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A COMPOUND POISSON LIMIT FOR STATIONARY SUMS, AND SOJOURNS OF GAUSSIAN PROCESSES[1]

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Let $\{X_{n,j} : j = 1, \cdots, n, n > 1\}$ be an array of nonnegative random variables in which each row forms a (finite) stationary sequence. The main theorem states sufficient conditions for the convergence of the distribution of the row sum $\Sigma_j X_{n,j}$ to a compound Poisson distribution for $n \to \infty$. This is applied to a stationary Gaussian process: it is shown that under certain general conditions the time spent by the sample function $X(s)$, $0 < s < t$, above the level u may be represented as a row sum in a stationary array, and so has, for t and $u \to \infty$, a limiting compound Poisson distribution. A second result is an extension to the case of a bivariate array. Sufficient conditions are given for the asymptotic independence of the component row sums. This is applied to the times spent by $X(s)$ above u and below $-u$.

0. Introduction. The purpose of this paper is to present a limit theorem for sums of nonnegative, stationary random variables, and apply it to the distribution of the time spent by the sample function of a stationary Gaussian process above a high level over a long time interval. The theorem states sufficient conditions for the convergence of the distribution of the sum of stationary random variables to a compound Poisson distribution. It is applied to the Gaussian process by showing that the occupation time of the set above the high level can be decomposed into a suitable sum of stationary random variables.

Limit theorems for the extreme values in a sequence of independent random variables are based, in part, on the Poisson limit theorem for sums of independent, Bernoulli random variables whose expected sum is relatively constant. Indeed, if $\{X_n\}$ is a sequence of independent random variables with the common distribution function $F(x)$, and if $\{u_n\}$ is a nondecreasing sequence of real numbers such that $1 - F(u_n) = 1/n$, then the number of random variables X_j such that $X_j > u_n$, $j = 1, \cdots, n$, has, for $n \to \infty$, a limiting Poisson distribution with mean 1. The limiting distributions of the extreme order statistics can then be derived from this result. (See, for example, [13], page 371.)

In recent years there has been much research on the extension of the Poisson limit theorem from independent to dependent Bernoulli random variables; see, for example, [12], [16], [26], [38], [40]. This has been accompanied by, and stimulated

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by the corresponding extension of the theory of extreme values from independent to dependent random variables [3], [4], [5], [14], [15], [19], [25], [31], [32], [37], [41], [42], [43]. A recent comprehensive review is contained in the monograph by Galambos [17].

The first result in our work is a generalization of the convergence to a Poisson distribution to the more general compound Poisson distribution. The latter is one whose Laplace-Stieltjes transform is of the form $\Omega(s) = \exp\{\int_0^\infty (1 - e^{-sx})\, dH(x)\}$ for some nonincreasing function H. Four conditions are put on the distributions of the summands. The first involves only the marginal distributions, and is a necessary and sufficient condition for convergence in the case of independent summands. The other three conditions constitute the mixing hypothesis which ensures the weakness of the dependence among terms which are mutually distant in the time index set. The mixing hypothesis is slightly different from others which have been used in the Poisson context. The first of the three mixing conditions is a local mixing condition: it implies that significant contributions to the sum can arise only from terms whose indices differ by a relatively large time quantity. It is a generalization of the last condition in Meyer's theorem [26] and of Leadbetter's condition $D(u_n')$ [19]. The last two conditions involve global mixing, placing an asymptotic independence condition on the finite-dimensional distributions of the random variables of the array whose indices are widely separated. It involves only the k-dimensional distributions for fixed but arbitrary $k > 1$. This mixing condition is slightly different from others of recent use, but appears to be easier to check when the process is specified by its finite-dimensional distributions (compare to Meyer's mixing condition [26] or Leadbetter's conditions $D(u_n)$ [20], [21]; recent work in this area has also been done by Serfozo [39].)

The main application of the Poisson limit theorem for dependent random variables has been to the distribution of the number of crossings of high level u by the sample function of a stochastic process $X(t)$, $0 < t < T$, where u and T both become infinite. This has been done in much detail for stationary Gaussian processes e.g., [2], [8]; however, it has been observed (see, for example, Leadbetter [19]) that the limiting Poisson distribution of the number of crossings holds for a general class of stationary stochastic processes, and not only Gaussian processes. The crossings theorem is based on the following considerations. It is customary to deal with the upcrossings alone. As u increases with the length T of the time interval, the upcrossings tend to become widely separated in time provided that there is a finite expected number in each interval. If $X(t)$ has a suitable mixing property, then the occurrences of upcrossings in widely separated intervals are asymptotically independent events. Then the Poisson theorem for dependent random variables implies that the number of upcrossings has a limiting Poisson distribution. The result has also been extended to "epsilon-upcrossings" for certain nondifferentiable processes where there is an infinite number of expected ordinary upcrossings in every interval (see Pickands [33]).

The main application of our theorem is to the limiting distribution of the time spent by the sample function in the set (u, ∞). We say that a sojourn above u begins at time t if $X(s) \leqslant u$ for all s in some nondegenerate interval with right endpoint t, and the set $\{s : X(s) > u\}$ has an intersection of positive measure with every nondegenerate interval with left endpoint t. An upcrossing of the level u marks the beginning of a sojourn. If there is a finite number of expected upcrossings in each interval, then the number of sojourns above u is the same as the number of upcrossings. As the level u increases, the duration of the sojourn tends to be small; however, when properly normalized, the duration has a limiting distribution, which is defined as a conditional limiting distribution. (The various definitions of this conditioning are discussed by Kac and Slepian [18], Cramer and Leadbetter [14], Chapter 11, and Berman [6].) It follows from the reasoning used in the proof of the Poisson limit for the distribution of upcrossings, that the time spent above u is the sum of a random number of nearly independent random variables (the durations of the sojourns) and where the random number is nearly Poisson distributed. This leads to the compound Poisson limit distribution.

The first result of this type is due to Volkonskii and Rozanov [41], who proved it under strong conditions on the local behavior of the covariance function, and under a strong mixing condition. These hypotheses were weakened by Cramér and Leadbetter [14] page 278; however, they still maintained the assumption of a finite expected number of upcrossings in each interval. The first step away from the identification of the sojourns with the upcrossings was taken by Berman [7], who considered a larger class of processes, with sample functions which were not necessarily differentiable, and with a possibly infinite expected number of upcrossings in each finite interval.

The purpose of this work is to extend the latter in several directions:

1. We show that the limit theorem for the sojourn time is a special case of a general compound Poisson convergence theorem for stationary sequences. This opens up the possibility of proving sojourn time theorems not only for Gaussian processes, but also for more general stationary processes, just as Leadbetter [18] has done for the Poisson limit for upcrossings.

2. In the last section we present a limit theorem for the bivariate sums taken from a bivariate stationary array, and apply it to the distribution of the times spent by the sample function of a stationary Gaussian process above a high level and below a low (negative) level, respectively. The theorem states sufficient conditions for the asymptotic independence of the two sums and with compound Poisson marginal distributions. The application to Gaussian processes represents an extension of Berman's [8] earlier result for the upcrossings and downcrossings of high and low levels. Later work in this area has also been done by Lindgren [22] and Lindgren, de Mare, and Rootzen [23].

3. The major application of the compound Poisson limit theorem in this paper is in the stationary Gaussian case, and so we wish to indicate how the results

obtained here are not only different from those obtained earlier, but are also more satisfactory. The mixing condition which is assumed here is that the covariance function $r(t)$ satisfies $r(t)\log t \to 0$ for $t \to \infty$. This is one of two conditions introduced by Berman [5] in the case of Gaussian sequences, and which has also been used in the continuous time case by many authors; for example, [5], [21], [22], [23], [27], [28], [30], [33], [34], [35], [36], [44]. However the method used in [7] was based on a mixing condition stated in terms of the mean smoothness of the spectral density function. It has not been used in subsequent work in this area, and we have not been able to directly compare this condition with the one mentioned above. For this reason, the earlier result on the sojourn time in [7] was mathematically isolated from other work in this area, and the current effort returns it to the context of naturally related theorems.

Finally we comment on the other condition of the mixing type which was introduced by the author in [5] in the discrete time case, namely, $\Sigma r_n^2 < \infty$, and used by him in [7] and in the continuous time case in the form $r(t) \in L_2(-\infty, \infty)$. Mittal kindly indicated to me that the argument used in [8], page 935, and again in [7], page 77, was incorrect, so that there was a gap in the derivation of the resulting limit theorems based on the above condition on r, including the sojourn time theorem. However, Mittal [29] and also Leadbetter, Lindgren and Rootzen [21] have fortunately reestablished and even improved the condition on r, so that the sojourn time theorem in [7] is valid under this mixing condition, and the additional condition on the spectral density mentioned there.

1. Assumptions for the limit theorem. Let $\{X_{n,j} : j = 1, \cdots, n, n \geqslant 1\}$ be a triangular array of nonnegative random variables. We assume that the n random variables $X_{n,j}, j = 1, \cdots, n$, form a stationary (though finite) sequence for each n. We make the following four assumptions about the array:

ASSUMPTION (I). *There is a nonincreasing function $H(x)$ such that*

$$\lim_{x\to\infty} H(x) = 0 \tag{1.1}$$

$$\int_0^1 x\, dH(x) > -\infty, \tag{1.2}$$

and such that

$$\lim_{n\to\infty} n\int_0^y x\, dP(X_{n,1} > x) = \int_0^y x\, dH(x) \tag{1.3}$$

at every point y of continuity of the limiting function. We also assume that $\int_1^\infty x\, dH(x) > -\infty$, and that (1.3) *holds for $y = \infty$, that is,*

$$\lim_{n\to\infty} nEX_{n,1} = -\int_0^\infty x\, dH(x) < \infty. \tag{1.4}$$

Condition (1.3) is necessary and sufficient for the convergence of the distribution of the sum to the compound Poisson distribution with H satisfying (1.1) and (1.2) when the summands are mutually independent [24], page 311. Later we will show that the theorem can be modified not to require (1.4) or even the finiteness of $EX_{n,1}$.

In the following assumption we require the finiteness of $EX_{n,i}X_{n,j}$ for $i \neq j$; and in the assumption (IV) below we also require the finiteness of $EX_{n,i_1} \cdots X_{n,i_k}$ for $i_1 < \cdots < i_k$. These requirements will be dropped in a later version of the theorem.

ASSUMPTION (II).

$$\lim_{k\to\infty}\limsup_{n\to\infty}\left[k\Sigma_{1\leqslant i<j\leqslant[n/k]}EX_{n,i}X_{n,j}\right] = 0. \tag{1.5}$$

The latter is a "local mixing" condition and is satisfied in the case where the summands are independent, because, by (1.4), $EX_{n,i}X_{n,j} = E^2X_{n,1} = 0(n^{-2})$.

ASSUMPTION (III). *For each $k \geqslant 2$, and positive number q,*

$$\lim_{n\to\infty}\sup_{1\leqslant j_1<\cdots<j_k\leqslant n,\ \min(j_{h+1}-j_h,\ 1<h<k)>qn} \left|\frac{P(X_{n,j_1} > x_1, \cdots, X_{n,j_k} > x_k)}{P(X_{n,j_1} > x_1) \cdots P(X_{n,j_k} > x_k)} - 1\right| = 0 \tag{1.6}$$

for every k-tuple of x's such that $H(x_i) > 0$ and x_i is a point of continuity of H, $i = 1, \cdots, k$. The latter is the first of our two "global mixing" conditions. It states that k random variables whose indices differ mutually by at least a positive multiple of n have a joint distribution which, for $n \to \infty$, is asymptotic to the product of the marginal distributions.

ASSUMPTION (IV). *For each $k \geqslant 2$, and positive number q, and each integer h, $1 < h < k$,*

$$\lim_{n\to\infty}\sup_{1\leqslant j_1<\cdots<j_k\leqslant n,\ j_h-j_{h-1}>qn} \left|\frac{EX_{n,j_1}\cdots X_{n,j_k}}{EX_{n,j_1}\cdots X_{n,j_{h-1}}EX_{n,j_h}\cdots X_{n,j_k}} - 1\right| = 0. \tag{1.7}$$

The latter is the second of the two global mixing conditions. On the one hand (1.7) requires the existence of moments and is a condition on the moments but not on the finite-dimensional distributions, as in (1.6). On the other hand the factorization in (1.7) is stronger than that in (1.6): In the latter we require factorization only if all k indices are sufficiently separated, but in the former we require factorization of the product moments even when subsets of the indices are separated.

There are several immediate consequences of these four assumptions. First of all, it follows directly from (1.3) by means of the weak compactness argument in the proof of Theorem 6.1 of [11] that

$$\lim_{n\to\infty} nP(X_{n,1} > x) = H(x) \tag{1.8}$$

on the continuity set of H. Since, by stationarity, the indices of the random variables in the denominator in (1.6) may be all replaced by $(n, 1)$, (1.8) implies that (1.6) is equivalent to

$$\lim_{n\to\infty}\sup_{1\leqslant j_1<\cdots<j_k\leqslant n,\ \min_h(j_{h+1}-j_h)>qn} |n^kP(X_{n,j_1} > x_1, \cdots, X_{n,j_k} > x_k) - H(x_1)\cdots H(x_k)| = 0. \tag{1.9}$$

Assumption (IV) has an extension which is formally stronger than itself. If $C_1, \cdots, C_k$ are sets of indices of fixed finite sizes which are mutually separated by at least qn time units, then assumption (IV) implies

$$(1.10) \qquad \lim_{n\to\infty} \sup_{C_1 \cdots C_k} \left| \frac{E\{\Pi_{j_1 \in C_1} X_{n,j_1} \cdots \Pi_{j_k \in C_k} X_{n,j_k}\}}{E\{\Pi_{j_1 \in C_1} X_{n,j_1}\} \cdots E\{\Pi_{j_k \in C_k} X_{n,j_k}\}} - 1 \right| = 0.$$

This is obtained by the successive application of (1.7) to the separated index sets.

2. Statement and proof of the compound Poisson limit theorem.

THEOREM 2.1. *Under assumptions* (I) – (IV) *above, the distribution of* $\Sigma_{j=1}^n X_{n,j}$ *converges for* $n \to \infty$ *to the distribution with the Laplace-Stieltjes transform*

$$(2.1) \qquad \Omega(s) = \exp\left[\int_0^\infty (1 - e^{-sx})\, dH(x)\right].$$

The proof of the theorem will begin with several lemmas which demonstrate that the distribution of the sum can be approximated by that of the sum of certain random variables forming a stationary array but which assume only finitely many values. Let $x_1 < \cdots < x_m$ be an arbitrary finite set of points of continuity of $H(x)$ in $(0, \infty)$. For each X in the array we define the function of X,

$$(2.2) \qquad Y = \Sigma_{j=1}^{m-1} x_j I_{[x_j < X < x_{j+1}]} + x_m I_{[x_m < X]},$$

where I is the indicator random variable, and put $X^c = \min(X, c)$.

LEMMA 2.1. *If* $c > 0$ *is a point of continuity of* H, *then*

$$(2.3) \qquad \limsup_{n\to\infty} |E(\exp(-s\Sigma_{j=1}^n X_{n,j}^c)) - E(\exp(-s\Sigma_{j=1}^n X_{n,j}))| \leqslant H(c).$$

PROOF. Since $X^c \leqslant X$, the absolute difference between the exponentials is at most

$$1 - \exp(-s\Sigma_{j=1}^n (X_{n,j} - X_{n,j}^c)).$$

This is at most equal to 1, and is positive only if some $X_{n,j}$ exceeds c; hence, the expected value of the expression displayed above is at most equal to

$$\Sigma_{j=1}^n P(X_{n,j} > c) = nP(X_{n,1} > c) \to H(c).$$

Let H^* be the distribution function tail obtained from H by the operation which is parallel to the transformation from X to Y:

$$(2.4) \qquad \begin{aligned} H^*(x) &= H(x_1), & 0 < x \leqslant x_1 \\ &= H(x_i), & x_{i-1} < x \leqslant x_i \\ &= 0, & x_m < x. \end{aligned}$$

LEMMA 2.2. *Let the array* $\{Y_{n,j}\}$ *be obtained from the array* $\{X_{n,j}\}$ *by means of the transformation* (2.2); *and put* $c = x_m$; *then*

$$(2.5) \qquad \lim_{n\to\infty} E|\Sigma_{j=1}^n X_{n,j}^c - \Sigma_{j=1}^n Y_{n,j}| = -\int_0^c x\, d(H(x) - H^*(x)).$$

PROOF. According to (2.2) we have $X_{n,j}^c \geqslant Y_{n,j}$, so that the absolute value signs may be removed from the left hand expression in (2.5), and then the expectation may be taken term by term. The relation (2.5) follows from assumption (I), formula (1.7), and the fact that the x's are points of continuity.

LEMMA 2.3. *Let the array* $\{Y_{n,j}\}$ *be obtained from the array* $\{X_{n,j}\}$ *by means of* (2.2). *If assumptions* (I), (II), *or* (III) *hold for the latter array, then the corresponding assumptions hold also for the former array with* H^* *in place of* H. *Furthermore, the limit relation* (1.6) *remains valid if the ratio of the probabilities for the* X's *is replaced by the ratio for the* Y's *with* $=$ *in place of* $>$:

$$\frac{P(Y_{n,j_i} = x_{l_i}, i = 1, \cdots, k)}{\Pi_{i=1}^k P(Y_{n,j_i} = x_{l_i})}.$$

PROOF. This is a consequence of (1.9), and the facts that Y defined in (2.2) is a nonincreasing function of X, that $X \geqslant Y$, and that the x's are points of continuity of H.

The proof of the theorem depends on the decomposition of the sum into nearly independent subsums, as is customary in proofs of limit theorems for strongly mixing sequences of random variables. For an arbitrary but fixed integer $k \geqslant 2$, decompose the variable index set $(1, \cdots, n)$ into consecutive ordered subsets $C_1, \cdots, C_k$: C_1 consists of the first $[n/k]$ integers, C_2 consists of the remaining integers up to index $[2n/k]$, etc. Let p and q be arbitrary positive numbers such that $p + q = 1$. For each j, decompose C_j into two subsets: A_j, consisting of the first $[pn/k]$ integers in C_j, and its complement B_j (which contains roughly $[qn/k]$ members). This yields the decomposition of the index set into $2k$ consecutive subsets $A_1, B_1, \cdots, A_k, B_k$, where each A_j has $[pn/k]$ members, and each B_j has approximately $[qn/k]$.

In the following calculations we consider the partial sums of the arrays $\{X_{n,j}\}$ and $\{Y_{n,j}\}$ over the subsets A_h, $h = 1, \cdots, k$ of indices. For simplicity we write

$$\Sigma_{A_h} Y \text{ and } \Sigma_{A_h} X$$

to represent

$$\Sigma_{j:j \in A_h} Y_{n,j} \text{ and } \Sigma_{j:j \in A_h} X_{n,j}, \text{ respectively.}$$

Put

$$W_h = \text{Indicator of the event } \{Y_{n,j} > 0 \text{ for at most one index } j \text{ in } A_h\}.$$

By (2.2) the event $Y_{n,j} > 0$ is equivalent to the event $X_{n,j} \geqslant x_1$. Therefore,

$$\begin{aligned} E(1 - \Pi_{h=1}^k W_h) &\leqslant \Sigma_{h=1}^k \Sigma_{i,j \in A_h, i \neq j} P(X_{n,i} \geqslant x_1, X_{n,j} \geqslant x_1) \\ &\leqslant (k/x_1^2) \Sigma_{i,j \in A_1, i \neq j} E X_{n,i} X_{n,j}, \end{aligned}$$

which, by (1.5), converges to 0 for $n \to \infty$ and then $k \to \infty$; thus,

$$\lim_{k\to\infty} \limsup_{n\to\infty} E(1 - \Pi_{h=1}^k W_h) = 0. \tag{2.6}$$

We prove the theorem first for the partial sums of the Y-array over A-sets of indices; a preliminary result is:

LEMMA 2.4. *Under assumptions* (I) *through* (IV):

$$\text{(2.7)}\quad \lim_{k\to\infty}\lim_{n\to\infty} E\Pi_{h=1}^{k}\{1+\Sigma_{A_h}[\exp(-sY_{n,j})-1]\} = \exp[p\textstyle\int_0^\infty (1-e^{-sx})\,dH^*(x)]$$

for every $s>0$, *where* H^* *is defined by* (2.4).

PROOF. The aim of the proof is to show that in passing to the limit over n we may take the expectation on the left hand side of (2.7) under the product sign, and then evaluate the product of the expected values of the factors.

The product in (2.7) may be expanded as a sum whose first term is 1, and whose typical subsequent term is of the form

$$\Sigma_{j_1\in A_1}\cdots\Sigma_{j_\alpha\in A_\alpha}\Pi_{h=1}^{\alpha}[\exp(-sY_{n,j_h})-1],$$

where α is an integer, $1\leqslant\alpha\leqslant k$. Take the expectation:

$$\text{(2.8)}\qquad \Sigma_{j_1\in A_1}\cdots\Sigma_{j_\alpha\in A_\alpha}E\Pi_{h=1}^{\alpha}[\exp(-sY_{n,j_h})-1].$$

The expectation of the product is a linear combination of probabilities $P(Y_{n,j_1}=x_{i_1},\cdots,Y_{n,j_\alpha}=x_{i_\alpha})$ with coefficients $\Pi_h[\exp(-sx_{j_h})-1]$. By assumption (III) and Lemma 2.3 the probabilities above are uniformly (in $j_h\in A_h$, $h=1,\cdots,\alpha$) asymptotic to the products of the marginal probabilities of the Y's. Therefore (2.8) is unchanged in the limit over n if the expectation is carried under the product sign. Therefore the same is true of the expected product in (2.7).

The proof is completed by computing the expected values of the factors in (2.7). By assumption (I) and Lemma 2.3:

$$\begin{aligned} E\{1+\Sigma_{A_h}[\exp(-sY_{n,j})-1]\} &= 1+\Sigma_{A_h}\Sigma_{i=1}^{m}(e^{-sx_i}-1)P(Y_{n,j}=x_i)\\ &\to 1+(p/k)\textstyle\int_0^\infty(1-e^{-sx})\,dH^*(x),\qquad \text{for } n\to\infty. \end{aligned}$$

The limit is the same for all k factors, and so the product of the factors is simply the kth power. The latter converges to the expression on the right-hand side of (2.7) as $k\to\infty$.

In the next lemma we consider the square of the random variable appearing under the expectation sign in (2.7).

LEMMA 2.5. *For every* $s>0$,

$$\text{(2.9)}\quad \limsup_{k\to\infty}\limsup_{n\to\infty} E\Pi_{h=1}^{k}\{1+\Sigma_{A_h}[\exp(-sY_{n,j})-1]\}^2<\infty.$$

PROOF. By virtue of the inequality $|e^{-x}-1|\leqslant x$, for $x>0$, it suffices to consider

$$E\Pi_{h=1}^{k}\{1+s\Sigma_{A_h}Y\}^2$$

in the place of the expectation in (2.9). Expand the square under the product sign:

$$E\Pi_{h=1}^{k}\{1 + 2s\Sigma_{A_h}Y + s^2\Sigma_{A_h}Y^2 + s^2\Sigma_{i,j\in A_h,\, i\neq j}Y_{n,i}Y_{n,j}\}.$$

Since the Y's are uniformly bounded by their largest value $x_m = c$, the expectation above cannot decrease if Y^2 is replaced by cY. Furthermore, the expectation cannot decrease if each $Y_{n,j}$ is replaced by $X_{n,j}$. Therefore it suffices to estimate

$$E\Pi_{h=1}^{k}\{1 + (2s + s^2c)\Sigma_{A_h}X + s^2\Sigma_{i,j\in A_h,\, i\neq j}X_{n,i}X_{n,j}\}. \tag{2.10}$$

Assumption (IV) (on the X's) permits us to take the expectation under the product sign in (2.10) before passing to the limit over n. The reasoning is the same as that permitting the interchange of product and expectation in (2.7); however, here we use the mixing condition (1.10) on the moments of $X_{n,j}$ in the place of the mixing condition (III) on the finite dimensional distributions of $Y_{n,j}$. To show how (1.10) is applied to (2.10) we consider the particular case $k = 2$ and note that the argument is directly extendable to $k > 2$. In the former case we note that (2.10) is equal to

$$\begin{aligned} E\Big\{1 &+ (2s + s^2c)(\Sigma_{A_1}X + \Sigma_{A2}X) + s^2\Sigma_{k=1}^{2}\Sigma_{i,j\in A_k i\neq j}X_{n,i}X_{n,j} \\ &+ (2s + s^2c)^2\Sigma_{A_1}X\Sigma_{A_2}X + s^4\Pi_{k=1}^{2}\Sigma_{i,j\in A_k i\neq j}X_{n,i}X_{n,j} \\ &+ (2s + s^2c)s^2\Sigma_{A_1}X\Sigma_{i,j\in A_2 i\neq j}X_{n,i}X_{n,j} \\ &+ (2s + s^2c)s^2\Sigma_{A_2}X\Sigma_{i,j\in A_1 i\neq j}X_{n,i}X_{n,j}\Big\}. \end{aligned}$$

Upon the taking of expectation under the signs of summations, we find that condition (1.10) permits the asymptotic factorization of the expectation of the products of random variables into products of expectations of random variables from the index sets A_1 and A_2, respectively; for example, if i and $j (i \neq j)$ belong to A_1, and k belongs to A_2, then $EX_{n,i}X_{n,j}X_{n,k} \sim EX_{n,i}X_{n,j}EX_{n,k}$. This implies that (2.10) is asymptotic to the product of expectations for $k = 2$. The calculations above are directly extendable to $k > 2$.

To complete the proof of the lemma we compute the expected values of the factors in (2.10):

$$\begin{aligned} \limsup_{n\to\infty} E\{1 + (2s + s^2c)\Sigma_{A_h}X + s^2\Sigma_{i,j\in A_h i\neq j}X_{n,i}X_{n,j}\} \\ = 1 + (2s + s^2c)(p/k)\left[-\textstyle\int_0^\infty x\, dH(x)\right] \\ + s^2\limsup_{n\to\infty}\Sigma_{i,j\in A_h,\, i\neq j}EX_{n,i}X_{n,j} \qquad \text{by (I).} \end{aligned}$$

By stationarity these k expressions are identical, and their kth powers are, by assumption (II), of the form

$$\left(1 + k^{-1}\mathit{constant} + o(k^{-1}\right)^k$$

and this converges to $\exp(\mathit{constant})$ for $k \to \infty$.

Having completed the preliminary calculations we turn to the main ideas of the proof of the theorem for the array $\{Y_{n,j}\}$. The Laplace-Stieltjes transform of the distribution of $\Sigma_h\Sigma_{A_h}Y$,

$$E\exp(-s\Sigma_h\Sigma_{A_h}Y), \qquad s > 0,$$

has the decomposition

$$E(1 - \Pi_{h=1}^k W_h)\exp(-s\Sigma_h\Sigma_{A_h}Y) + E\Pi_{h=1}^k\left[W_h\exp(-s\Sigma_{A_h}Y)\right]. \tag{2.11}$$

According to (2.6) the first term in (2.11) has the limit 0. The second term in (2.11) is representable as

$$E\Pi_{h=1}^k W_h\left(1 + \Sigma_{A_h}\left[\exp(-sY_{n,j}) - 1\right]\right). \tag{2.12}$$

Indeed, by the definition of W_h:

$$\begin{aligned} W_h\exp(-s\Sigma_{A_h}Y) &= W_h I_{[\Sigma_{A_h}Y=0]} + W_h\Sigma_{A_h}\exp(-sY_{n,j})I_{[Y_{n,j}>0]} \\ &= W_h\left\{1 + \Sigma_{A_h}\left[\exp(-sY_{n,j}) - 1\right]\right\}. \end{aligned}$$

Here we have used the fact that the intersections of $\{Y_{n,j} > 0\}$ with $\{W_h = 1\}$, $j \in A_h$, are disjoint.

The expression (2.12) is representable as the difference of two terms: the first is the expectation appearing in (2.7), and the second is

$$E(1 - \Pi_{h=1}^k W_h)\Pi_{h=1}^k\left\{1 + \Sigma_{A_h}\left[\exp(-sY_{n,j}) - 1\right]\right\}. \tag{2.13}$$

By Lemma 2.4, the former term has the limit given by (2.7). We will show that the second term, the expression (2.13), has the limit 0, and this will complete the proof that

$$\lim_{k\to\infty}\lim_{n\to\infty}E\exp(-s\Sigma_h\Sigma_{A_h}Y) = \exp\left[p\textstyle\int_0^\infty(1 - e^{-sx})\,dH^*(x)\right]. \tag{2.14}$$

By the Cauchy-Schwarz inequality, the expectation (2.13) is at most equal to the product of the square roots of $E(1 - \Pi_{h=1}^k W_h)$ and of

$$E\Pi_{h=1}^k\left\{1 + \Sigma_{A_h}\left[\exp(-sY_{n,j}) - 1\right]\right\}^2.$$

By (2.6) the former has the limit 0, and, by Lemma 2.5, the latter is bounded. This completes the proof of (2.14).

Let $\Sigma_h\Sigma_{B_h}Y$ represent the sum of the Y's taken over the index sets $B_1, \cdots, B_k$; then assumption (I) and Lemma 2.3 imply

$$\lim_{k\to\infty}\lim_{n\to\infty}E\Sigma_h\Sigma_{B_h}Y = -q\textstyle\int_0^\infty x\,dH^*(x). \tag{2.15}$$

(The integral is finite because it is actually taken over the domain $[x_1, x_m]$.)

Finally we combine our result for the limiting distribution of $\Sigma_h\Sigma_{A_h}Y$ with several other estimates to obtain the limiting distribution of the sum of the X's. The difference

$$E\left[\exp(-s\Sigma_{j=1}^n X_{n,j})\right] - \exp\left[\textstyle\int_0^\infty(1 - e^{-sx})\,dH(x)\right] \tag{2.16}$$

is equal to the sum of the intermediate differences,

$$E\left[\exp\left(-s\Sigma_{j=1}^{n}X_{n,j}\right)\right] - E\left[\exp\left(-s\Sigma_{j=1}^{n}X_{n,j}^{c}\right)\right], \tag{2.17}$$

$$E\left[\exp\left(-s\Sigma_{j=1}^{n}X_{n,j}^{c}\right)\right] - E\left[\exp\left(-s\Sigma_{j=1}^{n}Y_{n,j}\right)\right], \tag{2.18}$$

$$E\left[\exp\left(-s\Sigma_{j=1}^{n}Y_{n,j}\right)\right] - E\left[\exp\left(-s\Sigma_{h}\Sigma_{A_h}Y\right)\right], \tag{2.19}$$

$$E\left[\exp\left(-s\Sigma_{h}\Sigma_{A_h}Y\right)\right] - \exp\left[p\textstyle\int_0^\infty(1 - e^{-sx})\,dH(x)\right], \tag{2.20}$$

$$\exp\left[p\textstyle\int_0^\infty(1 - e^{-sx})\,dH(x)\right] - \exp\left[\textstyle\int_0^\infty(1 - e^{-sx})\,dH(x)\right]. \tag{2.21}$$

Pass to the limit over n and then over k. By Lemma 2.1, the limit of (2.17) is at most equal to $H(c)$. By Lemma 2.2 the limit of (2.18) is at most

$$-\,s\textstyle\int_0^c x\,d(H(x) - H^*(x)).$$

By (2.15), the limit of (2.19) is at most

$$-\,sq\textstyle\int_0^\infty x\,dH^*(x).$$

By (2.14) the limit of (2.20) is equal to

$$\exp\left[p\textstyle\int_0^\infty(1 - e^{-sx})\,dH^*(x)\right] - \exp\left[p\textstyle\int_0^\infty(1 - e^{-sx})\,dH(x)\right].$$

We infer from these estimates that the lim sup ($n \to \infty$ and then $k \to \infty$) of the difference (2.16) is at most equal to

$$H(c) - s\textstyle\int_0^c x\,d(H(x) - H^*(x)) - sq\textstyle\int_0^\infty x\,dH^*(x)$$
$$+\exp\left[p\textstyle\int_0^\infty(1 - e^{-sx})\,dH^*(x)\right] - \exp\left[\textstyle\int_0^\infty(1 - e^{-sx})\,dH(x)\right].$$

Since p and q are arbitrary positive numbers such that $p + q = 1$, we may let q tend to 0 and p tend to 1 in the expression displayed above:

$$H(c) - s\textstyle\int_0^c x\,d(H(x) - H^*(x)) + \exp\left[\textstyle\int_0^\infty(1 - e^{-sx})\,dH^*(x)\right]$$
$$-\exp\left[\textstyle\int_0^\infty(1 - e^{-sx})\,dH(x)\right].$$

Since $x_1, \cdots, x_m$ is an arbitrary set of points of continuity of H in the interval $(0, c]$, we may take H^* arbitrarily close to H in the sense of weak convergence over $(0, c]$; hence, all the terms except the first in the expression displayed above may be replaced by 0, so that we obtain the estimate $H(c)$. Since c may be taken arbitrarily large, we may pass to the limit $c \to \infty$, and apply (1.1) to get the limit 0. This completes the proof of Theorem 2.1.

Now we show that the theorem has a version not requiring the finiteness of moments.

COROLLARY 2.1. *Suppose there is a set C of positive real numbers with* $\sup(c : c \in C) = \infty$. *If* (I) *holds but not necessarily with* (1.4), *and if* (II), (III) *and* (IV) *hold for the array* $\{X_{n,j}^c\}$, *for all* $c \in C$, *then the conclusion of Theorem* 2.1 *remains valid.* (*As before*, $X^c = \min(X, c)$).

PROOF. All moments of $X^c_{n,j}$ are finite, and Theorem 2.1 holds for the partial sums of the array $\{X^c_{n,j}\}$. By Lemma 2.1, (1.1), and the fact that c may be taken arbitrarily large, the partial sum of the X's has the same limiting distribution.

Another variation is:

COROLLARY 2.2. *If assumption* (I), *but not necessarily including* (1.4) *holds for* $\{X_{n,j}\}$, *and if* (II) *and* (IV) *hold for the array* $\{X^c_{n,j}\}$ *for every* c *in the continuity set of* H *such that* $H(c) > 0$, *then the conclusion of Theorem* 2.1 *remains valid.*

PROOF. If (IV) holds for each indicated c, then (III) necessarily holds, and the result follows from Corollary 2.1.

In the special case where the random variables are indicators of events $A_{n,j}$ the compound Poisson limit becomes a simple Poisson limit. Assumptions (I) and (II) become

(i) $nP(A_{n,1}) \to \lambda$ for some $\lambda > 0$,

(ii) $\lim_{k\to\infty} \limsup_{n\to\infty} k\Sigma_{1 \leqslant i<j \leqslant [n/k]} P(A_{n,i} \cap A_{n,j}) = 0$,

respectively. The hypotheses of assumptions (III) and (IV) are implied by the single condition

(iii) $\lim_{n\to\infty} \sup_{1 \leqslant j_1 < \cdots < j_k \leqslant n,\, j_h - j_{h-1} > qn}$
$$\left| \frac{P(A_{n,j_1} \cap \cdots \cap A_{n,j_k})}{P(A_{n,j_1}) \cdots P(A_{n,j_k})} - 1 \right| = 0,$$
for each $k \geqslant 2$, and each h, $1 < h \leqslant k$.

These conditions are similar to those of Meyer [26] and Leadbetter [19] except that (iii) is different from their global mixing condition, and is formally similar to that of Sevastyinov [39].

3. Preliminary results on stationary Gaussian processes.

In this section we introduce some results about the distributions of Gaussian random variables and functionals of Gaussian processes. We use the standard notation for the Gaussian density and distribution function:
$$\phi(x) = (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2}x^2}, \qquad \Phi(x) = \textstyle\int_{-\infty}^{x} \phi(y)\, dy.$$
We recall the well-known relations
$$(3.1) \qquad 1 - \Phi(u) \leqslant \phi(u)/u, \text{ for } u > 0, \quad 1 - \Phi(u) \sim \phi(u)/u \text{ for } u \to \infty.$$

Let X and Y be random variables with a standard bivariate Gaussian density with correlation r. Since $\frac{1}{2}(X + Y)$ has a Gaussian density with mean 0 and variance $\frac{1}{2}(1 + r)$, we have, for every $u > 0$ and $b > 1$,
$$P(u < X < ub, u < Y < ub) \leqslant P\big(u < \tfrac{1}{2}(X + Y) < ub\big);$$
therefore,
$$(3.2) \qquad P(u < X < ub, u < Y < ub) \leqslant \Phi\left[\frac{ub2^{\frac{1}{2}}}{(1 + r)^{\frac{1}{2}}}\right] - \Phi\left[\frac{u2^{\frac{1}{2}}}{(1 + r)^{\frac{1}{2}}}\right].$$

The right-hand side can be estimated by means of (3.1); it is at most equal to

$$1 - \Phi\big(u(2/(1+r))^{\frac{1}{2}}\big) \leqslant \frac{(\phi(u)/u)}{(2/(1+r))^{\frac{1}{2}}}\exp\left[-\frac{u^2(1-r)}{2(1+r)}\right], \tag{3.3}$$

and the two sides above are asymptotically equal for $u \to \infty$.

Let $X(t)$, $-\infty < t < \infty$, be a stationary Gaussian process with mean 0, variance 1, and continuous covariance function $r(t)$. We assume that there exists α, $0 < \alpha \leqslant 2$, such that $1 - r(t)$ is regularly varying of index α for $t \to 0$:

$$\lim_{t\downarrow 0}\frac{1 - r(tx)}{1 - r(t)} = x^\alpha \quad \text{for} \quad x > 0. \tag{3.4}$$

For $u > 0$ define $v = v(u)$ as the largest solution of the equation

$$u^2(1 - r(1/v)) = 1. \tag{3.5}$$

As a consequence of (3.4) and (3.5) we obtain (see [10] page 369):

$$\lim_{u\to\infty} v^{\alpha'}/u^2 = 0, \quad \text{for every } \alpha' < \alpha. \tag{3.6}$$

We also note the following property of regularly varying functions: There exists $\delta > 0$ such that

$$\frac{1 - r(sh)}{1 - r(h)} \geqslant \tfrac{1}{4}s^{\alpha/2}, \quad \text{for all } 0 < h < \delta,\ 1 \leqslant s \leqslant \delta/h. \tag{3.7}$$

The proof, given in the appendix, is based on the Karamata representation.

Let $U(t)$, $t \geqslant 0$, be a Gaussian process with stationary increments such that

$$EU(t) \equiv 0,\ EU^2(0) = 0,\ E(U(t) - U(s))^2 = 2|t - s|^\alpha. \tag{3.8}$$

We state two lemmas which are extensions of results in previous work.

LEMMA 3.1. *For fixed y and $T > 0$, the process $u(X(s/v) - u)$, $0 \leqslant s \leqslant T$, conditioned by $u(X(0) - u) = y$, converges in distribution to the process $U(s) - s^\alpha + y$, $0 \leqslant s \leqslant T$, for $u \to \infty$. More generally, for fixed $y_1, \cdots, y_k$, the k-dimensional conditional vector process on $[0, T]^k$,*

$$\{u(X(t_i + s_i/v) - u),\ 0 \leqslant s_i \leqslant T,\ i = 1, \cdots, k\} \tag{3.9}$$

conditioned by $u(X(t_i) - u) = y_i$, $i = 1, \cdots, k$, where $t_1 < \cdots < t_k$ are arbitrary real numbers, converges in distribution as

$$u \to \infty \text{ and } \max_{i,j} u^2 r(t_i - t_j) \to 0 \tag{3.10}$$

to the vector process with independent components distributed as the processes

$$U(s_i) - s_i^\alpha + y_i, \quad 0 \leqslant s_i \leqslant T, \quad i = 1, \cdots, k. \tag{3.11}$$

PROOF. The first statement of the lemma is proved in [7].

Let us calculate the conditional expectation

$$E\big[u(X(t_i + s_i/v) - u) \mid u(X(t_i) - u) = y_i,\ u(X(t_j + s_j/v) - u) = y_j\big] \tag{3.12}$$

for $i \neq j$. The three random variables in the expression above have the covariance matrix

(3.13)

$$u^2\begin{bmatrix} 1 & r(s_i/v) & r(t_i + s_i/v - t_j - s_j/v) \\ r(s_i/v) & 1 & r(t_i - t_j - s_j/v) \\ r(t_i + s_i/v - t_j - s_j/v) & r(t_i - t_j - s_j/v) & 1 \end{bmatrix}.$$

Using the formula for the conditional expectation (see [1], page 28) and passing to the limit under the hypothesis (3.10), we find that (3.12) is asymptotic to $-u^2(1 - r(s_i/v)) + y_i$, which by (3.4) and (3.5), converges to $-s_i^\alpha + y_i$. But according to the first statement of the lemma, the latter is also equal to the limit of the conditional expectation (3.12) when the second conditioning variable is dropped.

Next we calculate the limit of

$$\text{(3.14)} \qquad \operatorname{Var}\left[u(X(t_i + s_i/v) - u)|X(t_i), X(t_j + s_j/v)\right].$$

According to the formula for the conditional variance, the expression (3.14) is equal to the ratio of the determinant of the matrix (3.13) to the determinant of the submatrix obtained by deleting the first row and the first column (see [1], page 28 and 42). By expansion of the determinant of the matrix (3.13) by minors of the last row, and passing to the limit under (3.10), we find that the ratio of the determinants is asymptotic by $u^2(1 - r^2(s_i/v))$, which, by (3.4) and (3.5), converges to $2|s_i|^\alpha$. But the latter is also the limit of (3.14) when the second conditioning variable is dropped.

We conclude that the conditional expectation (3.12) and the conditional variance (3.14) are asymptotically unchanged by the removal of the second conditioning variable $X(t_j + s_j/v)$. Therefore, $u(X(t_i + s_i/v) - u)$ is conditionally asymptotically independent of each $u(X(t_j + s_j/v) - u)$ for $j \neq i$. By a direct extension of the calculations above we also find that if $a_{h,j}$, $h = 1, \cdots, k, j = 1, \cdots, k'$ is an arbitrary set of real numbers, and $s_1, \cdots, s_k'$ are arbitrary distinct points in $[0, T]$, then $u(X(t_i + s_i/v) - u)$ is conditionally (given $u(X(t_i) - u) = y_i$) asymptotically independent of the linear combination

$$\Sigma_{h,j\,:\,h\neq i} a_{h,j} u\left(X(t_h + s_j/v) - u\right)$$

under the limiting operation (3.10). This implies the second statement of the lemma.

LEMMA 3.2. *For fixed y, $b > 1$, and $T > 0$, the conditional distribution of*

$$\textstyle\int_0^T I_{[u<X(s/v)<bu]}\, ds,$$

(where I is the indicator) given $u(X(0) - u) = y$, converges to the distribution of the random variable

$$\text{(3.15)} \qquad \textstyle\int_0^T I_{[U(s)-s^\alpha+y>0]}\, ds.$$

More generally, for fixed $y_1, \cdots, y_k$, the conditional joint distribution of the k

random variables

$$\int_0^T I_{[u<X(t_i+s/v)<bu]}\,ds, \qquad i = 1, \cdots, k,$$

given $u(X(t_i) - u) = y_i$, $i = 1, \cdots, k$, *converges under the limiting operation* (3.10) *to the joint distribution which is a product of the marginal distributions of the random variables* (3.15) *with* $y = y_1, \cdots, y_k$, *respectively.*

PROOF. This follows from Lemma 3.1 by computing the conditional joint moments of the random variables, and invoking the moment convergence theorem.

We present a new inequality comparing the probabilities of certain rectangles computed from two multivariate Gaussian distributions.

LEMMA 3.3. *Let* $(X_1, \cdots, X_k)$ *be a k-dimensional Gaussian random vector with means* 0 *and nonsingular covariance matrix* $\mathbf{R}$. *Let* $(Y_1, \cdots, Y_k)$ *be a similar random vector with nonsingular covariance matrix* $\mathbf{S}$. *Let m be the largest of the absolute values of the elements of the matrix* $\mathbf{R}^{-1} - \mathbf{S}^{-1}$. *Then, for every* $u > 0$ *and* $b > 1$,

$$(3.16)\quad (\det \mathbf{S}/\det \mathbf{R})^{\frac{1}{2}}\exp\left(-\tfrac{1}{2}b^2u^2k^2m\right) \leqslant \frac{P(u < X_i < bu,\ i = 1, \cdots, k)}{P(u < Y_i < bu,\ i = 1, \cdots, k)} \leqslant (\det \mathbf{S}/\det \mathbf{R})^{\frac{1}{2}}\exp\left(\tfrac{1}{2}b^2u^2k^2m\right).$$

PROOF. Let (a_{ij}) be the elements of $\mathbf{R}^{-1}$, and let (b_{ij}) be those of $\mathbf{S}^{-1}$; then, the probability in the numerator in (3.16) is

$$(2\pi)^{-k/2}(\det \mathbf{R})^{-\frac{1}{2}}\int_u^{bu} \cdots \int_u^{bu} \exp\left(-\tfrac{1}{2}\Sigma_{i,j}a_{ij}x_ix_j\right) dx_1 \cdots dx_k.$$

Write the quadratic form in the exponent as

$$\Sigma_{i,j}(a_{ij} - b_{ij})x_ix_j + \Sigma_{i,j}b_{ij}x_ix_j.$$

On the domain of integration the first term above is of magnitude at most $b^2u^2k^2m$; therefore, the multiple integral displayed above is at most equal to

$$(2\pi)^{-k/2}(\det \mathbf{R})^{-\frac{1}{2}}\exp\left(\tfrac{1}{2}b^2u^2k^2m\right)\int_u^{bu} \cdots \int_u^{bu} \exp\left(-\tfrac{1}{2}\Sigma_{i,j}b_{ij}x_ix_j\right) dx_1 \cdots dx_k$$

which is equal to the product of the last member in (3.16) and the probability in the denominator of the middle member. This proves the second inequality in (3.16). The proof of the first is similar.

Now we establish an elementary result relating the convergence of matrices to that of their inverses.

LEMMA 3.4. *Let* $\mathfrak{R}$ *be the class of covariance matrices* $\mathbf{R}$ *of order k having the following properties*:

(i) *The absolute values of the entries are bounded by* 1.

(ii) *For any* h, $1 \leqslant h < k$, *let* $\mathbf{R}$ *have the partitioned form*

$$\mathbf{R} = \begin{bmatrix} \mathbf{R}_{11} & \mathbf{R}_{12} \\ \mathbf{R}_{21} & \mathbf{R}_{22} \end{bmatrix}$$

where $\mathbf{R}_{11}$ *and* $\mathbf{R}_{22}$ *are square submatrices of orders h and* $k - h$, *respectively. There exists* w, $0 < w < 1$, *such that*

$$\inf_{\mathfrak{R}} \det \mathbf{R}_{11} \geq w, \qquad \inf_{\mathfrak{R}} \det \mathbf{R}_{22} \geq w. \tag{3.17}$$

Let ρ *be the maximum modulus of the entries of* $\mathbf{R}_{12}$, *and define* LIM *as the limiting operation*

$$\mathrm{LIM}(\cdots) = \lim_{\varepsilon\to 0} \limsup_{u\to\infty} \sup_{\mathfrak{R}\,:\, u^2\rho<\varepsilon}(\cdots). \tag{3.18}$$

Then

$$\mathrm{LIM}\{u^4[\det \mathbf{R} - \det \mathbf{R}_{11} \det \mathbf{R}_{22}]\} = 0 \tag{3.19}$$

and

$$\mathrm{LIM}\left(u^2\mathbf{R}^{-1} - \begin{bmatrix} u^2\mathbf{R}_{11}^{-1} & \mathbf{0} \\ \mathbf{0} & u^2\mathbf{R}_{22}^{-1} \end{bmatrix}\right) = \mathbf{0} \text{ (matrix of 0's)}. \tag{3.20}$$

PROOF. (3.19) follows from (3.17), (3.18) and the formula

$$\det \mathbf{R} = \det \mathbf{R}_{22} \det(\mathbf{R}_{11} - \mathbf{R}_{12}\mathbf{R}_{22}^{-1}\mathbf{R}_{21})$$

(see [1], page 42).

Let $\mathbf{R}^{-1}$ be partitioned in the same manner as $\mathbf{R}$, with submatrices $\mathbf{A}_{ij}$:

$$\mathbf{R}^{-1} = \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix}.$$

Under the assumptions of the lemma, the adjoint matrix of $\mathbf{R}$ differs from the adjoint matrix of

$$\begin{bmatrix} \mathbf{R}_{11} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_{22} \end{bmatrix}$$

by a matrix $\mathbf{J}$ whose entries are all of magnitude $O(\rho)$. Therefore, from the formula

adjoint = inverse · determinant,

we infer that

$$\det \mathbf{R} \begin{bmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{A}_{21} & \mathbf{A}_{22} \end{bmatrix} = \det \mathbf{R}_{11} \det \mathbf{R}_{22} \begin{bmatrix} \mathbf{R}_{11}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}_{22}^{-1} \end{bmatrix} + \mathbf{J};$$

hence, (3.20) follows from (3.19).

COROLLARY 3.4. *For given* w, $0 < w < 1$, *let* $\mathfrak{X}$ *be the family of all k-dimensional Gaussian random vectors* $(X_1, \cdots, X_k)$ *with* 0 *means and such that* $w \leq EX_i^2 \leq 1/w$, $i = 1, \cdots, k$. *Then, for every* $b > 1$,

$$\lim_{\varepsilon\to 0} \limsup_{u\to\infty} \sup_{\mathfrak{X}\,:\, u^2 \max_{i,j\; i\neq j}|EX_iX_j|<\varepsilon} \left| \frac{P(u < X_i < bu,\ i = 1, \cdots, k)}{\prod_{i=1}^k P(u < X_i < bu)} - 1 \right| = 0.$$

PROOF. For simplicity suppose that $EX_i^2 = 1$ for all i, and let $\mathbf{R}$ be the covariance matrix of the X's. By successive application of Lemma 3.4 to $\mathbf{R}$ and its principal submatrices, we find that

$$\text{LIM}(\det \mathbf{R} - 1) = 0, \text{ and } \text{LIM}\{u^2(\mathbf{R}^{-1} - \mathbf{I})\} = \mathbf{0}.$$

The conclusion of the corollary follows from Lemma 3.3.

For $t > 0$, define

$$L_t = \int_0^t I_{[u < X(s) < bu]}\, ds.$$

We state and prove a variation of [11], Lemma 6.2:

LEMMA 3.5. *For each sufficiently small $t > 0$, each $d > 1$, and each $b > 1$,*

$$\limsup_{u\to\infty} E(vL_{t-d/v} | u < X(t) < bu) \leqslant \int_d^\infty \exp(-s^\alpha/4)\, ds. \tag{3.21}$$

PROOF. By Fubini's theorem, the definition of conditional probability, and the stationarity of the process, the conditional expectation in (3.21) is equal to

$$v\int_0^{t-d/v} \frac{P(u < X(t-s) < bu, u < X(0) < bu)}{P(u < X(0) < bu)}\, ds.$$

Upon transformation of the variable of integration from s to $v(t - s)$, this may be written as

$$\int_d^{vt} \frac{P(u < X(s/v) < bu, u < X(0) < bu)}{P(u < X(0) < bu)}\, ds.$$

By (3.1), (3.2) and (3.3), this is at most asymptotically equal to

$$\int_d^{vt} \exp\left[-\tfrac{1}{2}u^2 \frac{1 - r(s/v)}{1 + r(s/v)}\right] ds.$$

By virtue of (3.5) and (3.7), and the inequality $r \leqslant 1$, the integrand above is dominated by the integrable function $\exp(-s^{\alpha/2}/16)$ as long as $t < \delta$, and converges everywhere to the function $\exp(-s^\alpha/4)$. Therefore, the integral converges to the expression on the right hand side of (3.21).

Next we extend the result in Lemma 3.5 to a more general conditioning operation:

LEMMA 3.6. *For each sufficiently small $t > 0$, each $d > 1$, each $b > 1$, and each integer $k > 1$,*

$$\begin{aligned} \limsup E(vL_{t-d/v} | u < X(t) < bu, u < X(t + t_i) < bu, i = 1, \cdots, k - 1) \\ \leqslant \int_d^\infty \exp(-s^\alpha/4), \end{aligned} \tag{3.22}$$

where the lim sup *operation is taken as in* (3.10), *with $t = t_k$.*

PROOF. Since the processes $\{X(s)\}$ and $\{X(t - s)\}$ are, for fixed t, identical in distribution, the conditional expectation in (3.22) is, after the change of variable

$s \to v(t-s)$, equal to

(3.23)

$$\frac{\int_d^{vt} P\left(u < X\left(\frac{s}{v}\right) < bu,\ u < X(0) < bu,\ u < X(-t_i) < bu,\ i = 1, \cdots, k-1\right) ds}{P(u < X(0) < bu,\ u < X(-t_i) < bu,\ i = 1, \cdots, k-1)}.$$

By (3.10) and Corollary 3.4, the denominator is asymptotic to

$$(P(u < X(0) < ub))^k. \tag{3.24}$$

The probability in the numerator is at most equal to

$$P\left(u < \frac{X(s) + X(0)}{2} < bu,\ u < X(-t_i) < bu,\ i = 1, \cdots, k-1\right).$$

By (3.10) and Corollary 3.4, this is asymptotic to

$$P\left(u < \tfrac{1}{2}(X(s) + X(0)) < bu\right) P^{k-1}(u < X(0) < bu), \tag{3.25}$$

uniformly in s, $d < s < vt$. Therefore, by (3.24) and (3.25), the ratio (3.23) is at most asymptotically equal to

$$\int_d^{vt} \frac{P\left(u < \frac{1}{2}(X(s) + X(0)) < bu\right)}{P(u < X(0) < bu)} ds$$

which, by the reasoning in the proof of Lemma 3.5, has a lim sup not exceeding the right-hand side of (3.22).

As a preliminary to the next result, we recall the identity used in [10] and [11]: *For any measurable stochastic process* $Y(t)$,

$$\int_A^B P\left(\int_0^T I_{[u < Y(s) < bu]}\, ds > x\right) dx = \int_0^T P\left(A < \int_0^t I_{[u<Y(s)<bu]}\, ds \leq B,\ u < Y(t) < bu\right) dt. \tag{3.26}$$

The following lemma is a version of one of the main results of [11]. Here we give a proof which is simpler than the one given there; the method will be used in the proof of the verification of assumption (III) for the occupation time of stationary Gaussian processes in Section 4.

LEMMA 3.7. *For every sufficiently small* $T > 0$, *and almost all* $x > 0$,

$$\lim_{u\to\infty} \frac{P(vL_T > x)}{E(vL_T)} = -F'(x), \tag{3.27}$$

where

$$F(x) = \int_0^\infty P\left(\int_0^\infty I_{[U(s) - s^\alpha + y > 0]}\, ds > x\right) e^{-y}\, dy. \tag{3.28}$$

PROOF. According to the reasoning in [11], page 1010, it suffices to show that

$$\frac{\int_x^\infty P(vL_T > y)\, dy}{E(vL_T)} \to F(x), \qquad x > 0, \tag{3.29}$$

where, by (3.26), the left-hand side is identical with

$$T^{-1}\int_0^T P(vL_t > x)|u < X(t) < bu)\, dt.$$

According to Lemma 3.5, in passing to the limit in the expression above, we may neglect the portion of the occupation time integral L_t over the domain $s < t - d/v$ if d is fixed but sufficiently large:

$$T^{-1}\int_0^T P\big(v\int_{t-d/v}^t I_{[u<X(s)<bu]}\, ds > x|u < X(t) < bu\big)\, dt.$$

By stationarity the integrand above is equal to

$$\frac{P\big(\int_0^d I_{[u<X(s/v)<bu]}\, ds > x,\, u < X(0) < bu\big)}{P(u < X(0) < bu)},$$

which, by application of Lemma 3.2 and conditioning by $u(X(0) - u) = y$, converges to

$$\int_0^\infty P\big(\int_0^d I_{[U(s)-s^\alpha+y>0]}\, ds > x\big)e^{-y}\, dy,$$

which is independent of t. Letting $d \to \infty$, we obtain the expression $F(x)$ in (3.28).

Equation (3.5) defined a functional relationship between u and v. Now we introduce a parameter t, and define u and v as functions of t, consistent with (3.5). For $t > 1$, let $v = v(t)$ be the largest solution of

$$(2 \log t)(1 - r(1/v)) = 1; \tag{3.30}$$

then define u in terms of t as

$$u = \left[2 \log \frac{tv/(2\pi)^{\frac{1}{2}}}{(2 \log t)^{\frac{1}{2}}}\right]^{\frac{1}{2}}. \tag{3.31}$$

(3.6) implies that $v^{\alpha'}/2 \log t \to 0$ for every $\alpha' < \alpha$ so that

$$u \sim (2 \log t)^{\frac{1}{2}}, \qquad t \to \infty. \tag{3.32}$$

Therefore, the relation (3.5) between u and v holds asymptotically for $t \to \infty$, and this is equivalent to (3.5) in the proofs of the limit theorems based on (3.4) and (3.5) (see [7], page 75).

As a consequence of (3.1), (3.31) and (3.32) we deduce

$$P(u < X(0) < bu) \sim P(X(0) > u) \sim \phi(u)/u \sim 1/tv, \tag{3.33}$$

for $t \to \infty$.

4. Application of the limit theorem to the time spent by a Gaussian process above a high level. Let $X(t)$ be a stationary Gaussian process satisfying the conditions up to and including (3.4). The object of this section is to apply the compound Poisson limit theorem to the limiting distribution of the random variable

$$v\int_0^t I_{[X(s)>u]}\, ds \tag{4.1}$$

for $t \to \infty$, where u and v are defined by (3.30) and (3.31). This is v times the time spent by X above the level u. It is asymptotically equal to the random variable

$$vL_t = v\int_0^t I_{[u<X(s)<bu]}\, ds,$$

considered in Section 3. Indeed, the latter is not greater than the former, and the expected difference between the two random variables is, by Fubini's theorem,

$$\begin{aligned} tv[P(X(0) > u) - P(u < X(0) < bu)] &= tvP(X(0) > bu) \\ &\sim tv\phi(bu)/bu \qquad \text{by (3.1)} \\ &\sim \phi(bu)/b\phi(u) \qquad \text{by (3.33)}, \end{aligned}$$

and the latter, by (3.32), converges to 0 for $t \to \infty$ for each $b > 1$. We have introduced the random variable vL_t in order to apply Lemma 3.3 to it.

THEOREM 4.1. *Under the stated conditions on $r(t)$, and under the additional condition*

$$\lim_{t\to\infty} r(t) \log t = 0, \tag{4.2}$$

the random variable (4.1), *or equivalently*, vL_t, *has a limiting distribution for $t \to \infty$ with a Laplace-Stieltjes transform of the form* (2.1), *where $H(x) = -F'(x)$, where the latter is defined by* (3.27) *and* (3.28).

The proof consists of showing that vL_t is representable as the sum of nonnegative random variables forming a stationary array, and that the conditions in the hypothesis of Theorem 2.1 hold.

Put $n = [t]$ for $t \geqslant 1$; then vL_t and vL_n have the same limiting distribution for $t \to \infty$. For, on the one hand, $vL_t - vL_n$ is nonnegative; and, on the other hand, its expectation converges to 0 for $t \to \infty$:

$$E(vL_t - vL_n) = v(t - [t])P(u < X(0) < bu),$$

and the latter, by (3.33), is asymptotically at most t^{-1} for $t \to \infty$.

Put

$$X_{n,j} = v(L_j - L_{j-1}), \qquad j = 1, \cdots, n, \tag{4.3}$$

so that

$$vL_n = \Sigma_{j=1}^n X_{n,j}.$$

We will verify that the array $\{X_{n,j}\}$ satisfies the conditions of Theorem 2.1.

It is clear from the stationarity of X that the array (4.3) is stationary. We will now show that the array satisfies the conditions in assumptions (I)–(IV) in Section 1.

ASSUMPTION (I). According to the relation

$$E(vL_1) = \int_0^\infty P(vL_1 > y)\, dy$$

the relation (3.29) is equivalent to

$$\int_0^x \frac{P(vL_1 > y)\, dy}{E(vL_1)} \to 1 - F(x), \qquad x > 0. \tag{4.4}$$

In order to justify (4.4) we would like to invoke Lemma 3.7 for $T = 1$. However, this lemma is stated for "all sufficiently small $T > 0$." The reason for this is that in proving Lemma 3.5, we had to invoke the inequality (3.7), which holds only for sufficiently small δ. However, the definition of the array $\{X_{n,j}\}$ in (4.3) can be modified so that the intervals of unit length are replaced by intervals of length T sufficiently small so that $T < \delta$. The random variable vL_n is then expressed as a sum of approximately n/T random variables, and the proof of the limit theorem is carried out exactly as when $T = 1$. We will continue to use the interval of length 1 in the course of the proof in order to make it as simple as possible.

We are now free to use (3.27) and (3.29) with $T = 1$. By integration by parts, the numerator in (4.4) is equal to $xP(vL_1 > x) - \int_0^x y\, dP(vL_1 > y)$, and the denominator is equal to $v(\Phi(bu) - \Phi(u))$ which, by (3.33), is asymptotic to $1/n$ for $n \to \infty$. Therefore, the left-hand side of (4.4) is asymptotic to

$$xP(vL_1 > x)/E(vL_1) - n\int_0^x y\, dP(vL_1 > y). \tag{4.5}$$

Put $H(x) = -F'(x)$; then we may write

$$1 - F(x) = \int_0^x H(y)\, dy$$

because $F(x) \to 1$ for $x \to 0$ (see (3.28)). (4.4) implies that the expression (4.5) converges to $\int_0^x H(y)\, dy$ for $n \to \infty$.

Lemma 3.7 implies that the first term in (4.5) converges to $xH(x)$. Therefore, it follows from what was proved above about (4.5) that the second term in (4.5) converges to a limit which satisfies

$$xH(x) - \lim_{n\to\infty} n\int_0^x y\, dP(vL_1 > y) = \int_0^x H(y)\, dy. \tag{4.6}$$

By integration by parts we find that

$$\int_0^x y\, dH(y) = xH(x) - \int_0^x H(y)\, dy$$

and from (4.6) it follows that

$$\lim_{n\to\infty} n\int_0^x y\, dP(vL_1 > y) = \int_0^x y\, dH(y),$$

which establishes (1.3). It has been shown (preceding (4.5)) that $nE(vL_1) \to 1$, so that (1.4) also holds. Thus, the conditions in assumption (I) are verified.

Next we verify the assumptions which state the mixing conditions for the array. $\{X_{n,j}\}$ was defined in (4.3) as the occupation times for successive intervals of unit length. Under the hypothesis (4.2) on r, the latter is necessarily bounded away from 1 for t bounded away from 0; therefore, there exists w, $0 < w < 1$, such that if s and t belong to nonadjacent intervals of unit length, then $r(s - t) < w$. In the calculations below we would like to extend this inequality on r to hold for any distinct intervals, even those which appear consecutively. We do this by modifying the definition of the array (4.3), and then showing that the modification hardly changes the limiting distribution of the sum. For arbitrary $\varepsilon > 0$, define $X^*_{n,j}$ as $v(L_{j-\varepsilon} - L_{j-1})$; this is the occupation time for the interval reduced by a subinterval of length ε. The distributions of $\Sigma_1^n X_{n,j}$ and $\Sigma_1^n X^*_{n,j}$ differ by at most a small quantity tending to 0 with ε:

$$E\Sigma_1^n(X_{n,j} - X_{n,j}^*) = \varepsilon n v P(u < X(0) < bu) \to \varepsilon, \text{ for } n \to \infty.$$

Therefore, it suffices to verify the mixing conditions for the modified X's. These have the desired property that if s and t are points of any two distinct time intervals over which the occupation times are defined, then

$$r(s - t) \leqslant w < 1. \tag{4.7}$$

For simplicity we write $X_{n,j}$ in place of $X_{n,j}^*$, but will be able to use the property (4.7).

ASSUMPTION (II). Let u and v be defined by (3.30) and (3.31), respectively. By the definition of $X_{n,j}$ in (4.3), the sum in (1.5) is, by stationarity, at most equal to

$$nv^2/k\int_0^1\int_0^1\Sigma_{j=1}^{[n/k]-1}P(u < X(s) < bu, u < X(t + j) < bu)\, dt\, ds. \tag{4.8}$$

For arbitrary c, $0 < c < 1$, split the sum in (4.8) into two subsums, one over indices $j \leqslant n^c$ and the other over indices $j > n^c$, where n is so large that $n^c < n/k$; then, by (3.32),

$$u^2 \sup_{s>n^c}|r(s)| \sim 2 \log n \sup_{s>n^c}|r(s)| \leqslant c^{-1} \sup_{s>n^c}(2 \log s|r(s)|),$$

which, by (4.2), converges to 0. Therefore, by Corollary 3.4, with $k = 2$,

$$\lim_{n\to\infty} \sup_{j>n^c,\, 0\leqslant s,\, t\leqslant 1}\left|\frac{P(u < X(s) < bu, u < X(t + j) < bu)}{P^2(u < X(0) < bu)} - 1\right| = 0.$$

Hence, the subsum of (4.8) over indices $j > n^c$ is at most asymptotic to $2n^2v^2P^2(u < X(0) < bu)/k^2$, which, by (3.33), is asymptotic to

$$2/k^2. \tag{4.9}$$

Next we consider the portion of (4.8) contributed by terms of index $j \leqslant n^c$. It is at most equal to

$$nv^2/k\Sigma_{j=1}^{n^c}\int_0^1\int_j^{j+1}P\big(u < \tfrac{1}{2}(X(s) + X(t)) < bu\big)\, dt\, ds,$$

which, by (3.3) and (4.7), is at most equal to

$$n^{1+c}v^2k^{-1}(\phi(u)/u) \exp\left[-\tfrac{1}{2}u^2(1 - w)/(1 + w)\right],$$

which, by (3.33), is asymptotic to

$$n^c v k^{-1} \exp\left[-\tfrac{1}{2}u^2(1 - w)/(1 + w)\right].$$

Since c is arbitrary, we may take $c < (1 - w)/(1 + w)$; then, by (3.6) and (3.32), the expression displayed above converges to 0 for $n \to \infty$.

We conclude that the lim sup of (4.8) for $n \to \infty$ is at most equal to the expression (4.9); therefore, (1.5) holds, and the condition for assumption (II) is verified.

ASSUMPTION (III). Let $J_1, \cdots, J_k$ be intervals of unit length on $[0, n]$ which are separated by at least qn units, where q is fixed, $0 < q < 1$. Put

$$X_{n,j_i} = v\int_{J_i} I_{[u<X(s)<bu]}\, ds, \qquad i = 1, \cdots, k.$$

The relation (1.6) is implied by

$$\lim n^k \int_{x_1}^{\infty} \cdots \int_{x_k}^{\infty} P(X_{n,J_i} > y_i, i = 1, \cdots, k)\, dy_k \cdots dy_1 = \Pi_{i=1}^k \int_{x_i}^{\infty} H(y)\, dy,$$

where the limiting operation is the same as that in (1.6). Indeed, under (1.8), the relation (1.6) follows from the relation above by the extension from 1 to k variables of the argument used in Section 1 to show that (1.3) implies (1.8). According to our definition of H as $-F'$, the limit relation above is equivalent to

$$\text{(4.10)}\quad \lim n^k \int_{x_1}^{\infty} \cdots \int_{x_k}^{\infty} P(X_{n,J_i} > y_i, i = 1, \cdots, k)\, dy_k \cdots dy_1 = \Pi_{i=1}^k F(x_i).$$

Let t_i be a point in J_i, and put

$$L_{t_i}^* = \int_{J_i \cap [0, t_i]} I_{[u < X(s) < bu]}\, ds.$$

By a direct extension of the identity (3.26) to k intervals the multiple integral in (4.10) is equal to

$$v^k \int_{J_1} \cdots \int_{J_k} P\big(vL_{t_i}^* > x_i, u < X(t_i) < bu, i = 1, \cdots, k\big)\, dt_k \cdots dt_1,$$

or, equivalently,

$$v^k \int_{J_1} \cdots \int_{J_k} P(u < X(t_i) < bu, i = 1, \cdots, k) \cdot P\big(vL_{t_i}^* > x_i, i = 1, \cdots, k \,|\, u < X(t_i) < bu, i = 1, \cdots, k\big)\, dt_k \cdots dt_1.$$

Multiply this by the factor n^k appearing in (4.10), and then introduce the factor $P^k(u < X(0) < bu)$ outside the integral, and the reciprocal of the factor inside the integral. According to (3.33):

$$\lim_{n\to\infty} (nv)^k P^k(u < X(0) < bu) = 1;$$

and according to (3.32) and (4.2), and Corollary 3.4,

$$\lim_{n\to\infty} \sup_{|t_i - t_{i-1}| > nq} \left| \frac{P(u < X(t_i) < bu, i = 1, \cdots, k)}{P^k(u < X(0) < bu)} - 1 \right| = 0;$$

therefore, (4.10) is equivalent to

$$\text{(4.11)}\quad \lim \int_{J_1} \cdots \int_{J_k} P\big(vL_{t_i}^* > x_i, i = 1, \cdots k \,|\, u < X(t_i) < bu, \; i = 1, \cdots, k\big) dt_k \cdots dt_1 = \Pi_{i=1}^k F(x_i).$$

Since the integrand in (4.11) is bounded (by 1), it suffices, for proving (4.11), to evaluate the limit of the integrand. Let us replace the occupation times over unit intervals by those over intervals of length of order $1/v$. We assert that if each $L_{t_i}^*$ in (4.11) is replaced by the occupation time of the subinterval $J_i \cap [t_i - d/v, t_i]$, of length at most d/v, then the limit in (4.11) is changed by at most $o(1)$ for $d \to \infty$. This is a consequence of Lemma 3.6, which states that the limiting conditional expectation of the occupation time of $J_i \cap [0, t_i - d/v]$, given $u < X(t_i) < bu$, $i = 1, \cdots, k$, is small relative to the latter event for large d. Thus, in verifying (4.11), we will suppose that the occupation times $L_{t_i}^*$ are now taken over $J_i \cap [t_i - d/v, t_i]$. Furthermore, since $v \to \infty$ for $n \to \infty$, and d is fixed, we may restrict ourselves to points t_i for which $[t_i - d/v, t_i] \subset J_i$, so that

$$vL_{t_i}^* = v\int_{t_i-d/v}^{t_i} I_{[u<X(s)<bu]}\, ds$$

$$= \int_0^d I_{[u<X(t_i-s/v)<bu]}\, ds.$$

According to Lemma 3.2 (with $-s$ in place of s in the integral) the random variables $vL_{t_i}^*$, $i = 1, \cdots, k$, displayed above are conditionally, given

$$u(X(t_i) - u) = y_i, \qquad i = 1, \cdots, k,$$

asymptotically independent and marginally distributed as

$$\int_0^d I_{[U(s)-s^\alpha+y_i>0]}\, ds, \qquad i = 1, \cdots, k. \tag{4.12}$$

The integrand in (4.11) may be written as

$$\int_0^{u^2(b-1)} \cdots \int_0^{u^2(b-1)} P\big(vL_{t_i}^* > x_i, i = 1, \cdots, k | u(X(t_i) - u) = y_i, i = 1, \cdots, k\big)$$

$$\frac{u^{-k}\left[\dfrac{\det(a_{ij})}{(2\pi)^k}\right]^{\frac{1}{2}} \exp\left[-\frac{1}{2}\Sigma_{i,j=1}^k a_{ij}(u + y_i/u)(u + y_j/u)\right] dy_1 \cdots dy_k}{P(u < X(t_i) < bu, i = 1, \cdots, k)} \tag{4.13}$$

where $\mathbf{A} = (a_{ij})$ is the inverse of the covariance matrix $\mathbf{R} = (EX(t_i)X(t_j))$. By (3.32) and (4.2) we have $u^2(\mathbf{R} - \mathbf{I}) \to \mathbf{0}$ (matrix of 0's); hence, by premultiplying this limit relation by $\mathbf{A}$, we obtain $u^2(\mathbf{A} - \mathbf{I}) \to \mathbf{0}$; therefore, the matrix $\mathbf{A}$ in (4.13) may be replaced by the identity before passing to the limit for $u \to \infty$; therefore the coefficient of the probability in the numerator in (4.13) may be replaced by

$$(\phi(u)/u)^k \exp\left(-\Sigma_{i=1}^k y_i - \tfrac{1}{2}\Sigma_{i=1}^k y_i^2/u^2\right).$$

It follows from this and the discussion leading to (4.12) that the expression (4.13) is asymptotically equal to

$$\frac{(\phi(u)/u)^k \Pi_{i=1}^k \int_0^\infty P\big(\int_0^d I_{[U(s)-s^\alpha+y>0]}\, ds > x_i\big) e^{-y}\, dy}{P(u < X(t_i) < bu, i = 1, \cdots, k)}.$$

It follows from the discussion preceding (4.11) that this converges to

$$\Pi_{i=1}^k \int_0^\infty P\big(\int_0^d I_{[U(s)-s^\alpha+y>0]}\, ds > x_i\big) e^{-y}\, dy.$$

Since d is arbitrary, we may let $d \to \infty$; then the expression above converges to $\Pi_{i=1}^k F(x_i)$. This completes the proof of (4.11), and so the condition for assumption (III) is verified.

ASSUMPTION (IV). The ratio in (1.7) is equal to

$$\frac{\int_{J_1}\cdots\int_{J_k}P(u<X(s_i)<bu,\, i=1,\cdots,k)\,ds_1\cdots ds_k}{\int_{J_1}\cdots\int_{J_k}P(u<X(s_i)<bu,\, i=1,\cdots,h-1)\, P(u<X(s_i)<bu,\, i=h,\cdots,k)\,ds_1\cdots ds_k}. \tag{4.14}$$

We apply Lemma 3.4 to the probability in the numerator. Let $\mathbf{R}$ be the covariance matrix of $X(s_1),\cdots,X(s_k)$, and let $\mathbf{S}$ be the covariance matrix obtained from $\mathbf{R}$ by replacing the covariances $EX(s_i)X(s_j)$, $1\leqslant i\leqslant h-1$, $h\leqslant j\leqslant k$, by 0's. Let $\mathbf{R}_{11}$ be the submatrix of covariances of $X(s_i)$, $1\leqslant i\leqslant h-1$; and let $\mathbf{R}_{22}$ be the submatrix of covariances of $X(s_i)$, $h\leqslant i\leqslant k$. According to the discussion preceding the verification of the conditions of assumption (II), we may treat the time intervals corresponding to the various occupation times $X_{n,j}$ as if they were separated by at least some $\varepsilon>0$. Under the hypothesis (4.2), every finite dimensional distribution of the process has a nonsingular covariance matrix; therefore, the determinant of the covariance matrix of any finite collection $X(s_1),\cdots,X(s_k)$ is bounded away from 0 on the set $\{s_1,\cdots,s_k : |s_i-s_j|>\varepsilon \text{ for every } i \text{ and } j\}$. It follows that $\det\mathbf{R}_{11}$ and $\det\mathbf{R}_{22}$ are both bounded away from 0 uniformly in $s_1,\cdots,s_k$. Condition (4.2) implies that $u^2EX(s_i)X(s_j)\to 0$ uniformly in $s_i\leqslant h-1$ and $s_j>h$; therefore, $u^2\mathbf{R}_{12}\to\mathbf{0}$ uniformly. We apply Lemma 3.4 to the covariance submatrices, and then invoke Lemma 3.3 to conclude that the probability in the numerator may be asymptotically factored into the product of probabilities appearing in the denominator. This concludes the proof that the conditions for assumption (IV) are valid.

The proof of Theorem 4.1 is complete.

5. A bivariate convergence theorem with asymptotic independence. In this section we consider a bivariate generalization of the stationary array: $\{(X_{n,j}, Y_{n,j}) : j=1,\cdots,n,\, n\geqslant 1\}$ where the pairs form a stationary sequence within each row. We state the conditions under which the pair of sums $\Sigma X_{n,j}$ and $\Sigma Y_{n,j}$ has a limiting distribution which is a product of compound Poisson distributions.

THEOREM 5.1. *Suppose that there exist monotone functions H_1 and H_2 satisfying the same conditions as does H in* (1.1) *and* (1.2); *and that* (1.3) *holds with $X_{n,1}$ and H_1 and with $Y_{n,1}$ and H_2 respectively. Suppose also that the statements of assumptions* (II), (III) *and* (IV) *remain valid whenever any set of X's in their statements is replaced by the set of Y's with the same indices. Then $\Sigma_{j=1}^n X_{n,j}$ and $\Sigma_{j=1}^n Y_{n,j}$ are asymptotically independent with compound Poisson distributions with functions H_1 and H_2 in* (2.1).

In order to illustrate the conditions of the theorem, we state the expanded form of assumption (II):

$$\lim_{k\to\infty}\Big[k\limsup_{n\to\infty}\Sigma_{1\leqslant i<j\leqslant[n/k]}EX_{n,i}X_{n,j}=0$$

$$\lim_{k\to\infty}\Big[k\limsup_{n\to\infty}\Sigma_{1\leqslant i<j\leqslant[n/k]}EX_{n,i}Y_{n,j}=0$$

$$\lim_{k\to\infty}\Big[k\limsup_{n\to\infty}\Sigma_{1\leqslant i<j\leqslant[n/k]}EY_{n,i}Y_{n,j}=0.$$

The proof of the bivariate theorem is a direct extension of that for the univariate theorem, and is omitted here. The proof, as in the univariate case, is based on the estimate of the Laplace-Stieltjes transform. It can be shown that the bivariate transform of the sums factors, in the limit, into the product of the transforms of the component sums. This is based on the fact that, according to assumption (II) in its expanded form, significant contributions to the X and Y sums can arise only from terms which are widely separated in time. According to the mixing conditions, such terms are asymptotically independent, and so the joint transform factors.

Let $X(t)$ be a stationary Gaussian process satisfying the conditions (3.4) and (4.2). For $t > 0$, define the pair of random variables

$$L_t = \int_0^t I_{[X(s)>u]}\, ds, \quad M_t = \int_0^t I_{[X(s)<-u]}\, ds,$$

the times spent above u and below $-u$, respectively. Then, as in Section 4, (vL_t, vM_t) may be represented as the sum of a bivariate stationary array $\{(X_{n,j}, Y_{n,j})\}$. It can be shown, by extensions and modifications of the arguments in the proof of Theorem 4.1, that the array satisfies the conditions in the hypothesis of Theorem 5.1. The major changes required in the proof involve replacing the inequality $-bu < X(t) < -u$ by $u < -X(t) < bu$, and changing the sign of the covariance.

APPENDIX

PROOF OF (3.7). Karamata's representation of $1 - r(t)$ is

$$1 - r(t) = c(t) \exp\left(-\int_t^1 a(y) y^{-1}\, dy\right), \qquad 0 \leqslant t \leqslant 1,$$

where $\lim_{t\to 0} c(t) = c_0$, for some $0 < c_0 < \infty$, and $\lim_{t\to 0} a(t) = \alpha$ (see, for example, [10]). Thus there exists a sufficiently small $\delta > 0$ such that $c_0/2 \leqslant c(t) \leqslant 2c_0$, and $a(t) \geqslant \alpha/2$, for all t, $0 < t < \delta$. Then, for the values of s and h in (3.7),

$$1 - r(sh) \geqslant \tfrac{1}{2} c_o \exp\left(-\int_{sh}^1 a(y) y^{-1}\, dy\right)$$

and

$$1 - r(h) \leqslant 2c_o \exp\left(-\int_h^1 a(y) y^{-1}\, dy\right),$$

so that

$$\frac{1 - r(sh)}{1 - r(h)} \geqslant \tfrac{1}{4} \exp\left(\int_h^{sh} a(y) y^{-1}\, dy\right) \geqslant \tfrac{1}{4} s^{\alpha/2}.$$

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REFERENCES

[1] ANDERSON, T. W. (1958). *Introduction to Multivariate Statistical Analysis*, Wiley, New York.

[2] BELJAEV, YU. K. (1967). On the number of intersections of a level by a Gaussian stochastic process II. *Teor. Verojatnosti. Primenen.* **13** 444–457.

[3] BERMAN, S. M. (1962). Limiting distribution of the maximum term in sequences of dependent random variables. *Ann. Math. Statist.* **33** 894–908.

[4] BERMAN, S. M. (1964a). Limiting distribution of the maximum of a diffusion process. *Ann. Math. Statist.* **35** 319–329.

[5] BERMAN, S. M. (1964b). Limit theorems for the maximum term in stationary sequences. *Ann. Math. Statist.* **35** 502–516.

[6] BERMAN, S. M. (1970). Excursions above high levels for stationary Gaussian processes. *Pacific J. Math.* **36** 63–79.

[7] BERMAN, S. M. (1971a). Maxima and high level excursions of stationary Gaussian processes. *Trans. Amer. Math. Soc.* **160** 65–85.

[8] BERMAN, S. M. (1971b). Asymptotic independence of the numbers of high and low level crossings of stationary Gaussian processes. *Ann. Math. Statist.* **42** 927–945.

[9] BERMAN, S. M. (1972). Maximum and high level excursion of a Gaussian process with stationary increments. *Ann. Math. Statist.* **43** 1247–1266.

[10] BERMAN, S. M. (1973). Excursions of stationary Gaussian processes above high moving barriers. *Ann. Probability* **1** 365–387.

[11] BERMAN, S. M. (1974). Sojourns and extremes of Gaussian processes. *Ann. Probability* **2** 999–1026.

[12] CHEN, L. H. Y. (1975). Poisson approximation for dependent trials. *Ann. Probability* **3** 534–545.

[13] CRAMÉR, H. (1946). *Mathematical Methods of Statistics*. Princeton Univ. Press.

[14] CRAMÉR, H. and LEADBETTER, M. R. (1967). *Stationary and Related Stochastic Processes*. Wiley, New York.

[15] DENZEL, G. E. and O'BRIEN, G. L. (1975). Limit theorems for extreme values of chain dependent processes. *Ann. Probability* **3** 773–779.

[16] FREEDMAN, D. (1974). The Poisson approximation for dependent events. *Ann. Probability* **2** 256–269.

[17] GALAMBOS, J. (1978). *The Asymptotic Theory of Extreme Order Statistics*. Wiley, New York.

[18] KAC, M. and SLEPIAN, D. (1959). Large excursions of Gaussian processes. *Ann. Math. Statist.* **30** 1215–1228.

[19] LEADBETTER, M. R. (1974). On extreme values in stationary sequences. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **28** 289–303.

[20] LEADBETTER, M. R. (1976). Weak convergence of high level exceedances by a stationary sequence. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **34** 11–15.

[21] LEADBETTER, M. R., LINDGREN, G. and ROOTZEN, H. (1977). Conditions for the convergence in distribution of maxima of stationary normal processes. *Stochastic Processes Appl.* **8** 131–139.

[22] LINDGREN, G. (1974). A note on the asymptotic independence of high level crossings for dependent Gaussian processes. *Ann. Probability* **2** 535–539.

[23] LINDGREN, G., DE MARE, J. and ROOTZEN, H. (1975). Weak convergence of high level crossings and maxima for one or more Gaussian processes. *Ann. Probability* **3** 961–978.

[24] LOÈVE, M. (1963). *Probability Theory, 3rd Ed.* Van Nostrand, Princeton.

[25] LOYNES, R. M. (1965). Extreme values in uniformly mixing stationary stochastic processes. *Ann. Math. Statist.* **36** 993–999.

[26] MEYER, R. M. (1973). A Poisson-type limit theorem for mixing sequences of dependent 'rare' events. *Ann. Probability* **1** 480–483.

[27] MITTAL, Y. (1974a). Limiting behavior of maxima in stationary Gaussian sequences. *Ann. Probability* **2** 231–242.

[28] MITTAL, Y. (1974b). Time revealed convergence properties of normalized maxima in stationary Gaussian processes. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **29** 181–192.

[29] MITTAL, Y. (1976). A technical lemma in the study of maxima of Gaussian processes. Tech. report #90, Depart. Statist. Stanford Univ.

[30] MITTAL, Y. and YLVISAKER, D. (1975). Limit distributions for the maxima of stationary Gaussian processes. *Stochastic Processes Appl.* **3** 1–18.

[31] O'BRIEN, G. L. (1974a). The maximum term of a uniformly mixing stationary process. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **30** 57–63.

[32] O'BRIEN, G. L. (1974b). Limit theorems for the maximum term of a stationary process. *Ann. Probability* **2** 540–545.

[33] PICKANDS III, J. (1969a). Upcrossing probabilities for stationary Gaussian processes. *Trans. Amer. Math. Soc.* **145** 51–75.

[34] PICKANDS III, J. (1969b). Asymptotic properties of the maximum in a stationary Gaussian process. *Trans. Amer. Math. Soc.* **145** 75–87.

[35] QUALLS, C. R. and WATANABE, H. (1971). An asymptotic 0 − 1 behavior of Gaussian processes. *Ann. Math. Statist.* **42** 2029–2035.

[36] QUALLS, C. R. and WATANABE, H. (1972). Asymptotic properties of Gaussian processes. *Ann. Math. Statist.* **43** 580–596.

[37] RESNICK, S. I. and NEUTS, M. F. (1970). Limit laws for maxima of a sequence of random variables defined on a Markov chain. *Advances in Appl. Probability* **2** 323–343.

[38] SERFLING, R. J. (1975). A general Poisson approximation theorem. *Ann. Probability* **3** 726–731.

[39] SERFOZO, R. (1978). High level exceedances of regenerative and semi-stationary processes. Unpublished manuscript.

[40] SEVAST'YANOV, B. A. (1972). Poisson limit law for a scheme of sums of dependent random variables. *Theor. Probability Appl.* **17** 695–699.

[41] VOLKONSKII, V. A. and ROZANOV, JU. A. (1961). Some limit theorems for random functions II. *Theor. Probability Appl.* **6** 186–198.

[42] WATSON, G. S. (1954). Extreme values in samples from *m*-dependent stationary processes. *Ann. Math. Statist.* **25** 798–800.

[43] WELSCH, R. E. (1972). Limit laws for extreme order statistics from strong-mixing processes. *Ann. Math. Statist.* **43** 439–446.

[44] WELSCH, R. E. (1973). A convergence theorem for extreme values from Gaussian sequences. *Ann. Probability* **1** 398–404.

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