(a_{n_i}')$ is the reciprocal of an integer n_i', i.e., $a_{n_i}' = a(n_i')$, and (a_n') is admissible for (n_i). By Corollary 1.3 for some $c > 1$

$$c^{-1} \leqslant n_i Q(a_{n_i}') \leqslant c$$

and since $Q(a'_{n_i}) = n_i'^{-1}$, the sequences (n_i) and (n_i') are equivalent. Choose k so that $n_i' \leqslant kn_i$ for all i. Note that $(n_i') \in \mathcal{N}_o$, and (2.14) of Theorem 2.8 shows that if $n_i'' \geqslant n_i'$ and (n_i'') is equivalent to (n_i'), then $(n_i'') \in \mathcal{N}_o$.

THEOREM 2.12. *Assume that F is in the domain of partial attraction of G, and that every distribution to whose domain of partial attraction F belongs is of the same type as G. Then F is in the domain of attraction of G.*

PROOF. Suppose first that $(n) \in \mathcal{N}$. By hypothesis every sequence (n') has a further subsequence (n'') such that for suitable sequences $(a_{n''})$, $(b_{n''})$ we have

$$\mathcal{L}\left(\frac{S_{n''}}{a_{n''}} - b_{n''}\right) \to_c G.$$

Let $d(H_1, H_2)$ denote the Lévy distance between pdf's H_1 and H_2. For each n let

$$\alpha_n = \inf\left\{ d\left(\mathcal{L}\left(\frac{S_n}{a_n} - b_n\right), G\right) : a_n > 0, b_n \text{ real} \right\}$$

and pick a_n and b_n so that

$$d\left(\mathcal{L}\left(\frac{S_n}{a_n} - b_n\right), G\right) \leqslant \alpha_n + \frac{1}{n}. \tag{2.17}$$

We claim that for such a_n and b_n we have

$$\mathcal{L}\left(\frac{S_n}{a_n} - b_n\right) \to_c G.$$

If not, then along a subsequence (n') the Lévy distance $\geqslant \delta > 0$. But then along a further subsequence (n'') we have $(a^*_{n''})$ and $(b^*_{n''})$ such that

$$d\left(\mathcal{L}\left(\frac{S_{n''}}{a^*_{n''}} - b^*_{n''}\right), G\right) \to 0$$

but this shows that the distance in (2.17) along n'' tends to zero, a contradiction.

Suppose then that $(n) \notin \mathcal{N}$. Now we use the fact (see the introduction) that the hypotheses imply that G is a stable law. Consider first the case where the index α of G is < 2. The Lévy-Klinchine spectral measure corresponding to G then takes a known form (see [3], for example) and on applying the convergence criterion (e.g., Proposition 1.1) we see that if

$$\mathcal{L}\left(\frac{S_{n_i}}{a_{n_i}} - b_{n_i}\right) \to_c G \tag{2.18}$$

then there is a positive constant c_o such that

$$\lim_{i\to\infty} n_i L(a_{n_i} x) = c_o x^{-\alpha}, \text{ uniformly for } x \geqslant \varepsilon > 0. \tag{2.19}$$

For $x > 0$ let

$$\begin{aligned} g_x(y) &= x^{-2}(1 + y^2), & y \leqslant x \\ &= y^{-2}(1 + y^2), & y > x, \end{aligned}$$

then g_x is a bounded continuous function and

$$Q(ax) = \int_{-\infty}^{\infty} g_{ax}(y) y^2 (1 + y^2)^{-1}\, dF(y) = \int_{-\infty}^{\infty} g_x(y) y^2 (1 + y^2)^{-1}\, dF(ay).$$

As a consequence of (2.18) we also have

$$\Psi_i(dy) = n_i y^2 (1 + y^2)^{-1}\, dF(a_{n_i} y) \to_c \Psi(dx);$$

hence

$$\int_{-\infty}^{\infty} g_x(y) \Psi_i(dy) = n_i Q(a_{n_i} x) \to \int_{-\infty}^{\infty} g_x(y) \Psi(dy). \tag{2.20}$$

Since Ψ is the Lévy-Khinchine measure of the limit stable law, (2.19) and (2.20) give

$$\lim_{i\to\infty} R(a_{n_i} x) = 2(2 - \alpha)^{-1}, \qquad x > 0. \tag{2.21}$$

Furthermore, Q and L decrease to zero, and we actually have for each $\varepsilon > 0$

$$\lim_{i\to\infty} \sup_{\varepsilon \leqslant x \leqslant \varepsilon^{-1}} |R(a_{n_i} x) - 2(2 - \alpha)^{-1}| = 0. \tag{2.22}$$

Since $\alpha < 2$, we have $\limsup_{x\to\infty} R(x) < \infty$. (See the remark after Corollary 1.9). Also $(n) \notin \mathfrak{N}$ by hypothesis, hence by Corollary 2.10 we must have

$$\liminf_{x\to\infty} R(x) = 1. \tag{2.23}$$

Let β satisfy $1 < \beta < 2(2 - \alpha)^{-1}$, and define

$$z_i = \sup\{t : t < a_{n_i} \text{ and } R(t) \leqslant \beta\}.$$

We now apply (1.22) of Proposition 1.12 with $[z_i, a_{n_i})$, $i \geqslant 1$, playing the role of the intervals $[x_n, y_n)$, $n \geqslant 1$, there. Clearly the second and the third conditions of (1.22) are fulfilled. Although $R(z_i) \geqslant \beta$, there exist z_i' arbitrarily close to the left of z_i such that $R(z_i') \leqslant \beta$. Dropping to a subsequence, if necessary, (2.22) shows that $a_{n_i}/z_i' \to \infty$, hence $a_{n_i}/z_i \to \infty$. We thus conclude that $\{z_i\}$ contains a set $\{\bar{z}_i\}$ of uniform decrease for L. If $\{\bar{z}_i\}$ is a set of uniform decrease, then so is $\{\bar{z}_i - \varepsilon_i\}$, $\varepsilon_i \to 0$, where ε_i may be chosen so that $R(\bar{z}_i - \varepsilon_i) \leqslant \beta$. We thus have a set $\{x_i\}$ of uniform decrease for L such that

$$R(x_i) \leqslant \beta, \qquad i \geqslant 1. \tag{2.24}$$

By Proposition 1.6 we can find a sequence (n_i') so that any sequence (a_k') with $a_{n_i}' = x_i$ is admissible. So one may assume (proceeding to a further subsequence, if necessary) that there exist b_k' such that

$$\mathfrak{L}\left(\frac{S_{n_i'}}{a_{n_i'}'} - b_{n_i'}\right) \to_c G',$$

where G' is, by assumption, stable of index α. So, as in (2.21), one must have

$$\lim_{i\to\infty} R(a_{n_i'}') = \frac{2}{2 - \alpha},$$

which contradicts (2.24). Therefore we must have $(n) \in \mathfrak{N}$, which in turn implies that F is in the domain of attraction of G, as we have already shown.

It remains to consider the case $\alpha = 2$, which by Lévy's theorem implies $\limsup_{x\to\infty} R(x) = \infty$. Again assume $(n) \notin \mathfrak{N}$. If $\lim R(x) = \infty$, then Corollary 2.9 shows that $(n) \in \mathfrak{N}$, a contradiction to the assumption $(n) \notin \mathfrak{N}$. Therefore we may assume

$$\liminf_{x\to\infty} R(x) < \infty, \qquad \limsup_{x\to\infty} R(x) = \infty. \tag{2.25}$$

One checks that (2.22) again holds with $\alpha = 2$. Using an argument similar to the one given above (via Proposition 1.12) one finds a set $\{x_i\}$ of uniform decrease for L such that $R(z_i) \leqslant \theta < \infty$, and obtains a contradiction as before.

REFERENCES

[1] Doeblin, W. (1940). Sur l'ensemble de puissances d'une loi de probabilité. *Studia Math.* **9** 71–96.

[2] Feller, W. (1966). *An Introduction to Probability Theory, Vol. II*. Wiley, New York.

[3] Gnedenko, B. V. and Kolmogorov, A. N. (1954). *Limit Distributions for Sums of Independent Random Variables*. Addison-Wesley, Cambridge, Mass.

[4] Lévy, P. (1937). *Theorie de l'Addition des Variables Aléatoires*. Gauthier-Villars, Paris.

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