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CLASSIFICATION OF COHARMONIC AND COINVARIANT FUNCTIONS FOR A LÉVY PROCESS[1]

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Excursion theory is applied to get identities for continuous time ladder variables. The identities are used to classify coharmonic and coinvariant functions for one dimensional Lévy processes.

1. Introduction. Let $\mathbf{X} = \{X_t, t \geqslant 0\}$ be a real valued Lévy-Khinchin process. This means that $\mathbf{X}$ has stationary independent increments and right continuous paths with left-hand limits. Let $\{\mathcal{P}_x, x \in \mathbf{R}\}$ be the associated sample space probabilities. A positive function $h(y)$ is *coinvariant* on $(0, \infty)$ if, for all $y > 0$,

$$h(y) = \mathcal{E}_{-y} I(t < \sigma^*) h(X_t^*). \tag{1.1}$$

Here $\mathbf{X}^* = \{X_t^*, t \geqslant 0\}$ is the dual process

$$X_t^* = -X_t$$

and σ, σ^* are the exit times

$$\sigma = \inf\{t > 0 : X_t \leqslant 0\}; \qquad \sigma^* = \inf\{t > 0 : X_t^* \leqslant 0\}. \tag{1.2}$$

A positive function h is *coharmonic* on $(0, \infty)$ if, for all $y > 0$,

$$h(y) = \mathcal{E}_{-y} I(T_M^* < \sigma^*) h[X^*(T_M^*)]. \tag{1.3}$$

Here T_M, T_M^* are the hitting times

$$T_M = \inf\{t > 0 : X_t \in M\}; \qquad T_M^* = \inf\{t > 0 : X_t^* \in M\}. \tag{1.4}$$

In (1.4) the set M can be any subset of the half line $(0, \infty)$ whose complement $(0, \infty) \setminus M$ is open and has compact closure contained in the open half line $(0, \infty)$. It follows from general principles that every positive coinvariant function is also coharmonic. In this paper we classify positive functions which are coharmonic on $(0, +\infty)$ and we identify those which are also coinvariant. Our results are valid only if we postulate.

Absolute continuity condition (ACC). For each $\alpha > 0$ there is an integrable function u_α such that, for Borel $f \geqslant 0$,

$$\mathcal{E}_x \int_0^\infty dt\, e^{-\alpha t} f(X_t) = \int dy\, u_\alpha(y) f(x + y).$$

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We take ACC for granted throughout the paper. If ACC were not true then we would have to formulate everything in terms of measures rather than functions.

To avoid consideration of uninteresting cases we assume throughout the paper that **X** is not a subordinator and also that it is not compound Poisson (with or without drift).

We often use without comment the standard terminology for Markov processes as set out in Blumenthal and Getoor (1968). When making explicit references to that book we will use the abbreviation B/G.

Unless explicitly stated otherwise the terms excessive and coexcessive will be understood to be relative to the absorbed process $\mathbf{X}^0$ defined by

$$(1.5)\qquad \begin{aligned} X_t^0 &= X_t \qquad \text{if} \quad t < \sigma \\ &= +\infty \qquad \text{if} \quad t \geqslant \sigma. \end{aligned}$$

The phrases "0 is regular for $(0, +\infty)$" etc., unless explicitly qualified, will be understood to be relative to the original process **X**.

Before giving the classification we need to introduce some machinery. The supremum functionals M_t, the dual supremum functionals M_t^*, the ladder variables L_t, and the dual ladder variables L_t^* are defined by

$$(1.6)\qquad \begin{aligned} &M_t = \sup\{X_s : 0 \leqslant s \leqslant t\}; \qquad M_t^* = \sup\{X_s^* : 0 \leqslant s \leqslant t\} \\ &L_t = \sup\{s \leqslant t : \max(X_s, X_{s-}) = M_t\}; \\ &L_t^* = \sup\{s \leqslant t : \max(X_s^*, X_{s-}^*) = M_t^*\}. \end{aligned}$$

By convention $X_{0-} = X_0$ and $X_{0-}^* = X_0^*$. The reflected process $\mathbf{Y} = \{Y_t, t \geqslant 0\}$ and dual reflected process $\mathbf{Y}^* = \{Y_t^*, t \geqslant 0\}$ are defined by

$$(1.7)\qquad Y_t = M_t^* + X_t: \qquad Y_t^* = M_t + X_t^*.$$

These processes are Markovian with stationary transition probabilities. Indeed the transition probabilities for **Y** are given explicitly by

$$(1.7')\qquad \begin{aligned} \mathcal{E}_0(f(Y_{t+s}) | Y_s = y) &= \mathcal{E}_0 f[X_t + y + (M_t^* - y)^+] \\ &= \mathcal{E}_0 f[X_t + \max(y, M_t^*)]. \end{aligned}$$

Thus **Y** is Feller and agrees with **X** up to time σ. Notice that the processes defined by (1.7) start at 0, independent of the starting position of **X**. Let $\mathbf{A} = \{A_t, t \geqslant 0\}$ be local time at 0 for the reflected process **Y** and let $\mathbf{B} = \{B_\tau, \tau \geqslant 0\}$ be the inverse process defined by

$$B_\tau = \inf\{t > 0 : A_t > \tau\}$$

with the understanding that $B_\tau = +\infty$ for $\tau \geqslant A_{+\infty}$. If 0 is regular for $(-\infty, 0)$ then 0 is regular for $\{0\}$ for **Y** and **A** is defined up to a multiplicative constant as in Section 1 of Blumenthal and Getoor (1964). (We will fix this constant in Section 2.) If 0 is not regular for $(-\infty, 0)$ then the zeroes of **Y** form a discrete set and we let **A** correspond to counting measure for that set. Then $B_0, B_1, \cdots$ are the successive members of that set. The processes $\mathbf{A}^*$, $\mathbf{B}^*$ are defined by analogy with $\mathbf{Y}^*$ playing the role of **Y**.

We will see in Section 3 that if 0 is regular for $(0, +\infty)$ then ACC guarantees the existence of a unique coexcessive function $\psi(y)$ for $\mathbf{X}^0$ such that

$$\int_0^\infty dy\, \psi(y) f(y) = \mathcal{E}_0 \int_0^\infty d\tau\, f(M \circ B_\tau^*) = \mathcal{E}_0 \int_0^\infty dA_t^*\, f(M_t) \tag{1.8}$$

for Borel $f \geqslant 0$ on $(0, +\infty)$. Our convention will be that $M_\infty = +\infty$ and $f(+\infty) = 0$ so that (1.8) is still true if $A_\infty < +\infty$. If 0 is not regular for $(0, +\infty)$ then we can still define $\psi(y)$ by

$$f(0) + \int_0^\infty dy\, \psi(y) f(y) = \mathcal{E}_0 \int_0^\infty d\tau\, f(M \circ B_\tau^*) = \mathcal{E}_0 \int_{0-}^\infty dA_t^* f(M_t) \tag{1.9}$$

where now f is defined on $[0, +\infty)$. The function ψ^* is defined by analogy.

Our first result is

THEOREM 1. (i) *If* 0 *is regular for* $(0, +\infty)$ *then* $\psi(y)$ *defined by* (1.8) *is coharmonic on* $(0, +\infty)$.

(ii) *If* 0 *is not regular for* $(0, +\infty)$ *then* $\psi(y)$ *defined by* (1.9) *is not coharmonic on* $(0, +\infty)$. *Indeed it does not dominate any nontrivial positive coharamonic function on* $(0, +\infty)$.

At the end of Section 10 we will use Theorem 7, below, to supplement Theorem 1(ii) by showing that if 0 is not regular for $(0, +\infty)$ then $\psi(y)$ can be represented

$$\psi(y) = \int_0^\infty \pi(dx) G(x, y) \tag{1.10}$$

where $G(x, y)$ is the Green's function defined by (1.16) and (1.17) below and π is the Lévy measure defined by (1.24) below.

Once we have proved Theorem 1 it will be clear that $\psi(y)$ is bounded and coharmonic if and only if "continuous passage to the right" is possible for $\mathbf{X}$. Thus we can view Theorem 1 as supplementing the results in Section 2 of Millar (1973) where it is shown that this is the case only if 0 is regular for $(0, +\infty)$.

The function $\psi(y)$ is naturally associated with "entrance from 0" and it is never coinvariant. However, there may be a coharmonic function associated with "entrance at $+\infty$" and this one is coinvariant. In stating results for this function we distinguish two cases, depending on whether or not $\mathbf{X}$ always returns to the left half line $(-\infty, 0)$; that is, depending on whether or not

$$\mathcal{P}_x(\sigma < \infty) = 1 \qquad \text{for} \quad x > 0. \tag{1.11}$$

THEOREM 2. *Assume that* (1.11) *is true and define*

$$g(x) = \int_0^x dy\, \psi(y). \tag{1.2}$$

(i) *If* 0 *is regular for* $(0, +\infty)$, *then* $g(x)$ *is coinvariant on* $(0, +\infty)$ *and, therefore, coharmonic on* $(0, +\infty)$.

(ii) *If* 0 *is not regular for* $(0, +\infty)$ *then conclusion* (i) *is true with* $g(x)$ *replaced by* $1 + g(x)$.

Theorem 2 is an analogue for P5 on page 212 in Spitzer (1964) and our proof amounts to a translation into continuous time of the argument given by Spitzer.

Theorem 2 is certainly applicable when $\mathbf{X}$ is recurrent and then one would suspect that $g(x)$ or $1 + g(x)$ corresponds somehow to the function $L_B(x)$ as defined in Port and Stone (1971) with $B = (-\infty, 0]$. However, we do not see how to make the connection precise since Port and Stone assume that B is relatively compact in their Theorems 23.2 and 23.3.

If (1.11) fails then there is a nontrivial positive coinvariant function on $(0, +\infty)$ if and only if there exists $q > 0$ such that

$$\mathcal{E}_0 e^{-qX_t} = 1 \qquad \text{for} \quad t \geqslant 0. \tag{1.13}$$

The precise positive result is

THEOREM 3. *Assume that* (1.11) *fails and that* (1.13) *is true for some* $q > 0$.

(i) *If* 0 *is regular for* $(0, +\infty)$, *then*

$$g_q(x) = \int_0^x dy\, e^{q(x-y)} \psi(y) \tag{1.14}$$

is coinvariant on $(0, +\infty)$ *and, therefore, coharmonic on* $(0, +\infty)$.

(ii) *If* 0 *is not regular for* $(0, +\infty)$ *then conclusion* (i) *is valid with* $g_q(x)$ *replaced by* $e^{qx} + g_q(x)$.

It is easy to see that (1.13) can be true only if (1.11) fails. There do exist examples for which both (1.11) and (1.13) fail and then there is no positive coinvariant function on $(0, +\infty)$.

We finish the classification in

THEOREM 4. *Every positive coharmonic function on* $(0, +\infty)$ *is a linear combination of the ones identified in Theorems* 1, 2 *and* 3.

The special case $\lambda_1 = 0$ in Theorem 9.1 in Fristedt (1974) gives the formula

$$\int_0^\infty dy\, \psi(y) e^{-\lambda y} = \exp \int_0^\infty \frac{dt}{t} \mathcal{E}_0 I(X_t > 0)\{e^{-\lambda X_t} - e^{-t}\}. \tag{1.15}$$

(In Section 2 we will normalize the local time $\mathbf{A}$ to eliminate a multiplicative constant.) For the strictly stable case Fristedt shows that one can explicitly invert (1.15) and identify $\psi(y)$ as an appropriate power function. For example, if $\mathbf{X}$ is symmetric stable with index α, then $\psi(y) = (\text{const.}) y^{\frac{1}{2}\alpha - 1}$. In order to show that $\psi(y)$ is coharmonic we supplement Fristedt's formula with a second formula, apparently new, which identifies $\psi(y)$ as the sojourn density for "typical excursions from a minimum." In Section 2 we will derive both formulae using excursion theory, and then in Section 3 we will prove Theorem 1. (An independent proof of Fristedt's formula can also be found in Greenwood and Pitman (1978).)

Our proof of Theorem 2 depends on establishing an analogue to the result of T. Harris (1956) concerning the existence of an invariant measure for any recurrent discrete (time and space) chain. The analogue is established in Section 4 and Theorem 2 itself is proved in Section 5.

Our proof of Theorem 3 depends on a different technique and it is given in Section 7. At the same time we will prove the following result which is perhaps more fundamental.

THEOREM 5. *If* (1.11) *fails then the following are equivalent for fixed* $q > 0$.

(i) e^{qx} *is coharmonic for* $\mathbf{X}$.

(ii) e^{qx} *is coinvariant for* $\mathbf{X}$.

(iii) e^{qx} *is coharmonic for* $-\mathbf{M}^* \circ \mathbf{B}$.

(iv) e^{qx} *is coinvariant for* $-\mathbf{M}^* \circ \mathbf{B}$.

If 0 is regular for $(-\infty, 0)$ then the process $\mathbf{M}^* \circ \mathbf{B} = \{M^* \circ B_\tau, \tau \geqslant 0\}$ is a subordinator (see Fristedt (1974)). If 0 is not regular for $(-\infty, 0)$ then we use an integral time scale for $\mathbf{M}^* \circ \mathbf{B}$ so that actually $\mathbf{M}^* \circ \mathbf{B} = \{M^*(B_j)\}_{j=0}^{\infty}$, a random walk. In either case conditions (iii) and (iv) in Theorem 5 make sense. However in the random walk case (iii) and (iv) are equivalent from general principles.

The *Green's operator* G is defined by

$$Gf(x) = \mathcal{E}_x \int_0^\sigma dt\, f(X_t) \tag{1.16}$$

for $x \geqslant 0$ and Borel $f \geqslant 0$ on $(0, +\infty)$. Condition ACC immediately implies, for the absorbed process $\mathbf{X}^0$, conditions (a) and (b) in Theorem 1.4 on page 254 in B/G, and from this theorem we easily deduce the existence of a unique *Green's function* $G(x, y)$ which is excessive in x and coexcessive in y and such that

$$Gf(x) = \int_0^\infty dy\, G(x, y) f(y). \tag{1.17}$$

In Section 3 we will establish the following formulae for $G(x, y)$.

THEOREM 6. (i) *If* 0 *is regular for both* $(0, +\infty)$ *and* $(-\infty, 0)$, *then*

$$G(x, y) = \int_0^{x \wedge y} du\, \psi^*(x - u)\psi(y - u). \tag{1.18}$$

(ii) *If* 0 *is regular for* $(0, +\infty)$ *but not for* $(-\infty, 0)$, *then*

$$G(x, y) = \int_0^{x \wedge y} du\, \psi^*(x - u)\, \psi(y - u) + \psi(y - x). \tag{1.19}$$

(iii) *If* 0 *is regular for* $(-\infty, 0)$ *but not for* $(0, +\infty)$, *then*

$$G(x, y) = \int_0^{x \wedge y} du\, \psi^*(x - u)\psi(y - u) + \psi^*(x - y). \tag{1.20}$$

Of course $\psi^*(x)$ and $\psi(y)$ are understood in Theorem 6 to be represented by the versions specified in Theorem 1 (and its dual).

Theorem 6 is an analogue to the formula on page 209 in Spitzer (1964) and again our proof amounts to a translation of his argument.

If $\mathbf{X}$ is symmetric stable with index α then (1.18) becomes

$$G(x, y) = (\text{const.}) \int_0^{x \wedge y} du (x - u)^{\frac{1}{2}\alpha - 1} (y - u)^{\frac{1}{2}\alpha - 1} \tag{1.21}$$

which could also be obtained from Corollary 4 in Blumenthal, Getoor and Ray (1961) after scaling and a passage to the limit. If X is completely asymmetric stable with index $\alpha > 1$ and no jumps downward, then

$$(1.22)\qquad \begin{aligned} G(x,y) &= (\text{const.})y^{\alpha-1} && \text{if } 0<y<x \\ &= (\text{const.})\big(y^{\alpha-1} - (y-x)^{\alpha-1}\big) && \text{if } 0<x\leqslant y. \end{aligned}$$

This could also be obtained from the formula on page 43 in Port (1970) after passage to the limit $b\uparrow\infty$.

Theorem 4 will be proved in Section 8 by combining Theorem 6 with the general theory of the Martin boundary.

Some of our results can be formulated directly in terms of integrodifferential equations. It is well known that the characteristic function for $\mathbf{X}$ can be written

$$(1.23)\qquad \mathcal{E}_0 e^{i\xi X_t} = e^{-t\Psi(\xi)}$$

where the *exponent* $\Psi(\xi)$ has a representation

$$(1.24)\qquad \Psi(\xi) = a\xi^2 - ib\xi + \int \pi(dy)\{1 - e^{i\xi y} + i\xi \sin y\}$$

where $a \geqslant 0$, where b is real, and where π is a measure on $\mathbf{R}$ which does not charge $\{0\}$, which has a finite second moment near 0, and which is bounded in the complement of any neighborhood of 0. The *local generator* $\mathcal{Q}$ is defined on $\varphi \in C^\infty_{\text{com}}(0, +\infty)$ (the collection of C^∞ functions with compact support in $(0, +\infty)$) by

$$(1.25)\qquad \mathcal{Q}\varphi(y) = a\varphi''(y) + b\varphi'(y) + \int \pi(dz)\{\varphi(y+z) - \varphi(y) - \varphi'(y)\sin z\}.$$

If $h \geqslant 0$ is coexcessive and locally integrable on $(0, +\infty)$ then Theorem 2.11 on page 272 in B/G guarantees that h has a unique representation

$$(1.26)\qquad h(y) = h_0(y) + \int \nu(dx)G(x,y)$$

where h_0 is coharmonic and ν is a Radon measure on $(0, +\infty)$. In Section 10 we will prove

THEOREM 7. *Let $h(x)$ be coexcessive and locally integrable on* $(0, +\infty)$ *with representation* (1.26). *Then for any* $\varphi \in C^\infty_{\text{com}}(0, +\infty)$ *the product* $h(x)\mathcal{Q}\varphi(x)$ *is integrable on* $(0, +\infty)$ *and*

$$(1.27)\qquad \int_0^\infty dy\, h(y)\mathcal{Q}\varphi(y) = -\int \nu(dy)\varphi(y).$$

Conversely let $h \geqslant 0$ on $(0, +\infty)$ *be locally integrable on* $(0, +\infty)$ *and suppose that for all* $\varphi \in C^\infty_{\text{com}}(0, +\infty)$ *the product* $h(y)\mathcal{Q}\varphi(y)$ *is integrable on* $(0, +\infty)$ *and* (1.27) *is valid with ν a Radon measure on* $(0, +\infty)$. *Then h has a version which is coexcessive and this version has the representation* (1.26).

Of course (1.27) is just the statement that $\mathcal{Q}^*h = -\nu$ on $(0, +\infty)$ in the distribution sense.

In Section 9 we use results for the half line $(0, +\infty)$ to classify coharmonic functions for a bounded interval.

Some preliminaries for approximate Markov processes are collected in Section 6.

2. Some identities. In this section we establish some formulae which are continuous time versions for well-known identities involving discrete time random walks. Some of the formulae were first discovered by Fristedt (1974), but others seem to be new. A special case of the formulae will be used in Section 3 to prove Theorems 1 and 6. It turns out that a different special case plays an important role in Monrad and Silverstein (1979).

Condition ACC could easily be suppressed in this section.

We begin by introducing machinery to treat excursions from 0 for the reflected process $\mathbf{Y}$ defined by (1.7). Let $\mathfrak{D}$ be the random open set

$$\mathfrak{D} = \{t > 0 : Y_t \neq 0 \quad \text{and} \quad Y_{t-0} \neq 0\}$$

and let $\mathfrak{L}$ be the set of left-hand endpoints of the components of $\mathfrak{D}$. For $i \in \mathfrak{L}$ let r^i denote the corresponding right-hand endpoint and let $\mathbf{w}^i = \{w_t^i, t \geqslant 0\}$ be the excursion from 0 starting at i for $\mathbf{Y}$. That is,

$$(2.1) \qquad \begin{aligned} w_t^i &= Y_{t+i} && \text{for} \quad 0 \leqslant t < r^i - i \\ &= \infty && \text{for} \quad t \geqslant r^i - i. \end{aligned}$$

Consider first the case when 0 is regular for $(0, +\infty)$ but not $(-\infty, 0)$. Then $\mathfrak{L}$ can be represented

$$(2.2) \qquad \mathfrak{L} = \{B_j\}_{j=0}^{\infty}$$

and it follows easily from the strong Markov property for $\mathbf{X}$ or $\mathbf{Y}$ that

$$(2.3) \quad \mathcal{E}_0 \Sigma_{i \in \mathfrak{L}} e^{-\beta i} e^{-\mu M_i^*} \xi(\mathbf{w}^i) = \mathcal{E}_0 \Sigma_{j=0}^{\infty} I(B_j < +\infty) e^{-\beta B_j} e^{-\mu M^*(B_j)} \mathcal{E}_0 \xi(\mathbf{X}^0)$$

for $\beta, \mu \geqslant 0$ and $\xi \geqslant 0$ any measurable function of a typical excursion $\mathbf{w}^i$. Here $\mathbf{X}^0 = \{X_t^0, t \geqslant 0\}$ is the absorbed process on $(0, +\infty)$ defined by (1.5). Alternatively

$$(2.4) \qquad \mathcal{E}_0 \Sigma_{i \in \mathfrak{L}} e^{-\beta i} e^{-\mu M_i^*} \xi(\mathbf{w}^i) = \mathcal{E}_0 \textstyle\int_{0-}^{\infty} dA_t e^{-\beta t} e^{-\mu M_t^*} \mathcal{E}_0 \xi(\mathbf{X}^0).$$

To make further progress we need an analogue for this formula which is valid when 0 is regular for $(-\infty, 0)$. Of course (2.4) itself cannot be correct since the second factor on the right collapses. A quick way to get a correct formula is to apply a slight extension of Theorem 4.1 in Maisonneuve (1975).

Assume that 0 is regular for $(-\infty, 0)$ for the original process $\mathbf{X}$ which means that 0 is regular for $\{0\}$ for the reflected process $\mathbf{Y}$ and so $\mathbf{A}$, local time at 0 for $\mathbf{Y}$, is well defined by the results of Blumenthal and Getoor (1964). In Maisonneuve's theorem the role of X_t is played by the reflected process $\mathbf{Y}$. The role of the closed random set M is played by $\{t \geqslant 0 : Y_t = 0 \text{ or } Y_{t-0} = 0\}$. The role of $\mathcal{F}_t$ there is played by the complete sigma-algebra generated by the coordinates $\{\mathbf{X}_s, s \leqslant t\}$. (Not the smaller sigma-algebra generated by $\{Y_s, s \leqslant t\}$. Thus we are actually using a slight extension of Maisonneuve's theorem. But this extension is obvious and we take it for granted.) By the uniqueness result on page 217 in B/G we can normalize so that the continuous additive functional K in Maisonneuve's Theorem 4.1 is identical with $\mathbf{A}$. We conclude then that there exists a uniquely determined measure $\mathcal{P}^{ex}$ with associated functional $\mathcal{E}^{ex}$, well defined on the sigma-algebra generated by the reflected process $\mathbf{Y}$ (and in particular on the sigma-algebra generated by the absorbed process $\mathbf{X}^0$) such that

$$\mathcal{E}_0 \Sigma_{i \in \mathcal{E}} e^{-\beta i} e^{-\mu M_i^*} \xi(\mathbf{w}^i) = \mathcal{E}_0 \int_0^\infty dA_t e^{-\beta t} e^{-\mu M_t^*} \mathcal{E}^{ex} \xi(\mathbf{X}^0). \tag{2.5}$$

This is the desired analogue to (2.4).

If $\beta, \mu \geqslant 0$, and if we rule out the case $\beta = \mu = 0$, then also

$$\mathcal{E}_0 \Sigma_0^\infty dA_t e^{-\beta t} e^{-\mu M_t^*} \mathcal{E}^{ex} \xi(\mathbf{X}^0) = \mathcal{E}_0 \int_0^\infty d\tau\, e^{-\beta B_\tau} e^{-\mu M^* \circ B_\tau} \mathcal{E}^{ex} \xi(\mathbf{X}^0). \tag{2.6}$$

(Recall our convention in Section 1 that $M_\infty^* = +\infty$.) We will assume always that $\beta, \mu \geqslant 0$ and we rule out the case $\beta = \mu = 0$. A similar convention will hold for the pair α, λ introduced below.

Next we collect some elementary identities which can be established by reversing the path from 0 to t exactly as in Chapter III of Feller (1950). To keep the notation as simple as possible we assume the underlying sample space Ω is the set of all right continuous trajectories from $[0, \infty)$ to $\mathbf{R}$. Then it is clear that for $t > 0$ there is a uniquely defined reversal operator ρ_t on Ω determined by the relations

$$\begin{aligned} X_s(\rho_t\omega) &= X_{t-0}(\omega) - X_{t-s-0}(\omega) \qquad \text{for} \quad 0 \leqslant s < t \\ X_t(\rho_t\omega) &= X_t(\omega). \end{aligned} \tag{2.7}$$

Certainly ρ_t preserves the probability $\mathcal{P}_0$ on the sigma-algebra generated by X_s, $s \leqslant t$. Also it is easy to check that at least if $X_0 = 0$ and $X_t = X_{t-0}$ then

$$M_t(\rho_t\omega) = M_t^*(\omega) + X_t(\omega) \tag{2.8}$$

$$(M_t - X_t)(\rho_t\omega) = M_t^*(\omega) \tag{2.9}$$

$$L_t(\rho_t\omega) = t - \inf\{s \leqslant t : \max(-X_s(\omega), -X_{s-0}(\omega)) = M_t^*(\omega)\}. \tag{2.10}$$

It follows that for fixed $t > 0$ these identities are valid for almost every ω relative to $\mathcal{P}_0$.

We continue to assume that 0 is regular for $(-\infty, 0)$. Following Fristedt (1974) we introduce the Laplace exponent $\varphi^*(\beta, \mu)$ by

$$\mathcal{E}_0 e^{-\beta B_\tau - \mu M^* \circ B_\tau} = e^{-\tau\varphi^*(\beta, \mu)}. \tag{2.11}$$

If 0 is not regular for $(0, +\infty)$, then with probability one the set of zeroes for $\mathbf{Y}$ has positive Lebesgue measure. This is because (2.8) guarantees

$$\mathcal{P}_0(Y_t = 0) = \mathcal{P}_0(M_t = 0). \tag{2.12}$$

Therefore we can normalize the local time $\mathbf{A}$ by

$$A_t = \int_0^t I(Y_s = 0) ds. \tag{2.13}$$

Taking

$$\xi(\mathbf{w}^i) = \int_0^{r^i - i} dt\, e^{-\alpha t} e^{-\lambda w_t^i}$$

we get

$$(2.14)\quad \mathcal{E}_0\int_0^\infty dt\, e^{-\beta L_t^*}e^{-\mu M_t^*}e^{-\alpha(t-L_t^*)}e^{-\lambda(X_t+M_t^*)} = \mathcal{E}_0\int_0^\infty dA_t e^{-\beta t-\mu M_t^*} + \mathcal{E}_0\Sigma_{i\in\mathfrak{L}}e^{-\beta i}e^{-\mu M_i^*}\xi(\mathbf{w}^i)$$

which combines with (2.5), (2.6) and (2.11) to give

$$(2.15)\quad \mathcal{E}_0\int_0^\infty dt\, e^{-\beta L_t^*}e^{-\mu M_t^*}e^{-\alpha(t-L_t^*)}e^{-\lambda(X_t+M_t^*)} = \varphi^*(\beta,\mu)^{-1}\left\{1+\mathcal{E}^{\text{ex}}\int_0^\sigma dt\, e^{-\alpha t}e^{-\lambda X_t^0}\right\}$$

valid if 0 is regular for $(-\infty, 0)$ but not for $(0, +\infty)$. If 0 is regular also for $(0, +\infty)$ then (2.12) implies that the set of zeroes for $\mathbf{Y}$ has Lebesgue measure 0 and the first term on the right in (2.14) should be omitted so that (2.15) is replaced by

$$(2.16)\quad \mathcal{E}_0\int_0^\infty dt\, e^{-\beta L_t^*}e^{-\mu M_t^*}e^{-\alpha(t-L_t^*)}e^{-\lambda(X_t+M_t^*)} = \varphi^*(\beta,\mu)^{-1}\mathcal{E}^{\text{ex}}\int_0^\sigma dt\, e^{-\alpha t}e^{-\lambda X_t^0}$$

valid if 0 is regular both for $(-\infty, 0)$ and $(0, +\infty)$.

Justifying (2.15) and (2.16) was the main step in our argument. The rest of this section depends on techniques which are standard for establishing identities of the kind considered here. (See, for example, Chapter IV in Spitzer (1964).)

First we return to the case when 0 is not regular for $(-\infty, 0)$ (and, therefore, is regular for $(0, +\infty)$ by the restrictions set forth in the introduction). In place of the Laplace exponent φ^* we introduce

$$(2.17)\quad \Phi^*(\beta,\mu) = \mathcal{E}_0 e^{-\beta\sigma}e^{-\mu X_\sigma^*}$$

and observe that the appropriate analogue for (2.15) or (2.16) is the elementary formula

$$(2.18)\quad \mathcal{E}_0\int_0^\infty dt\, e^{-\beta L_t^*}e^{-\mu M_t^*}e^{-\alpha(t-L_t^*)}e^{-\lambda(M_t^*+X_t)} = \{1-\Phi^*(\beta,\mu)\}^{-1}\mathcal{E}_0\int_0^\sigma dt\, e^{-\alpha t}e^{-\lambda X_t^0}$$

valid if 0 is regular for $(0, +\infty)$ but not for $(-\infty, 0)$.

The identities (2.15), (2.16) and (2.18) have obvious duals with φ^*, $\mathcal{E}^{\text{ex}}$, Φ^* etc., replaced by φ, $\mathcal{E}^{*\text{ex}}$, Φ etc., and we take these for granted below.

It is true that with probability one there do not exist times $0 < r < s < t$ such that $M_t = \max(X_r, X_{r-0}) = \max(X_s, X_{s-0})$. A weak version of this is proved in Pecherskii and Rogozin (1969) and this would suffice for present application. The version as stated can be deduced from results of Millar (1973) by considering cases as follows. By the formula (1.6) of Millar, the process $\mathbf{X}$ cannot jump into an old maximum. By Theorem 2.1 of Millar $\mathbf{X}$ can move continuously up to an old maximum only if 0 is regular for $(0, +\infty)$ and then $\mathbf{X}$ must immediately go above this maximum. Therefore, we can conclude from (2.10) that for almost every ω

$$(2.19)\quad L_t(\rho_t\omega) = t - L_t^*(\omega)$$

except for at most countably many t. This combines with (2.8) and (2.9) to give

$$(2.20)\quad \mathcal{E}_0\int_0^\infty dt\, e^{-\beta L_t^*}e^{-\mu M_t^*}e^{-\alpha(t-L_t^*)}e^{-\lambda(M_t^*+X_t)} = \mathcal{E}_0\int_0^\infty dt\, e^{-\alpha L_t}e^{-\lambda M_t}e^{-\beta(t-L_t)}e^{-\mu(M_t-X_t)}.$$

Suppose now that 0 is regular both for $(0, +\infty)$ and $(-\infty, 0)$. Then (2.20) combines with (2.16) and its dual version to yield

$$(2.21)\qquad \varphi(\alpha, \lambda)\mathcal{E}^{\text{ex}}\int_0^\sigma dt\, e^{-\alpha t}e^{-\lambda X_t^0} = \varphi^*(\beta, \mu)\mathcal{E}^{*\text{ex}}\int_0^{\sigma^*} dt\, e^{-\beta t}e^{-\mu X_t^{*0}}$$

which means that both sides must be constant. We choose our normalizations for the local times $\mathbf{A}$ and $\mathbf{A}^*$ so that both sides of (2.21) are 1 and so that also

$$(2.22)\qquad \varphi(1, 0) = 1,$$

and there follows

$$(2.23)\qquad \mathcal{E}^{\text{ex}}\int_0^\sigma dt\, e^{-\alpha t}e^{-\lambda X_t^0} = \varphi(\alpha, \lambda)^{-1}.$$

If 0 is regular for $(0, +\infty)$ but not for $(-\infty, 0)$ then

$$\{1 - \Phi^*(\beta, \mu)\}^{-1}\mathcal{E}_0\int_0^\sigma dt\, e^{-\alpha t}e^{-\lambda X_t} = \varphi(\alpha, \lambda)^{-1}\{1 + \mathcal{E}^{*\text{ex}}\int_0^{\sigma^*} dt\, e^{-\beta t}e^{-\mu X_t^*}\}$$

and, therefore,

$$(2.24)\quad \varphi(\alpha, \lambda)\mathcal{E}_0\int_0^\sigma dt\, e^{-\alpha t}e^{-\lambda X_t} = \{1 - \Phi^*(\beta, \mu)\}\{1 + \mathcal{E}^{*\text{ex}}\int_0^{\sigma^*} dt\, e^{-\beta t}e^{-\mu X_t^*}\}$$

and both sides equal a constant which has already been determined by the convention which is dual to (2.13). Indeed we can calculate

$$(2.25)\qquad \begin{aligned}\varphi(\alpha, \lambda)^{-1} &= \mathcal{E}_0\int_0^\infty dt\, e^{-\alpha t}I(M_t - X_t = 0)e^{-\lambda X_t}\\ &= \int_0^\infty dt\, e^{\alpha t}\mathcal{E}_0 I(M_t^* = 0)e^{-\lambda X_t}\\ &= \mathcal{E}_0\int_0^\sigma dt\, e^{-\alpha t}e^{-\lambda X_t}\end{aligned}$$

which identifies this constant as 1 and then by the dual of (2.24)

$$(2.26)\qquad 1 + \mathcal{E}^{\text{ex}}\int_0^\sigma dt\, e^{-\alpha t}e^{-\lambda X_t} = \{1 - \Phi(\alpha, \lambda)\}^{-1}$$

valid if 0 is regular for $(-\infty, 0)$ but not for $(0, +\infty)$.

To deduce the formula in Chapter 9 of Fristedt (1974), we need only combine the above with

$$(2.27)\quad \mathcal{E}_0\int_0^\infty dt\, e^{-\alpha L_t}e^{-\lambda M_t}e^{-\beta(t-L_t)}e^{-\mu(M_t-X_t)} = \exp\int_0^\infty \frac{dt}{t}\mathcal{E}_0\{I(X_t > 0)e^{-\alpha t-\lambda X_t} + I(X_t < 0)e^{-\beta t+\mu X_t} - e^{-t}\}$$

which is proved in Pecherskii and Rogozin (1969) by passage to the limit from discrete time skeletons. If 0 is regular for both $(-\infty, 0)$ and $(0, +\infty)$, then (2.16), (2.23), (2.20) and (2.27) combine to give

$$(2.28)\quad \varphi(\alpha, \lambda)\varphi^*(\beta, \mu) = \exp\int_0^\infty \frac{dt}{t}\mathcal{E}_0\{e^{-t} - I(X_t > 0)e^{-\alpha t-\lambda X_t} - I(X_t < 0)e^{-\beta t+\mu X_t}\}.$$

In particular $\varphi(1, 0)\varphi^*(1, 0) = 1$ since $\mathcal{P}_0(X_t = 0) = 0$ and so our normalization of $\mathbf{A}$ and $\mathbf{A}^*$ gives, together with (2.22), its dual

$$\varphi^*(1, 0) = 1 \tag{2.22*}$$

valid if 0 is regular for both $(-\infty, 0)$ and $(0, +\infty)$. Now plugging in $\beta = 1$ and $\mu = 0$ in (2.28) we deduce Fristedt's formula

$$\varphi(\alpha,\lambda) = \exp \int_0^\infty \frac{dt}{t} \mathcal{E}_0 I(X_t > 0)\{e^{-t} - e^{-\alpha t}e^{-\lambda X_t}\} \tag{2.29}$$

valid if 0 is regular for both $(-\infty, 0)$ and $(0, +\infty)$. If 0 is regular for $(-\infty, 0)$ but not for $(0, +\infty)$, then (2.15) combines with (2.26), (2.20) and (2.27) to give

$$\begin{aligned} &\{1 - \Phi(\alpha, \lambda)\}\varphi^*(\beta, \mu) \\ &\quad = \exp \int_0^\infty \frac{dt}{t} \mathcal{E}_0\{e^{-t} - I(X_t > 0)e^{-\alpha t - \lambda X_t} - I(X_t < 0)e^{-\beta t + \mu X_t}\}. \end{aligned} \tag{2.30}$$

We deduce immediately

$$\varphi^*(1, 0) = \{1 - \mathcal{E}_0 e^{-\sigma^*}\}^{-1} \tag{2.31}$$

$$1 - \Phi(\alpha, \lambda) = \{1 - \mathcal{E}_0 e^{-\sigma^*}\} \exp \int_0^\infty \frac{dt}{t} \mathcal{E}_0 I(X_t > 0)\{e^{-t} - e^{-\alpha t}e^{-\lambda X_t}\} \tag{2.32}$$

valid if 0 is regular for $(-\infty, 0)$ but not for $(0, +\infty)$. Similarly

$$\varphi(1, 0) = \{1 - \mathcal{E}_0 e^{-\sigma}\}^{-1} \tag{2.33}$$

$$\varphi(\alpha, \lambda) = \{1 - \mathcal{E}_0 e^{-\sigma}\}^{-1} \exp \int_0^\infty \frac{dt}{t} \mathcal{E}_0 I(X_t > 0)\{e^{-t} - e^{-\alpha t}e^{\lambda X_t}\} \tag{2.34}$$

valid if 0 is regular for $(0, +\infty)$ but not for $(-\infty, 0)$.

REMARK. We have used the fact that with the restriction on $\mathbf{X}$ set out in the introduction, $\mathcal{P}_0(X_t = 0) = 0$. Indeed it is well known that with our restrictions on $\mathbf{X}$

$$\mathcal{P}_0(X_t = x) = 0 \tag{2.35}$$

for all $t > 0$ and all real x (see Esseen (1968) or Rogozin (1961)), but we thought it worth including here a simple proof shown to the author by D. Monrad. It suffices to consider the special case when $\mathbf{X}$ is symmetric and $x = 0$ and the point is that for all $\varepsilon > 0$ we have $\mathcal{P}_0(M_t > \varepsilon) < 2\mathcal{P}_0(X_t \geq \varepsilon)$ and so $\mathcal{P}_0(X_t > 0) \geq \frac{1}{2}\mathcal{P}_0(M_t > 0) = \frac{1}{2}$. □

For convenient future reference we summarize the results of this section in

THEOREM 8. (i) *Assume that* 0 *is regular for both* $(-\infty, 0)$ *and* $(0, +\infty)$. *Let the excursion measure* $\mathcal{P}^{\text{ex}}$ *corresponding to excursions from a minimum be normalized by* (2.5). *Let* $\varphi^*(\mu, \beta)$ *be the Laplace exponent defined by* (2.11). *Let* $\mathcal{P}^{*\text{ex}}$ *and* $\varphi(\alpha, \lambda)$ *be the dual objects. Then the local times* $\mathbf{A}$ *and* $\mathbf{A}^*$ *can be normalized so that* (2.23), (2.29) *and their duals are valid.*

(ii) *Assume that* 0 *is regular for* $(0, +\infty)$ *but not for* $(-\infty, 0)$. *Let* $\Phi^*(\mu, \beta)$ *be defined by* (2.17) *and let the local time* $\mathbf{A}^*$ *be defined by the dual to* (2.13). *Then* (2.25), (2.34) *and the dual to* (2.26) *and* (2.32) *are valid.*

Of course if 0 is regular for $(-\infty, 0)$ but not for $(0, +\infty)$, then the dual to conclusion (ii) is valid.

3. **Proof of Theorems 1 and 6.** Suppose first that 0 is regular for both $(-\infty, 0)$ and $(0, +\infty)$. Taking $f(y) = e^{-\lambda y}$ and integrating with respect to τ the dual to (2.11) with $\alpha = 0$, we get

$$\varphi(0, \lambda)^{-1} = \mathcal{E}_0 \int_0^\infty d\tau\, e^{-\lambda M \circ B_\tau^*} \quad \text{for} \quad \lambda > 0 \tag{3.1}$$

and, after comparing with (2.23) for $\alpha = 0$, we get

$$\mathcal{E}^{ex} \int_0^\sigma dt\, f(X_t^0) = \mathcal{E}_0 \int_0^\infty d\tau\, f(M \circ B_\tau^*) \tag{3.2}$$

for Borel $f \geq 0$ and $(0, +\infty]$, with the convention $f(\infty) = 0$ as specified in the introduction. It follows then from Theorem 5.1 in Maisonneuve (1975), or more directly from Theorem 5 in Meyer (1971), that the measure on $(0, \infty)$ defined by the left side of (3.2) is excessive for the absorbed process $\mathbf{X}^0$. Then so is the measure defined by the right side and so Condition ACC together with Proposition 1.11 on page 258 in B/G guarantees that this measure has a unique coexcessive density $\psi(x)$ and (1.8) makes sense. Moreover (3.2) can be rewritten

$$\mathcal{E}^{ex} \int_0^\sigma dt\, f(X_t) = \int_0^\infty dy\, \psi(y) f(y) \tag{3.3}$$

and it is easy to check that in fact this is valid if 0 is regular for $(-\infty, 0)$, whether or not 0 is regular also for $(0, +\infty)$. Similarly

$$\mathcal{E}_0 \int_0^\sigma dt\, f(X_t) = \int_0^\infty dy\, \psi(y) f(y) \tag{3.4}$$

is valid if 0 is regular for $(0, +\infty)$ but not $(-\infty, 0)$.

The formulae (3.3) and (3.4) are continuous time analogues for the elementary "duality lemma" on page 378 in Feller (1966). They will be the main tools in this section. Actually (3.4) could be deduced directly from the elementary relation (2.9), but (3.3) seems to depend on the results in Section 2. We begin by proving Theorem 6.

If 0 is regular for both $(-\infty, 0)$ and $(0, +\infty)$ then the set of zeroes for the reflected process $\mathbf{Y}$ has measure zero and it follows that for $x > 0$

$$Gf(x) = \mathcal{E}_0 \Sigma_{i \in \mathcal{G}} I(M_i^* < x) \int_0^{r^i - i} dt\, f(x - M_i^* + w_t^i). \tag{3.5}$$

We are using the notation of Section 2 for excursions from 0 for $\mathbf{Y}$. We apply (4.9) in Maisonneuve (1975) to transform (3.5) into

(3.6)

$$Gf(x) = \int \mathcal{P}_0(d\omega) \int_0^\infty dA_t(\omega) I(M_t^*(\omega) < x) \int \mathcal{P}^{ex}(d\omega') \int_0^{\sigma(\omega')} ds\, f[x - M_t^*(\omega) + X_s^0(\omega')]$$

and now (3.3) combines with the dual of (1.8) to give

$$Gf(x) = \int_0^x dr\, \psi^*(r) \int_0^\infty ds\, \psi(s) f(x - r + s) \tag{3.7}$$

and (1.18) follows after the change of variables $u = x - r$ and $y = x - r + s$. If 0 is regular for $(-\infty, 0)$ but not for $(0, +\infty)$ then the set of zeroes for **Y** has positive Lebesgue measure and (3.7) must be replaced by

(3.8) $$Gf(x) = \int_0^x ds\, \psi^*(s) f(x - s) + \int_0^x dr\, \psi^*(r) \int_0^\infty ds\, \psi(s) f(x - r + s)$$

and (1.20) follows after a change of variables. Finally, if 0 is regular for $(0, +\infty)$ but not for $(-\infty, 0)$, the (3.6) is replaced by

(3.9) $$Gf(x) = \mathcal{E}_0 \Sigma_{j=0}^\infty I(M^*(B_j) < x) \int_{B_j}^{B_{j+1}} dt\, f(x + X_t)$$

and, after applying the strong Markov property and the dual of (1.9), we get

(3.10) $$\begin{aligned} Gf(x) &= \mathcal{E}_0 \int_0^\sigma dt\, f(x + X_t) + \int_0^x dr\, \psi^*(r) \mathcal{E}_0 \int_0^\sigma dt\, f(x - r + X_t) \\ &= \int_0^\infty ds\, \psi(s) f(x + s) + \int_0^x dr\, \psi^*(r) \int_0^\infty ds\, \psi(s) f(x - r + s) \end{aligned}$$

and (1.19) follows after a change of variables. This completes the proof of Theorem 6.

Our proof of Theorem 1 depends on Theorem 3.1 in Millar (1976). If 0 is regular for $(-\infty, 0)$ then Millar's theorem implies

(3.11) $$\mathcal{P}^{ex}(X^0_{0+} > 0) = 0 \qquad \text{if 0 is regular for } (0, +\infty)$$

(3.12) $$\mathcal{P}^{ex}(X^0_{0+} = 0) = 0 \qquad \text{if 0 is not regular for } (0, +\infty).$$

For $n \geqslant 2$ let $D_n = (1/n, n)$, let $T(D_n)$ be the hitting time defined as in (1.4) and let $l_n(dx)$ be the measure on the closure $[1/n, n]$ determined by

(3.13) $$\int l_n(dx) f(x) = \mathcal{E}^{ex} I(T(D_n) < \sigma) f[X^0(T_{D_n})]$$

for Borel $f \geqslant 0$ on $[1/n, n]$. Consider first the case when 0 is regular for both $(-\infty, 0)$ and $(0, +\infty)$ so that

$$\mathcal{P}^{ex}(T(D_n) = 0) = 0.$$

Now it is easy to apply Theorem 5.1 in Maisonneuve (1975) and deduce for Borel $f \geqslant 0$ on $(0, +\infty)$

$$\int l^n(dz) \int dy\, G(z, y) f(y) = \mathcal{E}^{ex} \int_{T(D_n)}^\sigma dt\, f(X_t^0).$$

Combining this with (3.3) we deduce

(3.14) $$\begin{aligned} &\int l^n(dz) G(z, y) \uparrow \psi(y) && \text{as } n \uparrow \infty \\ &\int l^n(dz) G(z, y) = \psi(y) && \text{for } y \in D_n. \end{aligned}$$

With M as in (1.3) we then have

$$\begin{aligned} &\int dy\, f(y) \mathcal{E}_{-y} I(T_M^* < \sigma^*) \psi(X^*(T_M^*)) \\ &\qquad = \operatorname{Lim}_{n \uparrow \infty} \int dy\, f(y) \int l^n(dz) \mathcal{E}_{-y} I(T_M^* < \sigma^*) G(z, X^*(T_M^*)) \end{aligned}$$

which, by Hunt's duality relation, as stated on page 261 in B/G,

$$\begin{aligned} &= \operatorname{Lim}_{n \uparrow \infty} \int l^n(dz) \mathcal{E}_z I(T_M < \sigma) Gf[X(T_M)] \\ &= \mathcal{E}^{ex} I(T_M < \sigma) \int_{T_M}^\sigma dt\, f(X_t^0) \end{aligned}$$

which, by (3.11) and (3.3),

$$= \int_0^\infty dy\, \psi(y) f(y)$$

and we have verified (1.3) with h replaced by ψ for almost every y and therefore for every y. This proves Theorem 1(i) in the case when 0 is regular also for $(-\infty, 0)$. The proof of Theorem 1(i) in the case when 0 is not regular for $(-\infty, 0)$ is simpler since then we can work with $\mathcal{E}_0$ in place of $\mathcal{E}^{\text{ex}}$ and we leave it to the reader.

If ψ dominated a positive coharmonic function h, then, except possibly for redefining the time scale, $\mathcal{P}^{\text{ex}}$ would dominate the approximate Markov process $\mathcal{P}_\infty$ constructed in Section 6 below for h. But then Proposition 6.2 would imply $\mathcal{P}^{\text{ex}}(X^0_{0+} = 0) > 0$ and from (3.12) we could conclude that 0 is regular for $(0, +\infty)$. This proves Theorem 1(ii).

4. Invariant measure for processes which must hit a point. In this section we prove an analogue to a well-known result of Harris (1956) for discrete time Markov chains.

Throughout this section we use $\mathcal{P}_x$ and $\mathcal{E}_x$ to denote the sample space probabilities and expectation functionals associated with a general Markov process $\mathbf{Z} = \{Z_t, t \geq 0\}$ instead of the Lévy process $\mathbf{X}$ studied in the rest of the paper. Also we use the symbol P_t to denote the transition operator

$$P_t f(x) = \mathcal{E}_x f(Z_t). \tag{4.1}$$

We assume that $\mathbf{Z}$ is strong Markov with right continuous paths and takes values in a locally compact metric space $\mathbf{E}$. Our basic assumption is that there exists a point $0 \in \mathbf{E}$ such that for every $x \in \mathbf{E}$

$$\mathcal{P}_x(T_0 < +\infty) = 1 \tag{4.2}$$

where T_0 is the hitting time

$$T_0 = \inf\{t > 0 : Z_t = 0\}. \tag{4.3}$$

We use the symbol P_t^0 for the absorbed transition operator

$$P_t^0 f(x) = \mathcal{E}_x I(t < T_0) f(Z_t). \tag{4.4}$$

We distinguish three cases.

CASE I. 0 is not regular for $\{0\}$. In this case we define the sojourn measure μ by

$$\int \mu(dx) f(x) = \mathcal{E}_0 \int_0^{T_0} dt\, f(Z_t) \tag{4.5}$$

for Borel $f \geq 0$ on $\mathbf{E}$.

CASE II. 0 is regular for $\{0\}$ but $\mathcal{P}_0(Z_t = 0) = 0$ for almost every $t > 0$. (It then follows from Kesten (1969) that $\mathcal{P}_0(Z_t = 0) = 0$ for every $t > 0$.) We fix a local time $\mathbf{A}$ at 0 and we define the excursion measure $\mathcal{P}^{\text{ex}}$ exactly as at the beginning of Section 2, using Maisonneuve (1975). The sojourn measure μ is now defined by

$$\int \mu(dx) f(x) = \mathcal{E}^{\text{ex}} \int_0^{T_0} dt\, f(Z_t). \tag{4.6}$$

CASE III. $\mathcal{P}_0(Z_t = 0) > 0$ for a set of $t > 0$ having positive Lebesgue measure. In this case we explicitly define the local time $\mathbf{A}$ by

$$A_t = \int_0^t ds I(Z_s = 0). \tag{4.7}$$

The excursion measure $\mathcal{P}^{\text{ex}}$ and the sojourn measure μ are defined exactly as for Case II.

The analogue to Harris' result is

THEOREM 9. *Assume that* (4.2) *is true. In Case* I *and* II *the sojourn measure* μ *is invariant for* P_t. *In Case* III *the measure* $\varepsilon_0 + \mu$ *is invariant for* P_t.

Of course ε_0 denotes the unit measure concentrated at 0. In Section 5 we will apply Theorem 9 in the special case when $\mathbf{Z} = \mathbf{Y}$, the reflected process, in order to prove Theorem 2.

Before proving Theorem 9 we emphasize that condition (4.2) is fundamental. This is easy to understand if one looks at Harris' original proof in the discrete time context.

Our goal is to prove that

$$\int \mu(dx) P_t f(x) = \int \mu(dx) f(x) \tag{4.8}$$

in Cases I and II and that

$$\int \mu(dx) P_t f(x) + P_t f(0) = \int \mu(dx) f(x) + f(0) \tag{4.9}$$

in Case III. Of course f can be any nonnegative Borel function on $\mathbf{E}$.

We first treat Cases II and III, beginning with

$$P_t f(x) = P_t^0 f(x) + \mathcal{E}_x \int_0^t dA_s \mathcal{E}^{\text{ex}} I(t - s < T_0) f(Z_{t-s}) + \mathcal{P}_x(Z_t = 0) f(0) \tag{4.10}$$

which can be deduced directly from Theorems 4.1 and 5.1 in Maisonneuve (1975). Then

$$\begin{aligned} \int \mu(dx) P_t f(x) = {} & \mathcal{E}^{\text{ex}} \int_0^{T_0} ds\, P_t^0 f(Z_s) \\ & + \int \mathcal{P}^{\text{ex}}(d\omega) \int_0^{T_0(\omega)} ds\, \mathcal{E}_{Z_s(\omega)} \int_0^t dA_p \mathcal{E}^{\text{ex}} I(t - p < T_0) f(Z_{t-p}) \\ & + \int \mathcal{P}^{\text{ex}}(d\omega) \int_0^{T_0(\omega)} ds\, \mathcal{P}_{Z_s(\omega)}(Z_t = 0) f(0). \end{aligned} \tag{4.11}$$

Certainly

$$\mathcal{E}^{\text{ex}} \int_0^{T_0} ds\, P_t^0 f(Z_s) = \mathcal{E}^{\text{ex}} I(t < T_0) \int_t^{T_0} ds\, f(Z_s). \tag{4.12}$$

After application of the Markov property and then the substitution $v = p + s$, the second term on the right in (4.11) becomes

$$\int \mathcal{P}^{\text{ex}}(d\omega) \int_0^{T_0(\omega)} ds \int_s^{s+t} dA_v(\omega) \mathcal{E}^{\text{ex}} I(t - v + s < T_0) f(Z_{t-v+s}).$$

There is no contribution for $s < T_0(\omega) - t$ and so after the substitution $r = T_0(\omega) - s$ we see that this equals

$$\int \mathcal{P}^{\text{ex}}(d\omega) \int_0^{T_0(\omega) \wedge t} dr \int_{T_0(\omega)}^{T_0(\omega) - r + t} dA_v(\omega) \int \mathcal{P}^{\text{ex}}(d\omega')$$
$$I(t - v + T_0(\omega) - r < T_0(\omega')) f(Z_{t-v+T_0(\omega)-r}(\omega')),$$

which, after the substitution $u = v - T_0(\omega)$ and an application of the strong Markov property for $\mathcal{P}^{\text{ex}}$, equals

$$\int \mathcal{P}^{\text{ex}}(d\omega)\int_0^{T_0(\omega)\wedge t} dr\, \mathcal{E}_0 \int_0^{t-r} dA_u \mathcal{E}^{\text{ex}} I(t-u-r < T_0) f(Z_{t-u-r})$$
$$= \int_0^t dr\, \mathcal{P}^{\text{ex}}(r < T_0)\mathcal{E}_0\int_0^t dA_u I(u+r<t)\mathcal{E}^{\text{ex}} I(t-r-u<T_0) f(Z_{t-r-u}).$$

(Before continuing we remark that if (4.2) were not true, then we would have to replace $\mathcal{P}^{\text{ex}}(r < T_0)$ by $\mathcal{P}^{\text{ex}}(r < T_0 < +\infty)$ at this point and the argument given below would fail.) Next we replace r by $s = u + r$ to get

$$\int_0^t ds\, \mathcal{E}_0\int_0^s dA_u \mathcal{E}^{\text{ex}} I(t-s<T_0) f(Z_{t-s})\mathcal{P}^{\text{ex}}(s-u<T_0)$$
$$= \int_0^t ds\, \mathcal{E}^{\text{ex}} I(t-s<T_0) f(Z_{t-s})\{\mathcal{E}_0\int_0^s dA_u \mathcal{P}^{\text{ex}}(s-u<T_0)\}.$$

Again it follows from Maisonneuve (1975) that

$$\mathcal{E}_0\int_0^s dA_u \mathcal{P}^{\text{ex}}(T_0 > s-u) = \mathcal{P}_0(Z_s \neq 0), \tag{4.13}$$

and so we have

$$\int \mathcal{P}^{\text{ex}}(d\omega)\int_0^{T_0(\omega)} ds\, \mathcal{E}_{z_s(\omega)}\int_0^t dA_p \mathcal{E}^{\text{ex}} I(t-p<T_0) f(Z_{t-p}) \tag{4.14}$$
$$= \int_0^t ds\, \mathcal{E}^{\text{ex}} I(t-s<T_0) f(Z_{t-s})\mathcal{P}_0(Z_s \neq 0)$$
$$= \mathcal{E}^{\text{ex}}\int_0^{t\wedge T_0} ds\, f(Z_s)\mathcal{P}_0(Z_{t-s}\neq 0).$$

Substituting (4.12) and (4.14) into (4.11) we get

$$\int \mu(dx) P_t f(x) = \mathcal{E}^{\text{ex}} I(t<T_0)\int_t^{T_0} ds\, f(Z_s) \tag{4.15}$$
$$+ \mathcal{E}^{\text{ex}}\int_0^{t\wedge T_0} ds\, f(Z_s)\mathcal{P}_0(Z_{t-s}\neq 0)$$
$$+ \int \mathcal{P}^{\text{ex}}(d\omega)\int_0^{T_0(\omega)} ds\, \mathcal{P}_{z_s(\omega)}(Z_t = 0) f(0).$$

The desired relation (4.8) follows directly in Case II since the third term on the right in (4.15) vanishes and since $\mathcal{P}_0(Z_{t-s}\neq 0) = 1$ for all $s > 0$. To prove (4.9) in Case III we apply the strong Markov property and transform the third term on the right in (4.15) into

$$\int \mathcal{P}^{\text{ex}}(d\omega)\int_0^{T_0(\omega)} ds\, I(s+t > T_0(\omega))\mathcal{P}_0(Z_{s+t-T_0(\omega)} = 0) f(0)$$

which, after the substitution $r = T_0(\omega) - s$,

$$= \int \mathcal{P}^{\text{ex}}(d\omega)\int_0^{T_0(\omega)\wedge t} dr\, \mathcal{P}_0(Z_{t-r} = 0) f(0)$$
$$= \int_0^t dr\, \mathcal{P}^{\text{ex}}(r < T_0(\omega))\mathcal{P}_0(Z_{t-r} = 0) f(0)$$
$$= \int_0^t dr\, \mathcal{P}_0(Z_r = 0)\mathcal{P}^{\text{ex}}(t-r < T_0(\omega)) f(0)$$
$$= \mathcal{E}_0\int_0^t dr\, I(Z_r = 0)\mathcal{P}^{\text{ex}}(t-r<T_0(\omega)) f(0)$$

which, by (4.13) and our normalization for local time in Case III,

$$= \mathcal{P}_0(Z_t \neq 0) f(0).$$

Thus, in Case III (but not in Case II) we can replace (4.15) by

$$(4.15') \qquad \int \mu(dx)P_t f(x) = \mathcal{E}^{\text{ex}} I(t < T_0)\int_t^{T_0} ds\, f(Z_s) + \mathcal{E}^{\text{ex}}\int_0^{t\wedge T_0} ds\, f(Z_s)\mathcal{P}_0(Z_{t-s} \neq 0) + \mathcal{P}_0(Z_t \neq 0)f(0).$$

Finally, in Case III,

$$(4.16) \qquad P_t f(0) = \mathcal{P}_0(Z_t = 0)f(0) + \mathcal{E}_0\int_0^t ds\, I(Z_s = 0)\mathcal{E}^{\text{ex}} I(t - s < T_0)f(Z_{t-s}) = \mathcal{P}_0(Z_t = 0)f(0) + \mathcal{E}^{\text{ex}}\int_0^{t\wedge T_0} ds\, f(Z_s)\mathcal{P}_0(Z_{t-s} = 0)$$

and (4.9) follows upon combining (4.15′) with (4.16).

In treating Case I we argue exactly as for Case II except that $\mathcal{P}^{\text{ex}}$ is replaced by $\mathcal{P}_0$ and the local time $\mathbf{A}$ is counting measure for the set of zeroes of $\mathbf{Z}$. Also we must keep track of the discontinuities of $\mathbf{A}$ and we will agree that the right hand limit is meant unless a minus sign is inserted. Then

$$(4.10') \qquad P_t f(x) = P_t^0 f(x) + \mathcal{E}_x\int_0^t dA_s \mathcal{E}_0 I(t - s < T_0)f(Z_{t-s})$$

and so

$$(4.11') \qquad \int \mu(dx)P_t f(x) = \mathcal{E}_0 I(t < T_0)\int_t^{T_0} ds\, f(Z_s) + \mathcal{E}_0\int_0^{T_0} ds\, \mathcal{E}_{Z_s}\int_0^t dA_p \mathcal{E}_0 I(t - p < T_0)f(Z_{t-p}).$$

Arguing exactly as in obtaining (4.14) we see that the second term on the right

$$\begin{aligned}
&= \int \mathcal{P}(d\omega)\int_0^{T_0(\omega)} ds \int_s^{s+t} dA_v(\omega)\mathcal{E}_0 I(t - v + s < T_0)f(Z_{t-v+s}).\\
&= \int \mathcal{P}_0(d\omega)\int_0^{T_0(\omega)\wedge t} dr \int_{T_0(\omega)-}^{T_0(\omega)-r+t} dA_v(\omega)\int \mathcal{P}_0(d\omega')I(t - v + T_0(\omega) - r\\
&\quad < T_0(\omega'))f\big(Z_{t-v+T_0(\omega)-r}(\omega')\big)\\
&= \int \mathcal{P}_0(d\omega)\int_0^{T_0(\omega)\wedge t} dr \mathcal{E}_0\int_{0-}^{t-r} dA_u \mathcal{E}_0 I(t - u - r < T_0)f(Z_{t-u-r})\\
&= \int_0^t dr\, \mathcal{P}_0(r < T_0)\mathcal{E}_0\int_{0-}^t dA_u I(u + r < t)\mathcal{E}_0 I(t - r - u < T_0)f(Z_{t-r-u})\\
&= \int_0^t ds\, \mathcal{E}_0\int_{0-}^s dA_u \mathcal{E}_0 I(t - s < T_0)f(Z_{t-s})\mathcal{P}_0(s - u < T_0)\\
&= \int_0^t ds\, \mathcal{E}_0 I(t - s < T_0)f(Z_{t-s})\{\mathcal{E}_0\int_{0-}^s dA_u \mathcal{P}_0(s - u < T_0)\}.
\end{aligned}$$

Now (4.13) can be replaced by

$$(4.13') \qquad \mathcal{E}_0\int_{0-}^s dA_u \mathcal{P}_0(s - u < T_0) = \mathcal{P}_0(Z_s \neq 0) = 1$$

and we have established

$$(4.14') \qquad \mathcal{E}_0\int_0^{T_0} ds\, \mathcal{E}_{Z_s}\int_0^t dA_p \mathcal{E}_0 I(t - p < T_0)f(Z_{t-p}) = \mathcal{E}_0\int_0^{t\wedge T_0} ds\, f(Z_s)$$

and (4.8) for Case II follows upon substituting (4.14′) into (4.11′).

This competes the proof of Theorem 9. □

REMARK. When we apply Theorem 9 in Section 5 it will be obvious that μ is Radon. It is important for the general theory that in fact μ is always σ-finite. This was observed in Getoor (1979), and we refer the reader to that place for a proof.

5. Proof of Theorem 2.

We begin with

LEMMA 5.1. *For $x > 0$ and $t > 0$*

$$\mathcal{P}_0(M_t = x) = 0 . \tag{5.1}$$

PROOF. If 0 is not regular for $(0, +\infty)$ we combine (2.35) with the strong Markov property to get

$$\mathcal{P}_0(M_t = x) = \mathcal{E}_0\Sigma_{j=1}^{\infty}I\big(B_j^*(\omega) < t;\ M\big(B_j^*(\omega)\big) = x\big)\mathcal{P}_0\big(\sigma^* > t - B_j^*(\omega)\big). \tag{5.2}$$

If 0 is regular for $(0, +\infty)$, we combine (2.35) with (4.2) in Maisonneuve (1975) to get

$$\mathcal{P}_0(M_t = x) = \mathcal{E}_0\textstyle\int_0^t dA_s^* I(M_s = x)\mathcal{P}^{*\mathrm{ex}}(\sigma > t - s). \tag{5.3}$$

In either case the lemma follows from the observation made at the beginning of Section 3 that the measure ν defined by

$$\textstyle\int \nu(dx)f(x) = \mathcal{E}_0\Sigma_{j=1}^{\infty}f\big[M(B_j^*)\big]I(B_j^* < +\infty)$$

or by

$$\textstyle\int \nu(dx)f(x) = \mathcal{E}_0\int_0^{\infty}dA_s^* f(M_s)$$

is excessive relative to the absorbed process $\mathbf{X}^0$ and therefore does not charge the singleton $\{x\}$. (This follows from (2.35) and does not depend on the smoothness condition ACC.) □

REMARK. Lemma 5.1 is proved on page 411 in Pecherskii and Rogozin (1969) under the assumption that 0 is regular for $(0, +\infty)$. Apparently their argument does not extend to the general case.

The basic tool for deducing Theorem 2 from Theorem 9 is

PROPOSITION 5.2. *For $x, y \geq 0$ and for $s, t > 0$*

$$\mathcal{P}_{-x}\big(y \leq X_t^{*0} < +\infty\big) = \mathcal{P}_0\big(Y_{t+s} \leq x \mid Y_s = y\big). \tag{5.4}$$

We have already remarked in the introduction that the right side of (5.4) is well defined and independent of $s > 0$.

PROOF. If $x > 0$, then the left side of (5.4)

$$= \mathcal{P}_0(x - X_t \geq y;\ M_t < x)$$

which, by Lemma 5.1,

$$= \mathcal{P}_0(x - X_t \geq y;\ M_t \leq x)$$

which, by (2.8) and (1.7′),

$$= \mathcal{P}_0(X_t + y \leq x;\ X_t + M_t^* \leq x)$$

$$= \mathcal{P}_0(X_t + \max(y, M_t^*) \leq x)$$

$$= \mathcal{P}_0(Y_{t+s} \leq x \mid Y_s = y).$$

For the case $x = 0$ we know that

$$\mathcal{P}_0(X_t^{*0} < +\infty) = \mathcal{P}_0(M_t = 0)$$

(see the discussion concerning (2.19)) and so the left side of (5.4)

$$\begin{aligned} &= \mathcal{P}_0(-X_t \geqslant y; M_t = 0) \\ &= \mathcal{P}_0(X_t + y \leqslant 0; X_t + M_t^* = 0) \\ &= \mathcal{P}_0(X_t + \max(y, M_t^*) = 0) \\ &= \mathcal{P}_0(Y_{t+s} = 0 \mid Y_s = y). \quad \square \end{aligned}$$

Now it is easy to prove Theorem 2. If 0 is regular for both $(-\infty, 0)$ and $(0, +\infty)$, then Case II in Section 4 is applicable with **Y** playing the role of **Z** there, and so, by Theorem 9 and the relation noted at the beginning of Section 3,

$$\int_0^\infty dy\, \psi(y)\mathcal{P}_0(Y_{t+s} \leqslant x \mid Y_s = y) = \int_0^x dy\, \psi(y) = g(x). \tag{5.5}$$

Therefore,

$$\begin{aligned} \mathcal{E}_{-x} I(t < \sigma^*) g(X_t^*) &= \mathcal{E}_{-x} I(X_t^{*0} < +\infty) \int_0^{X_t^{*0}} dy\, \psi(y) \\ &= \int_0^\infty dy\, \psi(y) \mathcal{P}_{-x}(y \leqslant X_t^{*0} < +\infty) \end{aligned}$$

which, by Proposition 5.2,

$$= \int_0^\infty dy\, \psi(y)\mathcal{P}_0(Y_{t+s} \leqslant x \mid Y_s = y),$$

and we need only appeal to (5.5).

If 0 is regular for $(0, +\infty)$ but not for $(-\infty, 0)$, then the argument goes the same way except that we use Case I rather than Case II in Section 4.

If 0 is regular for $(-\infty, 0)$ but not for $(0, +\infty)$, then Case III in Section 4 is applicable, and in place of (5.5) we have

$$\mathcal{P}_0(Y_t \leqslant x) + \int_t^\infty dy\, \psi(y)\mathcal{P}_0(Y_{t+s} \leqslant x \mid Y_s = y) = 1 + g(x), \tag{5.6}$$

and so

$$\begin{aligned} &\mathcal{P}_{-x}(X_t^{*0} < +\infty) + \mathcal{E}_{-x} I(t < \sigma^*) g(X_t^*) \\ &\qquad = \mathcal{P}_{-x}(X_t^{*0} < +\infty) + \int_0^\infty dy\, \psi(y)\mathcal{P}_{-x}(y \leqslant X_t^{*0} < +\infty) \\ &\qquad = \mathcal{P}_0(Y_t \leqslant x) + \int_0^\infty dy\, \psi(y)\mathcal{P}_0(Y_{t+s} \leqslant x \mid Y_s = y) \\ &\qquad = 1 + g(x). \end{aligned}$$

EXAMPLE. Suppose that **X** is a diffusion with local generator $\varphi = \frac{1}{2}\varphi'' + b\varphi'$. If $b = 0$, then clearly $\psi(x) = \text{const.}$ and Theorem 2 states that the function x is coinvariant on $(0, +\infty)$. This is easy to check directly. The point is that **X** is a martingale and so

$$\begin{aligned} \mathcal{E}_{-x} I(t < \sigma^*) X_t^{*0} &= \mathcal{E}_x I(t < \sigma) X_t \\ &= \mathcal{E}_x X_{t \wedge \sigma} = x. \end{aligned}$$

If $b > 0$, then still $\psi(x) = \text{const.}$, but Theorem 2 is *not* applicable, and it is *false* that x is coinvariant on $(0, +\infty)$. Indeed, x is not even coharmonic. However, Theorem 3, which will be proved in Section 8, is applicable, and since (1.13) is true with $q = 2b$, this states that $e^{2bx} - 1$ is coinvariant on $(0, +\infty)$. This can also be checked directly. Since $-X_t + bt$ is standard Brownian motion relative to $\mathcal{P}_x$, the process $\exp\{2b(-X_t + bt) - \frac{1}{2}(2b)^2 t\} = \exp\{-2bX_t + 2bx\}$ is a martingale relative to $\mathcal{P}_x$ (see page 13 in McKean (1969)). Thus

$$\mathcal{E}_{-x} I(t < \sigma^*)\{e^{2bX_t^*} - 1\} = \mathcal{E}_{-x} e^{-2bX_{t \wedge \sigma^*}} - 1$$
$$= e^{2bx} - 1.$$

(It was our desire to understand "where this coinvariant function comes from" that led us to formulate and prove Theorem 3.) If $b < 0$, then $\psi(x) = (\text{const.})e^{bx}$ and Theorem 2 is applicable and tells us that $1 - e^{2bx}$ is coinvariant on $(0, +\infty)$. This can be checked by the same calculation as above. If $b > 0$ and if $\mathbf{Z} = \mathbf{Y}$ in Section 4, then the sojourn measure μ is Lebesgue measure. However, we can be sure that Lebesgue measure is not an invariant measure for $\mathbf{Y}$. If it were, then the results in this section would guarantee that x is coinvariant on $(0, +\infty)$ for $\mathbf{X}$ and we have already noted that this is false.

6. Approximate Markov processes. Approximate Markov processes were used to prove Theorem 1(ii) and will play an important role in the remainder of the paper. They were first introduced by G. A. Hunt (1960) in his work on Martin boundaries for discrete time Markov chains. In this section we will outline the construction of the continuous time version. For more detail we refer to M. Weil (1970) or to Section 5 in Silverstein (1974) where symmetry is assumed.

Just as in Section 4 we let $\mathcal{P}_x$ and $\mathcal{E}_x$ correspond to a general strong Markov process $\mathbf{Z}$ which has right continuous paths in a locally compact metric space $\mathbf{E}$ with dead point ∂ adjoined. In addition, we postulate Hypothesis (2.2) and its dual at the top of page 266 in B/G. The transition operators and dual transition operators will be denoted by P_t and $P_t^{\wedge}$. Also we will use certain other notations which are consistent with B/G. The reference measure will be denoted by dx. The Green's function or 0-potential density will be denoted by $u(x, y)$. The sample space probabilities and expectation functional for the dual process will be denoted by $\mathcal{P}_x^{\wedge}$ and $\mathcal{E}_x^{\wedge}$. Potential and dual potential operators will be denoted by U and $U^{\wedge}$. Hitting and dual hitting operators will be denoted by P_A and $P_A^{\wedge}$, hitting times by T_A. The set of points regular (coregular) for A will be denoted by $A^r({}^rA)$.

Let E_n, $n \geq 1$, be an increasing sequence of open subsets of $\mathbf{E}$ with $\cup E_n = \mathbf{E}$ and such that for each n the closure $\overline{E}_n \subset E_{n+1}$. Let $h(x)$ be any locally integrable coexcessive function. There may or may not exist a sequence of Radon measures l_n, $n \geq 1$, satisfying

E.1. l_n is supported by E_n^-;

E.2. $\int l_n(dx)u(x, y) = h(y)$ for $y \in E_n$;

E.3. $\int l_n(dx)P_{E_m}f(x) = \int l_m(dx)f(x)$ for $m \leq n$ and $f \geq 0$ on E_m^-.

Any such sequence will be referred to as an *equilibrium system* for h. If such a sequence exists, it is unique. Indeed, each l_n is determined by the relation

(6.1) $$P_{E_n}^{\wedge} h(y) = \int l_n(dx)u(x, y).$$

The reason is that if $n < p$, then by E.2 we have $h(z) = \int l_p(dx)u(x, z)$ for $z \in \bar{E}_n$, and so the left side of (6.1) $= \int P^{\wedge}_{E_n}(y, dz)\int l_p(dx)u(x, z)$ which, by Hunt's duality relation as formulated on page 261 in B/G, $= \int l_p(dx)\int P_{E_n}(x, dz)u(z, y)$, and by E.3 this equals the right side of (6.1).

It is important for the general theory that an equilibrium system always exists when the E_n are relatively compact. (See Theorem 2.8 on page 271 in B/G.) More important for us is the following proposition which will enable us below to identify a given sequence of measures as an equilibrium system.

PROPOSITION 6.1. *Let $h \geqslant 0$ be locally integrable and let l_n, $n \geqslant 1$, be a sequence of Radon measures satisfying the following conditions.*

(i) *$h(x) = \int l_n(dz)u(z, x)$ whenever $x \in E_n \cup {}^rE_n$.*

(ii) *l_n is concentrated on E_n^r.*

Then h is coexcessive and the l_n form an equilibrium system for h.

PROOF. We begin by establishing (6.1). Since the measure $P^{\wedge}(y, d\cdot)$ is concentrated on $E_n \cup {}^rE_n$ we have by (i)

$$P^{\wedge}_{E_n}h(y) = \int l_n(dx)\int P^{\wedge}_{E_n}(y, dz)u(x, z)$$

which, by Hunt's duality relation,

$$= \int l_n(dx)\int P_{E_n}(x, dz)u(z, y).$$

But if $x \in E_n^r$, then $P_{E_n}(x, d\cdot)$ is the point mass at x and so (6.1) follows now from (ii). Now that (6.1) is established, the proposition follows directly. If $m < n$, then since $E_m^- \subset E_n$ we have $h(x) = \int l_n(dz)u(z, x)$ for $x \in E_m^-$ and, therefore,

$$\int l_m(dx)u(x, y) = \int P^{\wedge}_{E_m}(y, dz)h(z)$$
$$= \int P^{\wedge}_{E_m}(y, dz)\int l_n(dx)u(x, z),$$

which is enough to guarantee that the functions $\int l_n(dx)u(x, y)$ increase with n and so $h(y)$ is coexcessive. Properties E.1 and E.2 for the l_n are already contained in (i) and (ii). For E.3 observe that if $m < n$, then by Hunt's duality relation

$$\begin{aligned}\int l_n(dx)\int P_{E_m}(x, dz)u(z, y) &= \int l_n(dx)\int P^{\wedge}_{E_m}(y, dz)u(x, z)\\ &= P^{\wedge}_{E_m}P^{\wedge}_{E_n}h(y) = P^{\wedge}_{E_m}h(y)\\ &= \int l_m(dx)u(x, y),\end{aligned}$$

and so E.3 follows from the uniqueness result Proposition 1.15 on page 260 in B/G.

Now we are ready to outline the actual construction. Let h be locally integrable and coexcessive, and let l_n, $n \geqslant 1$, be an equilibrium system for h. Let Ω be the canonical sample space for the process $\mathbf{Z}$ (see paragraph 4.3 in Silverstein (1974)). For $n \geqslant 1$ let Ω_n be a copy of Ω and adjoint to Ω_n the dead trajectory δ_n, all of whose coordinates are the dead point ∂. Define a mapping π_n from Ω_{n+1} to Ω_n by setting

$$\begin{aligned}\pi_n\Omega &= \delta_n && \text{if} \quad T_{E_n}(\omega) = +\infty\\ \pi_n\omega(t) &= \omega(T_{E_n}(\omega) + t) && \text{if} \quad T_{E_n}(\omega) < +\infty.\end{aligned}$$

The inverse limit is the collection Ω_∞^0 of sequences $\{\omega_n\}_{n=1}^\infty$ with each $\omega_n \in \Omega_n$ and such that $\pi_n \omega_{n+1} = \omega_n$ for $n \geqslant 1$. We will work with the reduced inverse limit $\Omega_\infty = \Omega_\infty^0 - \{\delta\}$ where δ is the dead sequence whose components are the dead trajectories δ_n. Let $\pi_{\infty, n}$ be the natural projection of Ω_∞ onto Ω_n. The point is that there exists a unique countably additive measure $\mathcal{P}_\infty$ on the sigma-algebra on Ω_∞ generated by all pull backs of coordinate variables such that

$$\mathcal{E}_\infty I(\omega_n \neq \delta_n)\xi \circ \pi_{\infty, n} = \int l_n(dx)\mathcal{E}_x \xi \tag{6.2}$$

for $n \geqslant 1$ and $\xi \geqslant 0$ any random variable on Ω. (We are identifying $\Omega_n \setminus \{\delta_n\}$ with Ω.) The necessary consistency condition follows from E.3 and the strong Markov property. A verification of countable additivity depends on some nontrivial measure theory concerning which we refer the reader to the references cited above. A time scale, trajectory variables Z_t, the birth time $\zeta^\wedge$, the death time ζ, and hitting times T_A are introduced exactly as in Silverstein (1974) (although with slightly different symbols). The main point is that the 0 of the time scale is $\sup\{T_{E_n} : T_{E_n} < +\infty\}$.

Next we state two propositions which relate properties of $h(y)$ to initial behavior for $(\Omega_\infty, \mathcal{P}_\infty)$.

PROPOSITION 6.2. *h is coharmonic if and only if*

$$\mathcal{P}_\infty(T_M > \zeta^\wedge) = 0 \tag{6.3}$$

whenever D is open with compact closure and $M = \mathbf{E} \setminus D$. This is the case if and only if

$$\operatorname{Lim}_{n\uparrow\infty} l_n(K) = 0 \tag{6.4}$$

for every compact subset K of $\mathbf{E}$.

PROPOSITION 6.3. *h is coinvariant if and only if*

$$\mathcal{P}_\infty(\zeta^\wedge > -\infty) = 0.$$

By the argument at the end of Section 3,

$$\int dy\, P_M^\wedge h(y) f(y) = \mathcal{E}_\infty I(T_M < \infty)\int_{T_M}^{\zeta} ds\, f(Z_s) \tag{6.5}$$

for Borel $f \geqslant 0$ on $\mathbf{E}$ and Proposition 6.2 follows easily. For Proposition 6.3 it suffices to observe that

$$\begin{aligned}
\int dy\, P_t^\wedge h(y) f(y) \\
&= \lim_{n\uparrow\infty} \int l_n(dx) \int dy\, f(y) \int P_t^\wedge(y, dz) u(x, z) \\
&= \operatorname{Lim}_{n\uparrow\infty} \int l_n(dx) \int dy\, P_t f(y) u(x, y) \\
&= \operatorname{Lim}_{n\uparrow\infty} \int l_n(dx)\, U P_t f(x) \\
&= \operatorname{Lim}_{n\uparrow\infty} \mathcal{E}_\infty I(T_{E_n} < \infty) \int_{T_{E_n}+t}^{\zeta} ds\, f(Z_s).
\end{aligned}$$

We finish this section by establishing a convenient generalization of (6.4). Suppose that h is the copotential of a Radon measure ν. That is, $h(x) =$

$\int \nu(dz)u(z, x)$. Then

$$P_{E_n}^{\hat{}}h(y) = \int P_{E_n}^{\hat{}}(y, dx)\int \nu(dz)u(z, x)$$
$$= \int \nu(dz)\int P_{E_n}(z, dx)u(x, y),$$

and so there always exists an equilibrium system l_n, $n \geqslant 1$, and indeed,

$$l_n(dx) = \int \nu(dz)P_{E_n}(z, dx). \tag{6.6}$$

Now if f is continuous with compact support in $\mathbf{E}$, we have

$$\begin{aligned} \mathrm{Lim}_{n\uparrow\infty}\int l_n(dx)f(x) &= \mathrm{Lim}_{n\uparrow\infty}\int \nu(dx)P_{E_n}f(x) \\ &= \mathrm{Lim}_{n\uparrow\infty}\int \nu(dx)\mathcal{E}_x f[X(T_{E_n})] \\ &= \int \nu(dx)f(x). \end{aligned} \tag{6.7}$$

The passage to the limit is legitimate since $\int \nu(dx)\mathcal{P}_x(T_K < +\infty)$ is finite for any compact set K (see Theorem 2.8 on page 271 in B/G.) Combining (6.7) with (6.4), we have

PROPOSITION 6.4. *Let $h(x)$ be coexcessive and locally integrable with a representation*

$$h(x) = h_0(x) + \int \nu(dz)u(z, x) \tag{6.8}$$

where h_0 is coharmonic and ν is a Radon measure on $\mathbf{E}$. *If l_n, $n \geqslant 1$, is an equilibrium system for h, then $l_n \to \nu$ vaguely as $n\uparrow\infty$.*

7. **Proof of Theorems 3 and 5.** Throughout this section we assume that (1.11) fails. We begin by using Laplace transforms to establish some results for the minimum subordinator $\mathbf{M}^* \circ \mathbf{B}$, assuming that 0 is regular for $(-\infty, 0)$. Since $\mathbf{M}^* \circ \mathbf{B}$ is a nonconservative subordinator, its Laplace exponent $\varphi^*(0, \mu)$ (see (2.11)) can be represented

$$\varphi^*(0, \mu) = c^* + b^*\mu + \int_0^\infty l^*(dz)\{1 - e^{-\mu z}\} \tag{7.1}$$

with $c^* > 0$, $b^* \geqslant 0$ constants and with l^* a Radon measure on $(0, +\infty)$ which is bounded near $+\infty$ and has a finite first moment near 0. We prove

PROPOSITION 7.1. *If* (1.11) *fails and if* 0 *is regular for* $(-\infty, 0)$, *then there are two possibilities.*

(i) *There exists unique $q > 0$ satisfying*

$$b^*q + \int_0^\infty l^*(dz)\{e^{qz} - 1\} = c^*. \tag{7.2}$$

In this case

$$\mathcal{E}_0 e^{qM^* \circ B_\tau} = 1 \tag{7.3}$$

for all $\tau > 0$. In particular, e^{qx} is coharmonic for $-\mathbf{M}^* \circ \mathbf{B}$ *on* $\mathbf{R}$ *and every positive coharmonic function is a multiple of e^{qx}.*

(ii) *There is no $q > 0$ satisfying* (7.2). *In this case very positive coharmonic function for* $-\mathbf{M}^* \circ \mathbf{B}$ *on* $\mathbf{R}$ *is trivial.*

Moreover, if e^{qx} *is coharmonic for* $-\mathbf{M}^* \circ \mathbf{B}$, *then there exists a system of equilibrium measures corresponding to the intervals* $I_n = (-\infty, n)$.

For the proof, suppose first that $q > 0$ satisfies (7.2). Then initially for $\mu > q$ and then for all $\mu > 0$

$$-\log \mathcal{E}_0 e^{(q-\mu)M^* \circ B_\tau} = \tau\varphi^*(0, \mu - q)$$
$$= b^*\mu + \int_0^\infty l^*(dz)e^{qz}\{1 - e^{-\mu z}\}$$

and (7.3) follows directly. By the results of Doob, Snell and Williams (1960) every positive harmonic function for $\mathbf{M}^* \circ \mathbf{B}$ on $\mathbf{R}$ is a multiple of e^{qx} for some $q > 0$. Thus the dichotomy (i) versus (ii) will be established once we show that if e^{qx} is harmonic, then (7.2) is satisfied. But this follows easily from the following known identity for ladder variables

$$(7.4)\quad \int_0^\infty dx\, e^{-\lambda x}\mathcal{E}_0 e^{-\alpha t(x) - \mu H(x)} I(t(x) < +\infty)$$
$$= (\mu - \lambda)^{-1}\{\varphi^*(0, \mu) - \varphi^*(0, \lambda)\}\{\alpha + \varphi^*(0, \lambda)\}^{-1}$$

valid for $\alpha, \lambda, \mu \geqslant 0$. We are using the notation

$$(7.5)\quad t(x) = \inf\{\tau > 0 : M^* \circ B_\tau > x\}$$
$$H(x) = M^* \circ B_{t(x)} - x.$$

We will first deduce (7.2) with the help of (7.4) and then we will outline a derivation of (7.4). The point is that (7.4) gives for $\mu > q$, after a passage to the limit $\lambda \downarrow 0$,

$$(7.6)\quad q\int_0^\infty dx\, \mathcal{E}_0 e^{(q-\mu)H(x)} I(t(x) < +\infty)$$
$$= q(\mu - q)^{-1}\{\varphi^*(0, \mu - q) - c^*\}(1/c^*)$$
$$= q\{(\mu - q)c^*\}^{-1}\{b^*(\mu - q) + \int_0^\infty l^*(dz)(1 - e^{-(\mu - q)z})\}.$$

If e^{qx} is harmonic for $\mathbf{M}^* \circ \mathbf{B}$, then certainly

(7.7)

$$q\int_0^\infty dx \mathcal{E}_0 e^{qH(x)} I(t(x) < +\infty) = q\int_0^\infty dx\, e^{-qx}\mathcal{E}_0 e^{q(x + H(x))} I(t(x) < +\infty) = 1$$

and (7.2) follows after comparing (7.6) with (7.7), extending (7.6) to all $\mu > 0$, and then passing to the limit $\mu \downarrow 0$.

Before completing the proof of Proposition 7.1, we outline a derivation of (7.4). Indeed, (7.4) follows from

$$(7.8)\quad \int_0^\infty dx\, e^{-\lambda x}\mathcal{E}_0 e^{-\alpha t(x) - \mu H(x)} I(t(x) < +\infty)$$
$$= (\mu - \lambda)^{-1}\exp\left[\int_0^\infty dt\, t^{-1}e^{-\alpha t}\mathcal{E}_0 I(0 < Z_t < +\infty)\{e^{-\lambda Z_t} - e^{-\mu Z_t}\}\right] - (\mu - \lambda)^{-1}$$

valid for any Levy process $\mathbf{Z}$, possibly not conservative. (We agree here that $Z_t = +\infty$ after it "disappears".) This follows in turn after a routine passage to the limit from

$$(7.9)\quad \int_0^\infty dx\, e^{-\lambda x}\mathcal{E}_0 e^{-\alpha t(x)-\mu H(x)} I(t(x) < +\infty)$$
$$= (\mu-\lambda)^{-1}\exp\left[\Sigma_{n=1}^\infty n^{-1}e^{-n\alpha}\mathcal{E}_0(0 < S_n < +\infty)\{e^{-\lambda S_n} - e^{-\mu S_n}\}\right]$$
$$-(\mu-\lambda)^{-1}$$

valid for any random walk $\mathbf{S} = \{S_n, n \geqslant 0\}$. The latter is stated as formula (3.1) in Pecherskii and Rogozin where the reader is referred to Borovkov (1962) and Presman (1967) for a proof. We insert here a quick derivation which depends only on the familiar identity of Baxter (1958) and Spitzer (1956)

$$(7.10)\quad 1 - \mathcal{E}_0 I(T_1 < +\infty)e^{-\alpha T_1}d^{-\lambda S(T_1)}$$
$$= \exp\left[-\Sigma_{n=1}^\infty n^{-1}e^{-n\alpha}\mathcal{E}_0 I(0 < S_n < +\infty)e^{-\lambda S_n}\right]$$

where $T_1, T_2, \cdots$ are the strict ladder variables

$$T_1 = \min\{n > 0 : S_n > 0\}; \qquad T_2 = \min\{n > T_1 : S_n > S(T_1)\}, \text{ etc.}$$

Putting $T_0 = 0$ we see that the left side of (7.9)

$$= \Sigma_{r=1}^\infty \mathcal{E}_0 I(T_r < +\infty)e^{-\alpha T_r}\int_{S(T_{r-1})}^{S(T_r)} dx\, e^{-\lambda x}e^{-\mu[S(T_r)-x]}$$
$$= (\mu-\lambda)^{-1}\Sigma_{r=1}^\infty \mathcal{E}_0 I(T_r < +\infty)e^{-\alpha T_r}\{e^{-\lambda S(T_r)}$$
$$-\exp(-\lambda S(T_{r-1}) - \mu[S(T_r) - S(T_{r-1})])\},$$

and, after plugging in (7.10), we get the right side of (7.9).

To complete the proof of Proposition 7.1 we assume that e^{qx} is coharmonic for $-\mathbf{M}^* \circ \mathbf{B}$ and verify that

$$(7.11)\qquad l_n(dx) = b^* e^{nq}\varepsilon_n(dx) + e^{nq}l_q^*(n-x)I(x \leqslant n)dx$$

is the equilibrium measure for I_n. Here

$$(7.12)\qquad l_q^*(x) = e^{-qx}\int_{x+}^\infty l^*(dz)e^{qz},$$

and ε_n is the unit point mass at n. For this it suffices to verify

$$(7.13)\qquad b^* e^{nq}\psi^*(n-x) + e^{nq}\int_x^n dy\, l_q^*(n-y)\psi^*(y-x) = e^{qx}$$

for $x < n$. (This follows from Proposition 6.1. The point n is regular but not coregular for I_n relative to $-\mathbf{M}^* \circ \mathbf{B}$.) By the dual to (3.1)

$$(7.14)\qquad \varphi^*(0,\mu)^{-1} = \int_0^\infty dx\, \psi^*(x)e^{-\mu x}.$$

Starting from (7.2) we get for $\mu \geqslant 0$

$$(7.15)\qquad (q+\mu)^{-1}\varphi^*(0,\mu) = b^* + (q+\mu)^{-1}\int_0^\infty l^*(dz)\{e^{qz} - e^{-\mu z}\}$$
$$= b^* + \int_0^\infty l^*(dz)e^{qz}\int_0^z dy\, e^{-(q+\mu)y}$$
$$= b^* + \int_0^\infty dy\, e^{-\mu y}l_q^*(y).$$

Comparing (7.14), with (7.15), we conclude that

$$(7.16)\qquad b^*\psi^*(x) + \int_0^x dy\, l_q^*(y)\psi^*(x-y) = e^{-qx}$$

for almost every $x > 0$. Replacing x by $n - x$ with $x < n$, we get (7.13) for almost every $x < n$ and since both sides of (7.13) are coexcessive relative to $-\mathbf{M}^* \circ \mathbf{B}$ we conclude that (7.13) is valid for all $x < n$ (but not $x = n$). This completes the proof of Proposition 7.1.

Next we consider the case when 0 is not regular for $(-\infty, 0)$. Then there exists $L^*(z) \geqslant 0$ on $(0, \infty)$ satisfying $\int_0^\infty dz L^*(z) < 1$ such that

$$\int_0^\infty dz L^*(z) e^{-\mu z} = \Phi^*(0, \mu) \tag{7.17}$$

$$\psi^*(z) = \Sigma_{r=1}^\infty (L^*)^{*r}(z). \tag{7.18}$$

In (7.18) we are using the superscript $*r$ to denote r-fold convolution. (The dual to (2.26) together with ACC guarantees that in (7.17) we have a density $L_*(z)$ rather than a general measure $L^*(dz)$. See the first paragraph in Section 3.) The analogue of Proposition 7.1 is

PROPOSITION 7.2. *If* (1.11) *fails and if* 0 *is not regular for* $(-\infty, 0)$, *then there are two possibilities.*

(i) *There exists unique* $q > 0$ *satisfying*

$$\int_0^\infty dz L^*(z) e^{qz} = 1. \tag{7.19}$$

In this case

$$\mathcal{E}_0 I(B_j < +\infty) e^{qM^*(B_j)} = 1 \tag{7.20}$$

for all $j \geqslant 0$ *which means that* e^{qx} *is coharmonic for the random walk* $\{-M^*(B_j)\}_{j=0}^\infty$. *Also every positive coharmonic function is a multiple of* e^{qx}.

(ii) *There is no* $q > 0$ *satisfying* (7.19). *In this case every positive coharmonic function for* $\{-M^*(B_j)\}_{j=0}^\infty$ *is trivial.*

Moreover, if e^{qx} *is coharmonic for* $\{-M^*(B_j)\}_{j=0}^\infty$, *then there exists a system of equilibrium measures corresponding to the intervals* $I_n = (-\infty, n)$.

The concepts coharmonic function and equilibrium measure for random walks are exactly the same as for continuous time processes. However, it is worth noting that the distinction between coharmonic functions and coinvariant functions disappears for random walks.

Verification of the dichotomy (i) versus (ii) in Proposition 7.2 is completely routine and we take it for granted. To prove the last paragraph we assume that e^{qx} is coharmonic for $\{-M^*(B_j)\}_{j=0}^\infty$ and we verify that the equilibrium measure for I_n is

$$l_n(dx) = e^{nq} L_q^*(n - x) I(x < n) dx \tag{7.21}$$

where now

$$L_q^*(x) = e^{-qx} \int_x^\infty dy\, e^{qy} L^*(y). \tag{7.22}$$

Instead of (7.13) we must prove

$$e^{nq}\{L_q^*(n - x) + \int_x^n dz L_q^*(n - z) \psi^*(z - x)\} = e^{qx} \tag{7.23}$$

for $x < n$. (In fact, (7.23) is true also for $x = n$.) The formulae (7.17) and (7.18) together give

$$\{1 - \Phi^*(0, \mu)\}^{-1} = 1 + \int_0^\infty dz\, \psi^*(z) e^{-\mu z}. \tag{7.24}$$

By (7.19) we get for $\mu \geqslant 0$

(7.25)

$$(q + \mu)^{-1}\{1 - \Phi^*(0, \mu)\} = (q + \mu)^{-1} \int_0^\infty dz L^*(z)\{e^{qz} - e^{-\mu z}\} = \int_0^\infty dy\, e^{-\mu y} L_q^*(y)$$

and, after combining with (7.24), we conclude that

$$L_q^*(x) + \int_0^x dy L_q^*(y) \psi^*(x - y) = e^{-qx}$$

for almost every $x \geqslant 0$ and therefore every $x \geqslant 0$ since both sides are continuous. The rest of the argument goes exactly as for Proposition 7.1. This takes care of Proposition 7.2.

Now we are ready to prove Theorem 5. Suppose first that (i) in Theorem 5 is true. Then it is easy to check that $Z_t = e^{-qX_t}$ is a local martingale in the sense of Meyer (1967). Let $t(-x) = T_{(-\infty, -x)}$ for $x > 0$. By Fatou's lemma $\mathcal{E}_0 Z_{t(-x)} < +\infty$ and from this it follows that the process $\{Z_{t \wedge t(-x)}\}_{t \geqslant 0}$ is uniformly integrable and therefore $\mathcal{E}_0 I(t(-x) < +\infty) Z_{t(-x)} = 1$ which is enough to guarantee (iii) which by Proposition 7.1 or 7.2 is equivalent to (iv). Thus (i) implies (iii) and (iv). It follows easily from the Riesz decomposition of page 272 in B/G that (ii) implies (i). Thus Theorem 5 will be proved if we can show that (iii) and (iv) imply (ii).

We will work with the potential kernel $u(x)$ defined as in ACC in Section 1 but with $\alpha = 0$. For $x \neq 0$ this has a representation

$$u(x) = \int_0^\infty du\, \psi^*(u) \psi(x + u) \tag{7.26}$$

$$u(x) = \int_0^\infty du\, \psi^*(u) \psi(x + u) + \psi(x) \tag{7.27}$$

$$u(x) = \int_0^\infty du\, \psi^*(u) \psi(x + u) + \psi^*(-x) \tag{7.28}$$

according to whether 0 is regular for $(-\infty, 0)$ and $(0, +\infty)$, 0 is regular for $(0, +\infty)$ but not $(-\infty, 0)$, or 0 is regular for $(-\infty, 0)$ but not $(0, +\infty)$. (It is understood that $\psi(u) = 0$ and $\psi^*(u) = 0$ for $u \leqslant 0$.) The argument given in Section 3 when we proved Theorem 6 suffices to establish these identities for almost every x. Identity for all $x > 0$ follows since the right side is coexcessive for the absorbed process $\mathbf{X}^0$ and since also

$$u(x) = \operatorname{Lim}_{\alpha \uparrow \infty} \alpha \mathcal{E}_x \int_0^{\sigma^*} dt\, e^{-\alpha t} u(X_t^*).$$

A similar argument works for $x < 0$. In general, for $x = 0$ we can only apply Fatou's lemma and conclude that

$$u(0) \geqslant \int_0^\infty du\, \psi^*(u) \psi(u). \tag{7.29}$$

The argument for $x \neq 0$ can be used to establish equality if 0 is not regular for both half lines.

Consider first the case when 0 is regular for both $(-\infty, 0)$ and $(0, +\infty)$. Then (7.13) is true for $x < n$, and we can combine this with (7.26) to deduce

$$b^* e^{nq} u(x-n) + e^{nq} \int_{-\infty}^{n} dz\, l_q^*(n-z) u(x-z) = e^{qx}\{\int_0^\infty du\, \psi(u) e^{-qu}\} \tag{7.30}$$

for $x < n$. If $b^* = 0$ then the same argument gives (7.30) for $x = n$. However, if $b^* > 0$ then, since we only have the inequality (7.29), the same argument only gives

$$b^* e^{nq} u(0) + e^{nq} \int_{-\infty}^{n} dz\, l_q^*(n-z) u(n-z) \geqslant e^{qn}\{\int_0^\infty du\, \psi(u) e^{-qu}\}. \tag{7.31}$$

But then equality in (7.31) follows since the right side in (7.30) is continuous in x and the left side is lower semicontinuous in x. Thus (7.30) is always valid for $x \leqslant n$ and we have established property (i) in Proposition 6.1. Property (ii) is automatic since $n \in I_n^r$ and therefore we can apply Proposition 6.1 and deduce that e^{qx} is coexcessive for **X** and that the l_n form an equilibrium system. The criterion of Proposition 6.2 then guarantees that e^{qx} is actually coharmonic for **X**. To deduce that e^{qx} is actually coinvariant we must work with the approximate Markov process $(\Omega_\infty, \mathcal{P}_\infty)$ associated with e^{qx} and the l_n and with **X** playing the role of **Z** in Section 6. It is easy to see that $\mathcal{P}_\infty(X_{\zeta\wedge} < +\infty) = 0$ and therefore the minimum process $\mathbf{M}^*$ and the reflected process **Y** are well defined. The local times A_t at 0 for **Y** need not be well defined but the increments $A_{t+s} - A_t$ are and so we can still make sense out of the time changed process $\mathbf{M}^* \circ \mathbf{B}$. The function e^{qx} is also coharmonic for $-\mathbf{M}^* \circ \mathbf{B}$ and it is clear that $(\Omega_\infty, \mathcal{P}_\infty)$ is the associated approximate Markov process if we view the $-\mathbf{M}^* \circ \mathbf{B}_\tau$ as coordinate variables. Since e^{qx} is coinvariant for $-\mathbf{M}^* \circ \mathbf{B}$ we can deduce from Proposition 6.3 that for every n

$$\mathcal{P}_\infty(A[T(I_n) \wedge \zeta] - A[\zeta^\wedge] < +\infty) = 0 \tag{7.32}$$

Since **B** is a subordinator if we start counting from any of the times $T(I_m)$ for $m > n$, it follows that

$$\mathcal{P}_\infty(T(I_n) \wedge \zeta - \zeta^\wedge < +\infty) = 0 \tag{7.33}$$

and now Proposition 6.3 guarantees that also e^{qx} is coinvariant for **X**. This completely proves Theorem 5 in the case when 0 is regular for both $(-\infty, 0)$ and $(0, +\infty)$.

Theorem 5 is proved in essentially the same way when 0 is regular for $(-\infty, 0)$ but not for $(0, +\infty)$. In place of (7.30) we establish

$$\begin{aligned} b^* e^{nq} u(x-n) &+ e^{nq} \int_{-\infty}^{n} dz\, l_q^*(n-z) u(x-z) \\ &= e^{qx}\{1 + \int_0^\infty du\, \psi(u) e^{-qu}\}. \end{aligned} \tag{7.34}$$

This time we only need (7.34) for $x < n$. In fact, (7.34) is true also for $x = n$ and indeed is easier to prove than in the previous case for $x = n$.

Finally we consider the case when 0 is regular for $(0, +\infty)$ but not for $(-\infty, 0)$. Now (7.23) is true and we combine this with (7.27) to deduce

$$e^{qn} \int_{-\infty}^{n} dz\, L_q^*(n-z) u(x-z) = e^{qx}\{\int_0^\infty du\, \psi(u) e^{-qu}\}$$

for $x \geqslant n$. Proposition 6.1 allows us to conclude that e^{qx} is coexcessive for **X** and that the measure l_n defined by (7.21) form an equilibrium system. We conclude exactly as before that e^{qx} is coharmonic. To establish coinvariance we again introduce the approximate Markov process but in place of the subordinator $\mathbf{M}^* \circ \mathbf{B}$ we work with the discrete time process $\{M^*(B_j)\}$ where the B_j, with some convenient labeling, are the times at which $\mathbf{X}$ assumes a new minimum. The crucial argument for establishing coinvariance is exactly analogous to the one given above and we leave the details to the reader.

We turn now to Theorem 3. If 0 is regular for $(-\infty, 0)$ then the measure l_n defined by (7.11) satisfies

$$\int_u^{n+} l_n(dy)\psi^*(y-u) = e^{qu} \qquad \text{for} \quad 0 \leqslant u \leqslant y. \tag{7.35}$$

If 0 is regular also for $(0, +\infty)$ then we plug in (1.18) to get

$$\begin{aligned}
&\int_0^{n+} l_n(dy) G(y, x) \\
&\qquad = \int_0^{n+} l_n(dy) \int_0^{y \wedge x} du\, \psi^*(y-u)\psi(x-u) \\
&\qquad = \int_0^x du\, \psi(x-u) \int_0^{n+} l_n(dy)\psi^*(y-u) \\
&\qquad = \int_0^x du\, \psi(x-u) e^{qu}
\end{aligned}$$

and, with the help of Proposition 6.1, we conclude that the function $g_q(x)$ defined by (1.14) is coharmonic on $(0, +\infty)$. If 0 is not regular for $(0, +\infty)$ then instead we plug in (1.20) and conclude that $e^{qx} + g_q(x)$ is coharmonic on $(0, +\infty)$. If 0 is not regular for $(-\infty, 0)$ then (7.35) is replaced by

$$e^{nq}\left\{L_q^*(n-u) + \int_u^n dy L_q^*(n-y)\psi^*(y-u)\right\} = e^{qu} \tag{7.35$'$}$$

and we plug in (1.19) to get

$$\begin{aligned}
&e^{nq}\int_0^n dy L_q^*(n-y) G(y, x) \\
&\qquad = e^{nq}\int_0^n dy L_q^*(n-y)\psi(x-y) \\
&\qquad\quad + e^{nq}\int_0^n dy L_q^*(n-y)\int_0^{x \wedge y} du\, \psi^*(y-u)\psi(x-u) \\
&\qquad = \int_0^x du\, \psi(x-u)\left\{e^{nq}L_q^*(n-u) + \int_u^n dy L_q^*(n-y)\psi^*(y-u)\right\} \\
&\qquad = \int_0^x du\, \psi(x-u) e^{qu}
\end{aligned}$$

and again we conclude that $g_q(x)$ is coharmonic on $(0, +\infty)$. To prove that the functions identified in Theorem 3 are actually coinvariant on $(0, +\infty)$ we can argue exactly as above when we deduced (ii) in Theorem 5 from (iii) and (iv). The main point is that Theorem 5 allows us to replace (1.13) by (7.3) or (7.20). We leave the details for this to the reader.

8. Proof of Theorem 4. Fix a function $g(y)$ positive and continuous on the full line such that $Gg(x) = \int_0^\infty dy\, G(x, y)g(y)$ is bounded and continuous. It is clear from the general theory of the Martin boundary, as presented, for example, in Kunita and Watanabe (1965), that Theorem 4 will be proved if we can establish the following result.

Suppose that for all f continuous with compact support in $(0, +\infty)$

$$I(f) = \operatorname{Lim} Gf(x_n)/Gg(x_n) \tag{8.1}$$

exists for some fixed sequence $x_n \to 0$ or $x_n \to +\infty$. Then I can be represented

$$I(f) = \int dy\, h(y)f(y) + \int \nu(dz)Gf(z) \tag{8.2}$$

where h is one of the coharmonic functions identified in Theorems 1, 2 or 3 and where ν is a Radon measure on $(0, +\infty)$.

We will check this explicitly only when 0 is regular for both half lines so that (1.18) is valid, and when (1.11) is true. The case $x_n \downarrow 0$ follows immediately from the fact that

$$Gf(x)/\int_0^x du\, \psi^*(x-u) \to \int_0^\infty dy\, \psi(y)f(y)$$

as $x\downarrow 0$. The case $x_n \to \infty$ is slightly more complicated. We can certainly restrict our attention to the case when also

$$J(f) = \operatorname{Lim}\{\int dy\, f(y)\psi^*(y+x_n)/\int dy\, g(y)\psi^*(y+x_n)\}$$

exists and then, since 1 is the only positive harmonic function for a conservative subordinator, it is clear that

$$J(f) = \alpha \int dy\, f(y) + \int \nu(dt)\int dy\, \psi^*(y+t)f(y)$$

for some $\alpha \geqslant 0$ and some Radon measure ν. But for sufficiently large n

$$\begin{aligned} Gf(x_n) &= \int_0^\infty dy\, f(y)\int_0^y dv\, \psi^*(x_n+v-y)\psi(v) \\ &= \int_0^\infty dv\, \psi(v)\int_v^\infty dy\, f(y)\psi^*(x_n+v-y) \\ &= \int_0^\infty dv\, \psi(v)\int_{-\infty}^0 dy\, f(v-y)\psi^*(y+x_n) \end{aligned}$$

and, therefore, since we need only take the v-integral over a bounded interval where $\psi(v)$ is integrable and since the ratios $\{\int_{-\infty}^0 dy\, f(v-y)\psi^*(x_n+y) /\int dy\, g(y)\psi^*(y+x_n)\}$ are uniformly bounded, we conclude from the dominated convergence theorem that

$$\begin{aligned} &\operatorname{Lim}\{Gf(x_n)/\int dy\, g(y)\psi^*(y+x_n)\} \\ &\quad = \int_0^\infty dv\, \psi(v)\{\alpha\int_{-\infty}^0 dy\, f(v-y) + \int \nu(dt)\int_{-\infty}^0 \psi^*(y+t)f(v-y)\} \\ &\quad = \alpha\int_0^\infty dy\, f(y)\int_0^y dv\, \psi(v) + \int \nu(dt)\int_0^\infty dy\, f(y)\int_{-\infty}^{t\wedge y} du\, \psi^*(t-u)\psi(y-u) \end{aligned}$$

and this is good enough. (The function $f(v-y)I(y \leqslant 0)$ may be discontinuous at $y = 0$, but this causes no problem since the relevant limiting measures have no atoms.)

REMARK. It is clear from the example of Dyson, as presented, for example, on page 59 in Chung (1967), that we cannot expect the limit in (8.1) to exist independent of the choice of $x_n \to \infty$, nor can we expect $\nu = 0$ in (8.2). (I am grateful to H. Kesten for pointing this out to me.) Fortunately, this is irrelevant for us since we do not care about controlling the full Martin boundary.

9. The interval (0, 1). In this section we use our results for the half line to classify coharmonic functions on the bounded open interval $\mathbf{I} = (0, 1)$. First we introduce some convenient terminology.

Let $\mathbf{X}^{00}$ be the absorbed process on $\mathbf{I} = (0, 1)$. That is,

$$\begin{aligned} X_t^{00} &= X_t && \text{if } t < \tau \\ &= +\infty && \text{if } t \geqslant \tau \end{aligned}$$

where $\tau = \inf\{t > 0 : X_t \geqslant 1 \text{ or } X_t \leqslant 0\}$. Let h^{00} be positive and coharmonic for $\mathbf{X}^{00}$ and let $(\Omega_\infty, \mathcal{P}_\infty)$ be an approximate Markov process associated with h^{00} as in Section 6. We will say h^{00} is *associated with the boundary point* 0 if $\mathcal{P}_\infty(\mathrm{Lim}_{t\downarrow\zeta^-} X_t^{00} = 1) = 0$ and it is *associated with* 1 if $\mathcal{P}_\infty(\mathrm{Lim}_{t\downarrow\zeta^-} X_t^{00} = 0) = 0$.

The classification is given in

THEOREM 10. (i) *If* 0 *is regular for* $(0, +\infty)$, *then up to multiples there is exactly one positive coharmonic function for* $\mathbf{X}^{00}$ *which is associated with the boundary point* 0.

(ii) *If* 0 *is regular for* $(-\infty, 0)$, *then up to constant multiples there is exactly one positive coharmonic function for* $\mathbf{X}^{00}$ *which is associated with* 1.

(iii) *Every positive coharmonic function is a linear combination of the ones identified in* (i) *and* (ii).

It will be clear from our proof that there is never a nontrivial positive coharmonic function for $\mathbf{X}^{00}$ which is also coinvariant.

If 0 is regular for $(0, +\infty)$, then the argument at the end of Section 3 shows that $h^{00}(x)$ defined modulo null sets by

$$\int dx\, h^{00}(x) f(x) = \mathcal{E}^{\mathrm{ex}} \int_0^{\tau \wedge \sigma} dt\, f(X_t^0) \tag{9.1}$$

where $\tau = \inf\{t > 0 : X_t^0 \geqslant 1\}$, has a version which is coharmonic for $\mathbf{X}^{00}$ and is associated with the boundary point 0. The existence part of Theorem 10 follows from this and a similar argument for the boundary point 1.

With the help of the associated approximate Markov process $(\Omega_\infty, \mathcal{P}_\infty)$ it is easy to see that every positive coharmonic function on (0, 1) is a linear combination of ones associated with 0 and 1. Therefore, to complete the proof of Theorem 10 we need only to consider nontrivial $h^{00}(x)$ positive and coharmonic for $\mathbf{X}^{00}$ and associated with 0, and show that 0 is regular for $(0, +\infty)$ and that up to a constant multiple h is the function determined by (9.1).

Consider first the case when the Levy measure π does not charge (0, 1). Then for $n = 2, 3, \cdots$ and for $1/n < x < 1$, clearly

$$h^{00}(x) = \mathcal{P}_{-x}(\inf\{t > 0 : X_t^* = 1/n\} < \inf\{t > 0 : X_t^* \geqslant 1/n\}) h^{00}(1/n)$$

and the desired uniqueness follows easily. Therefore, we can restrict our attention to the case when

$$\int_0^{1-} \pi(dz) > 0. \tag{9.2}$$

For $n = 2, 3, \cdots$ let $I_n = (1/n, +\infty)$ and let $l_n^{00}(dx)$, $n \geqslant 1$, be the equilibrium system for h^{00} associated with the sequence $I_n \cap \mathbf{I} = (1/n, 1)$. (Since h^{00} is associated with 0 we can be sure that l_n^{00} is defined even though $I_n \cap \mathbf{I}$ does not have compact closure in $\mathbf{I}$.) For a given m the measures

$$l_m^{(n)}(dy) = \int l_n^{00}(dz)P_m(z, dy)$$

increase with n and agree with l_m^{00} on $\bar{I}_m \cap \mathbf{I}$. (Here $P_m^0(x, dy)$ is the hitting measure of I_m relative to $\mathbf{X}^0$, the absorbed process on the full half line $(0, +\infty)$.) Thus

$$l_m^0(dy) = \operatorname{Lim}_{n\uparrow\infty} l_m^{(n)}(dy)$$

is a well-defined measure on $(0, +\infty)$ and $l_m^0 = l_m^{00}$ on $\bar{I}_m \cap \mathbf{I}$. Also for $n > m$

$$\int l_n^0(dz)P_m^0(z, dy) = l_m^0(dy). \tag{9.3}$$

We show next that each l_m^0 is bounded.

Fix $m < n$ and let $G_m(x, y)$ be the Green's function for the interval $(0, 1/m)$. Clearly,

$$\begin{aligned} l_m^{00}(I_m) &\geqslant \int l_n^{00}(dz)\int_0^{1/m} G_m(z, y)dy \int_{(1/m)-y}^{1-y} \pi(du) \\ &\geqslant \int l_n^{00}(dz)\int_0^{1/m} G_m(z, y)dy\ \pi((1/m, 1 - 1/m)), \end{aligned} \tag{9.4}$$

$$\begin{aligned} l_m^{(n)}(I_m) &\leqslant l_{m+1}^{00}(I_{m+1}) \\ &\quad + \int l_n^{00}(dz)\int_0^{1/(m+1)} G_m(z, y)dy\ \pi((1/m - 1/(m + 1), +\infty)). \end{aligned} \tag{9.5}$$

By (9.2) we have $\pi((1/m, 1 - 1/m)) > 0$ for m sufficiently large and so (9.4) and (9.5) give an estimate for $l_{(m)}^n(I_m)$ which is independent of n and this implies that l_m^0 is bounded. Now (9.3) guarantees that l_m^0, $m \geqslant 1$, is an equilibrium system for some function h^0 which is coexcessive for $\mathbf{X}^0$, the absorbed process on the full half line $(0, +\infty)$. If we knew that h^0 was coharmonic for $\mathbf{X}^0$, then by Theorems 1 and 4 we could be sure that 0 is regular for $(0, +\infty)$ and also that $h^0 = \text{const.}\ \psi$ which would be enough to establish uniqueness for h^{00}. (This is because $l_m^0 = l_m^{00}$ on $\bar{I}_m \cap \mathbf{I}$.) Thus by Proposition 6.2 we will be done if we can show that

$$\operatorname{Lim}_{m\uparrow\infty} l_m^0(K) = 0 \tag{9.6}$$

for every compact subset K of $(0, +\infty)$.

Since $l_n^0 = l_n^{00}$ on $\mathbf{I}$ and since $l_n^{00} \to 0$ vaguely on $\mathbf{I}$ by Proposition 6.2 applied to h^{00}, we know that (9.6) is true whenever $K \subset (0, 1)$ and therefore

$$\operatorname{Lim}_{m\uparrow\infty} \operatorname{Lim}_{n\uparrow\infty} \int l_n^{00}(dz)\int_0^{1/m} G_m(z, y)dy\ \pi(K \cap [1/m, 1 - y) = 0$$

for any such K. From (9.2) it then follows that actually

$$\operatorname{Lim}_{m\uparrow\infty} \operatorname{Lim}_{n\uparrow\infty} \int l_n^{00}(dz)\int_0^{1/m} G_m(z, y)dy = 0.$$

But this is enough to imply that

$$\operatorname{Lim}_{m\uparrow\infty} l_m^0([1, \infty)) = 0$$

and we are done.

10. Proof of Theorem 7. Before proving Theorem 7 we establish a preliminary result which gives '*a priori*' control over coexcessive functions h and the measures ν which can occur in (1.26).

PROPOSITION 10.1. *Let $h \geqslant 0$ be coexcessive and locally integrable on* $(0, +\infty)$. *Then h is integrable near* 0 *and the measure ν in* (1.26) *satisfies*

$$\int_0^{1^+} \nu(dx) \int_0^x du\, \psi^*(u) + \int_{3^+}^{\infty} \nu(dx) \int_0^1 du\, \psi^*(x-u) < +\infty. \tag{10.1}$$

Moreover, if 0 *is regular for* $(0, +\infty)$ *but not for* $(-\infty, 0)$, *then* (10.1) *can be replaced by*

$$\int_0^{1^+} \nu(dx) + \int_{3^+}^{\infty} \nu(dx) \int_0^1 du\, \psi^*(x-u) < +\infty. \tag{10.2}$$

PROOF. We know from Theorems 1 through 4 that every coharmonic function is integrable near 0. Therefore, we can assume that $h_0 = 0$ in the representation (1.26). Fix $y > 0$ for which $h(y) < +\infty$. Then surely

$$\int_0^{\frac{1}{2}y} \nu(dx) \int_0^x du\, \psi^*(x-u)\psi(y-u) < +\infty$$

and since $\psi(y-u)$ is bounded away from 0 for $0 < u \leqslant \frac{1}{2}y$, we conclude that the first term in (10.1) converges. Also

$$\int_{3^+}^{\infty} \nu(dx) \int_1^2 dy \int_0^1 du\, \psi^*(x-u)\psi(y-u) < +\infty,$$

and from this we conclude that the second term in (10.1) converges. If 0 is regular for $(0, +\infty)$ but not for $(-\infty, 0)$, then also

$$\int_0^{1^+} \nu(dx)\psi(y-x) < +\infty$$

for some $y > 1$ and (10.2) follows from this. Finally, the estimates (10.1) and (10.2) immediately imply

$$\int_0^{\infty} \nu(dx) \int_0^1 dy\, G(x,y) < +\infty$$

(that is, integrability near 0) and we are done. ▯

Now we are ready to prove Theorem 7. Observe first that if $\varphi \in C_{\text{com}}^{\infty}(0, +\infty)$ then $\varphi(X_t) - \int_0^t ds\, \mathcal{Q}\varphi(X_s)$ is a martingale relative to any $\mathcal{P}_x$ and from this it follows that φ has the representation

$$\varphi(x) = -\int G(x,y)\mathcal{Q}\varphi(y)dy. \tag{10.3}$$

Now let coexcessive and locally integrable h have the representation (1.26), let $E_n = (1/n, n)$ for $n \geqslant 2$ and let l_n, $n \geqslant 2$, be the associated equilibrium system. (This exists by Theorem 2.8 on page 271 in B/G.) Then, by (10.3) and Proposition 6.4, we have, for any such φ,

$$\operatorname{Lim}_{n\uparrow\infty} \int l_n(dx) \int dy\, G(x,y)\mathcal{Q}\varphi(y) = -\int \nu(dx)\varphi(x) \tag{10.4}$$

where, for now, the integral on the left is understood in the iterated sense. For any

$\varphi \in C^{\infty}_{\text{com}}(0, +\infty)$ the function

$$(10.5) \quad \mathcal{Q}_1\varphi(y) = a\varphi''(y) + b\varphi'(y) + \int \pi(dz)I(|z| \leqslant 1)\{\varphi(y+z) - \varphi(y) + \varphi'(y)\sin z\} - \varphi(y)\int \pi(dz)I(|z| > 1) + \varphi'(y)\int \pi(dz)I(|z| > 1)\sin z$$

is bounded and has compact support in **R**. By Proposition 10.1 the function h is integrable near 0 and so we can apply the dominated convergence theorem to get

$$(10.6) \quad \operatorname{Lim}_{n\uparrow\infty}\int l_n(dx)\int dy\; G(x,y)\mathcal{Q}_1\varphi(y) = \int dy\; h(y)\mathcal{Q}_1\varphi(y).$$

Now (10.4) and (10.6) together guarantee that also

$$\operatorname{Lim}_{n\uparrow\infty}\int l_n(dx)\int dy\; G(x,y)\int \pi(dz)I(|z| > 1)\varphi(y+z)$$

exists and is finite. Putting all this together, we conclude that for $\varphi \geqslant 0$ in $C^{\infty}_{\text{com}}(0, +\infty)$ the product $h(x)\mathcal{Q}\varphi(x)$ is integrable and (1.27) is valid. The restriction on φ is easy to remove, and we have proved the direct part of Theorem 7.

Now let $h \geqslant 0$ be locally integrable on $(0, +\infty)$ and suppose that $h(y)\mathcal{Q}\varphi(y)$ is integrable and (1.27) is valid for all $\varphi \in C^{\infty}_{\text{com}}(0, +\infty)$. Fix $\varepsilon, R > 0$ with $R > 5\varepsilon$ and define $\mathcal{Q}_\varepsilon\varphi(y)$ as in (10.5) except that 1 is replaced by ε. Clearly, $\mathcal{Q}_\varepsilon\varphi(y)$ is bounded and has compact support in $(0, +\infty)$ when φ is supported in $(2\varepsilon, R+\varepsilon)$. Thus local integrability of $h(y)$ on $(0, +\infty)$ guarantees integrability of $h(y)\mathcal{Q}_\varepsilon\varphi(y)$. Since also $h(y)\mathcal{Q}\varphi(y)$ is integrable we conclude that $h(y)\int \pi(dz)I(|z| > \varepsilon)\varphi(y+z)$ is integrable for all such φ and this guarantees

$$(10.7) \quad \int_0^\infty dy\; h(y)\int \pi(dz)I(|z| > \varepsilon)I(3\varepsilon \leqslant y+z \leqslant R) < +\infty.$$

If φ is supported in $(4\varepsilon, R-\varepsilon)$ then (1.27) implies

$$(10.8) \quad \int_0^\infty dy\; h(y+x)\mathcal{Q}\varphi(y) = -\int \nu(dy)\varphi(y-x)$$

whenever $|x| \leqslant \varepsilon$. Let $\eta \geqslant 0$ be C^∞ with support contained in $(-\varepsilon, +\varepsilon)$. With the help of (10.7) we can multiply both sides of (10.8) by $\eta(x)$ and integrate with respect to x to get

$$(10.9) \quad \int_0^\infty dy\; h_\varepsilon(y)\mathcal{Q}\varphi(y) = -\int dy\; \nu_\varepsilon(y)\varphi(y)$$

where

$$h_\varepsilon(y) = \int dx\; \eta(x)h(x+y); \; \nu_\varepsilon(y) = \int \nu(dx)I(|x| > \varepsilon)\eta(x-y).$$

Clearly, $h_\varepsilon(y)$ is smooth and satisfies

$$(10.10) \quad \int_0^\infty dy\; h_\varepsilon(y)\int \pi(dz)I(|z| > \varepsilon)I(4\varepsilon \leqslant y+z \leqslant R-\varepsilon) < +\infty$$

and, therefore,

$$\mathcal{Q}^*h_\varepsilon(y) = ah_\varepsilon''(y) - bh_\varepsilon'(y) + \int \pi(dz)\{h_\varepsilon(y-z) - h_\varepsilon(y) + h_\varepsilon'(y)\sin z\}$$

is well defined for $4\varepsilon \leqslant y \leqslant R-\varepsilon$. Moreover, it follows from (10.9) after varying φ that $\mathcal{Q}^*h_\varepsilon(y) = -\nu_\varepsilon(y)$ for $4\varepsilon < y < R-\varepsilon$. Now fix D open with compact support contained in $(4\varepsilon, R-\varepsilon)$ and let $M = (0, +\infty) \setminus D$. Then

$$h_\varepsilon[X^*(T_M^* \wedge t)] + \int_0^{t\wedge T_M^*} ds\; \nu_\varepsilon(X_s^*), \qquad t \geqslant 0,$$

is a martingale relative to $\mathcal{P}_x$ for any $x \in D$. (Use the fact that $\theta h_\varepsilon(X_t^*) - \int_0^t ds\, \mathcal{Q}^*(\theta h_\varepsilon)(X_s^*)$, $t \geq 0$, is a martingale for $\theta \geq 0$ in $C_{\text{com}}^\infty(\mathbf{R})$ such that $\theta = 1$ on a neighborhood of $[4\varepsilon, R - \varepsilon]$, truncate at time T_M^*, and pass to the limit in θ with the help of (10.10).) In particular, h_ε is coexcessive on D. Letting η run through an approximation to the identity, we conclude that h has a version which is coexcessive on D and Theorem 7 follows upon varying D.

We finish by showing how Theorem 7 can be combined with some simple Fourier analysis to give an independent proof of Theorem 1(i) and also to prove (1.10).

Suppose first that 0 is regular for both $(0, +\infty)$ and $(-\infty, 0)$. The formula (2.29) and its dual immediately imply

$$\Psi(\xi) = \varphi(0, -i\xi)\varphi^*(0, i\xi). \tag{10.11}$$

For $\eta \in C_{\text{com}}^\infty(0, +\infty)$ define the Fourier transform $\eta\hat{}(\xi)$ by

$$\eta\hat{}(\xi) = \int dx\, e^{-ix\xi}\eta(x).$$

Extending the definition of the Fourier transform to tempered distributions in the usual way, we get

$$(\mathcal{Q}\eta)\hat{}(\xi) = -\Psi(\xi)\eta\hat{}(\xi)$$

and, since by (2.11) and (1.9),

$$\psi\hat{}(\xi) = 1/\varphi(0, i\xi) \tag{10.12}$$

we get

$$\begin{aligned} \int dx\, \psi(x)\mathcal{Q}\eta(x) &= (2\pi)^{-1}\int d\xi \{1/\varphi(0, i\xi)\}^- (\mathcal{Q}\eta)\hat{}(\xi) \\ &= -(2\pi)^{-1}\int d\xi\, \varphi^*(0, i\xi)\eta\hat{}(\xi) \\ &= -b^*\eta'(0) + c^*\eta(0) - \int_0^\infty l^*(dz)\{\eta(-z) - \eta(0)\} \\ &= 0 \end{aligned} \tag{10.13}$$

and Theorem 7 guarantees that $\psi(x)$ is coharmonic on $(0, +\infty)$.

Essentially the same argument works if 0 is regular for $(0, +\infty)$ but not $(-\infty, 0)$.

If 0 is regular for $(-\infty, 0)$ but not $(0, +\infty)$ then (10.11) must be replaced by

$$\Psi(\xi) = \{1 - \Phi(0, -i\xi)\}\varphi^*(0, i\xi), \tag{10.14}$$

(10.12) must be replaced by

$$\psi\hat{}(\xi) = \{1 - \Phi(0, i\xi)\}^{-1} - 1 \tag{10.15}$$

and the calculation (10.13) is replaced by

$$\begin{aligned} \int dx\, \psi(x)\mathcal{Q}\eta(x) &= -(2\pi)^{-1}\int d\xi \{\varphi^*(0, i\xi) - \Psi(\xi)\}\eta\hat{}(\xi) \\ &= -\mathcal{Q}\eta(0) \\ &= -\int_0^\infty \pi(dz)\eta(z) \end{aligned} \tag{10.16}$$

and (1.10) follows from Theorem 7 and Theorem 1(ii).

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