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OPTIMAL STOPPING IN AN URN[1]

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An urn contains N objects, labelled with the integers $1, \cdots, N$. One object is removed at a time, without replacement. If after n draws the largest number which has been observed is m_n, and the process is terminated, we receive a payoff $f(n, m_n)$. For payoff functions f in a certain class, the optimal time to stop is with draw

$$\tau_f = \inf\{n > 0: m_n - n > j_n\}$$

where the j_n are computable from a simple algorithm, which permits also exact computation of the value

$$V_f = E\{f(\tau_f, m_{\tau_f})\}.$$

We also study the behavior of V_f when N is large in special cases.

1. Introduction. We consider a problem of optimal stopping for urn sampling, the general outlines of which may be described in the language of the classical secretary problem. A company has a single job opening for which there are N potential applicants. Before recruitment begins these individuals are compared and ranked with respect to their qualifications. Each of the N candidates will eventually seek the job, but only one at a time, and in a random order. The company may terminate the recruitment process at any stage by hiring one of the individuals who has already sought the position. Given that the resulting payoff is an increasing function of the rank of the person hired and a decreasing function of the number of job seekers interviewed, our objective is to describe the recruitment policy for which the expected payoff is maximized. We remark that our problem is distinguished from other versions of the secretary problem (for example, [1]) by the assumption that the *absolute* rank of each applicant is known, and because a person who had previously applied and been rejected may be recalled.

We will introduce our notation in the context of an urn formulation. Thus, consider an urn which contains $N \geqslant 1$ objects which for sampling purposes are indistinguishable, but which otherwise are labelled in some arbitrary fashion with the integers 1 through N. We sample the urn by selecting one object at a time, *without* replacement. Let x_j denote the label on the jth object drawn, so that the random variables $x_1, \cdots, x_N$ are a random permutation of the integers

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$1, \cdots, N$. Set $m_0 = 0$ and for each $1 \leqslant j \leqslant N$ define

$$m_j = \max(x_1, \cdots, x_j).$$

We suppose that if the selection process is terminated with draw n, and $m_n = m$, we will receive a payoff $f(n, m)$. We wish to determine the stopping policy for which our expected payoff is a maximum.

More formally, let $\mathfrak{F}_0$ denote the trivial σ-algebra, let $\mathfrak{F}_j = \mathfrak{F}(x_1, \cdots, x_j)$, $1 \leqslant j \leqslant N$, and define a finite valued ($\mathfrak{F}_n$-measurable) payoff sequence

$$z_n = f(n, m_n), \qquad 0 \leqslant n \leqslant N.$$

By a stopping time for the stochastic sequence $\{z_n, \mathfrak{F}_n\}_0^n$ we mean an integer valued random variable τ, $0 \leqslant \tau \leqslant N$, for which $\{\tau \leqslant n\} \in \mathfrak{F}_n$, $0 \leqslant n \leqslant N$. We seek the stopping time τ_f (referred to as optimal for the payoff function f) for which

$$V_f = \sup E(z_\tau) = E(z_{\tau_f})$$

where the supremum is taken with respect to all stopping times τ.

A complete solution to our problem is provided *in principle* by the method of backward induction. Let f be given. Set

$$V(N, N) = f(N, N) \tag{1.1}$$

and for each $n = N - 1, \cdots, 1, 0$ and $\forall m \geqslant n$ recursively define

$$V(n, m) = \max\{f(n, m), \gamma(n, m)\} \tag{1.2}$$

where for $n < N$ we have let

$$\begin{aligned} \gamma(n, m) &= \sup_{\tau > n} E(z_\tau | \mathfrak{F}_n) = \sup E(z_\tau | m_n = n) \\ &= \frac{1}{N - n}\{(m - n)V(n + 1, m) + \Sigma_{j=1}^{N-m} V(n + 1, m + j)\} \end{aligned} \tag{1.3}$$

and adopted the convention that $V(m + 1, m) = 0$. Then

$$\tau_f = \inf\{n \geqslant 0 : V(n, m_n) = f(n, m_n)\} \tag{1.4}$$

is optimal for f, and

$$V_f = E(z_{\tau_f}) = V(0, 0). \tag{1.5}$$

In the next section we will attempt to determine the solution in a more transparent and useable form. To accomplish this it will obviously be necessary to impose conditions on f. The conditions we require employ a function ϕ which may be described in a simple way as part of a statement of the *myopic strategy*, with which we complete this section.

Let f be given. For $0 \leqslant n < N$, $m \geqslant n$ define the function

$$\begin{aligned} \rho(n, m) &= E(z_{n+1} | m_n = m) \\ &= \frac{1}{N - n}\left\{(m - n)f(n + 1, m) + \sum_{j=1}^{N-m} f(n + 1, m + j)\right\}, \end{aligned} \tag{1.6}$$

where the convention is $f(m + 1, m) = 0$. Set $\phi(N, N) = 0$ and for $n < N$ let

$\phi(n, m) = f(n, m) - \rho(n, m)$; then ϕ is simply the difference between our payoff if we terminate sampling after n draws, with $m_n = m$, and our conditional expected payoff if we proceed to make one more draw and then stop. The *myopic strategy* is defined by the stopping time

$$\sigma_f = \inf\{n \geq 0 : \phi(n, m_n) \geq 0\}. \tag{1.7}$$

It is obvious that $\gamma(n, m) \geq \rho(n, m)$ so that

$$V(n, m) = f(n, m) \text{ implies } \phi(n, m) \geq 0 \tag{1.8}$$

and $\sigma_f \leq \tau_f$.

2. Some results for a given class of payoff functions. In the spirit of our preface we shall suppose throughout that the payoff function $f(n, m)$ satisfies the condition

$$\mathcal{G}_1 : f \text{ is decreasing } (= \text{nonincreasing}) \text{ in } n$$
$$\text{and increasing } (= \text{nondecreasing}) \text{ in } m.$$

We may observe at once that the functions V and γ, defined by (1.1) through (1.3), inherit these properties of f, in the sense of

THEOREM 2.1. *For* $\forall n \leq m$

$$\text{A. } \gamma(n, m + 1) \geq \gamma(n, m) \qquad \text{B. } V(n, m + 1) \geq V(n, m) \tag{2.1}$$

and

$$\text{A. } \gamma(n, m) \leq \gamma(n - 1, m) \qquad \text{B. } V(n, m) \leq V(n - 1, m). \tag{2.2}$$

PROOF. From $\mathcal{G}_1$ and (1.2) it is obvious that

$$A \Rightarrow B \tag{2.3}$$

in both (2.1) and (2.2).

To prove (2.1), simple algebra on (1.3) yields

$$\gamma(n, m + 1) - \gamma(n, m) = \frac{m - n}{N - n}\{V(n + 1, m + 1) - V(n + 1, m)\}, \tag{2.4}$$

so that in particular for $\forall n < N$

$$\gamma(n, n + 1) = \gamma(n, n), \tag{2.5}$$

proving (2.1) in the case $m = n$. To prove (2.1) for $m > n$, notice first that from $\mathcal{G}_1$, $\gamma(N - 2, N - 1) = \frac{1}{2}\{\max[f(N - 1, N - 1), f(N, N)] + \max[f(N - 1, N), f(N, N)]\} \leq f(N - 1, N) = \gamma(N - 2, N)$. Next make the induction assumption that for given $n \leq N - 2$ and $\forall m > n$, A holds. In consequence, from (2.5) we may assume that A holds for $\forall m \geq n$. Thus, from (2.4) we have, for $m > n - 1$, $\gamma(n - 1, m + 1) - \gamma(n - 1, m) = \frac{m - n + 1}{N - n + 1}\{V(n, m + 1) - V(n, m)\}$, which is nonnegative by (2.3), completing the induction on A. B is immediate. Moreover, a similar induction proves (2.2), where now we make use of the fact that by definition, for $\forall m \geqslant n$,

$$(N-n)\{\gamma(n-1,m)-\gamma(n,m)\} = (m-n)\{V(n,m)-V(n+1,m)\} + \Sigma_{j=1}^{N-1}\{V(n,m+j)-V(n+1,m+j)\} + V(n,m) - \gamma(n-1,m)$$

so that

$$(N-n+1)\{\gamma(n-1,m)-\gamma(n,m)\} \geqslant (m-n)\{V(n,m)-V(n+1,m)\} + \Sigma_{j=1}^{N-m}\{V(n,m+j)-V(n+1,m+j)\}.$$

Another simple property (following from $\mathcal{G}_1$) which we shall want to refer to later is recorded as

THEOREM 2.2. $\forall n < N$

$$\gamma(n-1,n-1) \geqslant \gamma(n,n). \tag{2.6}$$

PROOF. By definition (1.3)

$$(N-n)\{\gamma(n-1,n-1)-\gamma(n,n)\} = \Sigma_{j=1}^{N-n}\{V(n,n+j)-V(n+1,n+j)\} + V(n,n) - \gamma(n-1,n-1);$$

hence $(N-n+1)\{\gamma(n-1,n-1)-\gamma(n,n)\} \geqslant \Sigma_{j=1}^{N-n}\{V(n,n+j)-V(n+1,n+j)\}$, and the theorem follows from (2.2B).

Our principle objective in this section is to further exploit properties of the dynamic program in order to arrive at a more definite characterization of τ_f and V_f. To do this, we shall further restrict the class of payoff functions under consideration by imposing two additional conditions on f.

$$\mathcal{G}_2 : \exists \text{ an integer } n^* = n_f^*, \qquad 0 \leqslant n^* \leqslant N,$$

for which

$$\phi(n,n) \geqslant 0 \quad \forall n \geqslant n^* \tag{2.7}$$

$$\phi(n,n) < 0 \quad \forall n < n^* \tag{2.8}$$

$\mathcal{G}_3$: For $\forall m \geqslant n$

$$\phi(n,m) \geqslant 0 \qquad \text{implies} \qquad \phi(n-1,m) \geqslant 0.$$

We shall denote by $\mathcal{G}$ the class of payoff functions satisfying $\mathcal{G}_1$, $\mathcal{G}_2$, and $\mathcal{G}_3$, and assume henceforth that $f \in \mathcal{G}$. We note that since $\rho(n,n) \geqslant f(n+1,n+1)$, (2.7) implies

$$f(n,n) \geqslant f(n+1,n+1), \qquad n^* \leqslant n < N. \tag{2.9}$$

There does not seem to be any uninvolved or transparent motivation for adopting $\mathcal{G}_2$ or $\mathcal{G}_3$. However, (among others) the following simple examples, for which $f \in \mathcal{G}$, are of sufficient interest to suggest that this class of payoff functions is not overly restrictive.

EXAMPLE 2.1. Suppose that we receive a reward equal to the largest number that we draw, but that we incur a cost of c units for each draw that we make. Then our payoff g if we stop with n draws, and $m_n = m$, is

$$g(n, m) = m - cn \qquad c > 0.$$

That $\mathcal{G}_1$ holds is obvious, and since

$$\phi(n, m) = c - \frac{(N - m)(N - m + 1)}{2(N - n)},$$

it is simple to verify that $\mathcal{G}_2$ and $\mathcal{G}_3$ hold also. For later reference, adopting the standard notation $x^+ = \max(0, x)$ and $[y]$ = the smallest integer $\geqslant y$, we note that

$$n^* = [(N + 1 - 2c)^+] \wedge N. \tag{2.10}$$

EXAMPLE 2.2. Suppose that as in the previous example we will receive a reward m_n, but that this reward is discounted by a fixed amount γ for each draw that we make. Thus if we stop with draw n, and $m_n = m$, our payoff is

$$h(n, m) = \gamma^n m \qquad 0 < \gamma < 1.$$

For this example

$$\phi(n, m) = m\gamma^n\left\{1 - \gamma - \gamma\frac{(N - m)(N - m + 1)}{2m(N - n)}\right\}$$

and it is simple to check that $h \in \mathcal{G}$; here we have

$$n^* = \left[\frac{\gamma(N + 1)}{2 - \gamma}\right] \wedge N. \tag{2.11}$$

The principle result in this section may now be stated. Suppose that $f \in \mathcal{G}$ is given. Define for each $n = 0, \cdots, N$ and $k = 0, \cdots, N - n$

$$B_n(k) = \Sigma_{j=n+k}^N f(n, j), \tag{2.12}$$

and let $n^* = n_f^*$ be defined by $\mathcal{G}_2$. For each $n \geqslant n^*$, set

$$A_n = f(n, n) \quad \text{and} \quad j_n = 0$$

and then successively compute the numbers

$$A_{n^*-1}, j_{n^*-1}, \cdots, A_2, j_2, A_1, j_1, A_0, j_0$$

by appealing for $n < n^*$ to the formulae

$$A_n = \frac{1}{N - n}\{j_{n+1}A_{n+1} + B_{n+1}(j_{n+1})\} \tag{2.13}$$

and

$$j_n = \text{smallest integer } j \geqslant 0 \in f(n, n + j) \geqslant A_n. \tag{2.14}$$

Our result is

THEOREM 2.3. *Suppose that $f \in \mathcal{G}$; then*

$$\tau_f = \inf\{n \geqslant 0: m_n - n \geqslant j_n\} \tag{2.15}$$

is optimal for f, and

$$V_f = A_0. \tag{2.16}$$

We shall prove the theorem by exploiting the conditions $\mathcal{G}_2$ and $\mathcal{G}_3$. The proof, although somewhat involved, is interesting in its own right, for it exposes several fundamental characteristics of the dynamic program and the optimal stopping time. We begin with

LEMMA 2.1. *For* $m \geqslant n \geqslant n^*$

$$\text{(2.17)} \qquad \text{A. } \gamma(n, m) = \rho(n, m) \quad \textit{and} \quad \text{B. } V(n, m) = f(n, m).$$

PROOF. Suppose A is true for a given pair (n, m), with $n \geqslant n^*$. Then by definition $V(n, m) = \max\{f(n, m), \rho(n, m)\}$, so that

$$\text{(2.18)} \qquad \mathrm{A} \Rightarrow \mathrm{B}$$

by (2.17). We will prove A by induction. For $m \geqslant N - 1$, $\gamma(N - 1, m) = f(N, N) = \rho(N - 1, m)$. Suppose that A holds for given n, $n^* < n \leqslant N - 1$, and $\forall m \geqslant n$; then from (2.18) B holds also. Thus for $\forall m \geqslant n - 1$

$$\begin{aligned}\gamma(n-1, m) &= \frac{1}{N-n+1}\{(m-n+1)V(n, m) + \Sigma_{j=1}^{N-m}V(n, m+1)\}\\ &= \frac{1}{N-n+1}\{(m-n+1)f(n, m) + \Sigma_{j=1}^{N-m}f(n, m+j)\}\\ &= \rho(n-1, m)\end{aligned}$$

and the induction is completed.

As an immediate corollary, we have

$$\sigma_f \leqslant n^*.$$

Notice that this property is reflected in our statement of (2.15), since we have there set $j_{n^*} = 0$.

LEMMA 2.2. *For* $\forall n < n^*$

$$V(n, n) > f(n, n).$$

PROOF. For $n < n^*$, $\phi(n, n) < 0$ by (2.8); the result is immediate because of (1.8).

As a corollary to Lemma 2.1B and Lemma 2.2, we have

$$\sigma_f > 0 \quad \text{if and only if} \quad n^* > 0.$$

Parts B and C of the next lemma together imply that the optimal rule is

$$\tau_f = \inf\{n \geqslant 0: m_n > k_n\}$$

where the k_n are a nondecreasing sequence of constants; A asserts that if $\tau_f > n$, then $\gamma(n, m)$, the expected payoff given that the present maximum is $m_n = m$, does not depend on m, and is the vital observation in the explicit determination of the k_n sequence.

LEMMA 2.3. *Let* n *and* $m \geqslant n$ *be given; then*

$$\begin{aligned}&\text{A. } V(n, m) > f(n, m) \Rightarrow V(n, m') = \gamma(n, n), \forall n \leqslant m' \leqslant m;\\ \text{(2.19)} \quad &\text{B. } V(n, m) = f(n, m) \Rightarrow V(n, m') = f(n, m'), \forall m' \geqslant m;\\ &\text{C. } V(n, m) = f(n, m) \Rightarrow V(n-1, m) = f(n-1, m).\end{aligned}$$

Before proceeding with the proof, we remark that the assertions of B and C may *seem* obvious for any payoff function satisfying $\mathcal{G}_1$ only. That B and C do not hold in this generality is exhibited by an example. Let $N = 4$ and define $f(n, m)$ by the following table:

n \ m	0	1	2	3	4
0	0				
1		0	16	21	25
2			0	20	24
3				0	0
4					0

Then f is decreasing in n and increasing in m, but by direct computation $V(1, 2) = f(1, 2) = 16$, $V(2, 3) = f(2, 3) = 20$, and $V(1, 3) = 21\frac{1}{3} > f(1, 3)$ so that neither B nor C holds. Note that f does not satisfy either $\mathcal{G}_2$ or $\mathcal{G}_3$.

PROOF OF THE LEMMA. Notice first that A and B hold trivially for all $n \geqslant n^*$ by Lemma 2.1B. Similarly C holds for $n > n^*$. Now for $n = n^*$, we know that $V(n, m) = f(n, m)$ by Lemma 2.1B, so that for C to hold at $n = n^*$, we must establish that for $m \geqslant n^*$

$$V(n^* - 1, m) = f(n^* - 1, m). \tag{2.20}$$

But since it follows directly from the definitions of γ, ρ, and Lemma 2.1B that $\gamma(n^* - 1, m) = \rho(n^* - 1, m)$, we have $V(n^* - 1, m) = \max\{f(n^* - 1, m), \rho(n^* - 1, m)\}$. (2.20) now follows since $\mathcal{G}_2$ implies $\phi(n^*, m) \geqslant 0$, so that $\phi(n^* - 1, m) \geqslant 0$ by $\mathcal{G}_3$. Thus A, B, and C hold for $\forall n \geqslant n^*$.

It remains to prove the three statements for $n < n^*$. To do this we employ a somewhat nonstandard induction by making an induction assumption (I.A.) that the statements hold simultaneously at a given $n \leqslant n^*$, and establishing then that each holds also at $n - 1$. Thus suppose that A, B, and C hold for some $n \leqslant n^*$; then for $m \geqslant n + 1$

$$\begin{aligned}
&V(n-1, m) > f(n-1, m) \\
&\Rightarrow V(n, m) > f(n, m) && \text{by I.A. applied to C} \\
&\Rightarrow V(n, m-1) > f(n, m-1) && \text{by I.A. applied to B} \\
&\Rightarrow V(n, m) = V(n, m-1) && \text{by I.A. applied to A} \\
&\Rightarrow \gamma(n-1, m) = \gamma(n-1, m-1) && \text{by (2.4),}
\end{aligned}$$

and the last implication holds for $n = m$ also by (2.5). Thus for given $n \leqslant n^*$ and $m \geqslant n$, the I.A. implies

$$\begin{aligned}
&V(n-1, m) > f(n-1, m) \\
&\Rightarrow V(n-1, m) = \gamma(n-1, m) \\
&\Rightarrow V(n-1, m) = \gamma(n-1, m-1).
\end{aligned} \tag{2.21}$$

Moreover, since $f(n-1, m-1) \leqslant f(n-1, m)$ by $\mathcal{G}_1$, the I.A. implies that

$$\begin{aligned} & V(n-1, m) > f(n-1, m) \\ (2.22) \qquad & \Rightarrow f(n-1, m) < \gamma(n-1, m) = \gamma(n-1, m-1) \\ & \Rightarrow V(n-1, m-1) > f(n-1, m-1). \end{aligned}$$

Iterating successively in (2.21) and (2.22) now completes the induction on A and B.
Finally, for C the I.A. on $n \leqslant n^*$ implies that for $m \geqslant n-1$

$$\begin{aligned} & V(n-1, m) = f(n-1, m) \\ (2.23) \qquad & \Rightarrow V(n-1, m') = f(n-1, m') \; \forall m' \geqslant m && \text{by B} \\ & \Rightarrow \gamma(n-2, m) = \rho(n-2, m) && \text{by defn.} \\ & \Rightarrow V(n-2, m) = \max\{f(n-2, m), (n-2, m)\} && \text{by defn.} \end{aligned}$$

But since

$$\begin{aligned} & V(n-1, m) = f(n-1, m) \\ & \Rightarrow \phi(n-1, m) \geqslant 0 && \text{by (1.7)} \\ & \Rightarrow \phi(n-2, m) \geqslant 0 && \text{by } \mathcal{G}_3 \end{aligned}$$

it follows from (2.23) that $V(n-1, m) = f(n-1, m) \Rightarrow V(n-2, m) = f(n-2, m)$, completing the induction for C, and the proof of the lemma.

As a corollary to Lemma 2.1 and Lemma 2.3 we have

$$m_n \geqslant n^* \Rightarrow \tau_f \leqslant n.$$

More importantly, setting

$$A_n = V(n, n), \qquad n = 0, \cdots, N$$

we obtain

$$(2.24) \qquad \tau_f = \inf\{n \geqslant 0: f(n, m_n) \geqslant A_n\}$$

is optimal for f. For by Lemma 2.3A $V(n, m) > f(n, m) \Rightarrow A_n > f(n, m)$, and conversely, by $\mathcal{G}_1$ and (2.1A), we have $A_n > f(n, m) \Rightarrow \gamma(n, m) > f(n, m) \Rightarrow V(n, m) > f(n, m)$; thus (2.24) defines the same stopping time as (1.4). If we now recall definition (2.14), viz.

$$j_n = \text{smallest integer } j \geqslant 0 \ni f(n, n+j) \geqslant A_n,$$

we have clearly

$$f(n, m_n) \geqslant A_n \qquad \text{if and only if} \quad m_n \geqslant n + j_n.$$

Thus, τ_f defined by (2.15) is optimal. Moreover, for $n \geqslant n^*$, Lemma 2.1B yields

$j_n = 0$ and $A_n = f(n, n)$, whereas for $n < n^*$ we obtain from Lemma 2.3A

$$A_n = V(n, n) = \frac{1}{N - m} \Sigma_{j=1}^{N-n} V(n + 1, n + j)$$
$$= \frac{1}{N - n} \left\{ \Sigma_{j=1}^{j_{n+1}} V(n + 1, n + j) + \Sigma_{j=j_{n+1}+1}^{N-n} V(n + 1, n + j) \right\}$$
$$= \frac{1}{N - n} \left\{ \Sigma_{j=1}^{j_{n+1}} A_{n+1} + \Sigma_{j=j_{n+1}}^{N-n} f(n + 1, n + j) \right\}$$
$$= \frac{1}{N - n} \{ j_{n+1} A_{n+1} + B_{n+1}(j_{n+1}) \}$$

verifying (2.13). Recalling (1.5) completes the proof of Theorem 2.1.

Table 1 below gives the values of j_n for the payoff sequence of Example 2.1; namely

$$g(n, m_n) = m_n - cn$$

where we have let $c = 10$ and $N = 100$. For this case the value of n^* (defined by (2.10)) is 81. For convenience, we include a list of $n + j_n$ the least value of m_n for which we would stop with draw n.

More generally, concerning the constants A_n and j_n we can observe that for arbitrary $f \in \mathcal{G}$

$$V_f = A_0 \geqslant A_1 \geqslant \cdots \geqslant A_N \tag{2.25}$$

TABLE 1.

Values of j_n when $N = 10$ and the payoff sequence is defined by Example 2.1, with $c = 10$.

n	j_n	$n + j_n$	n	j_n	$n + j_n$	n	j_n	$n + j_n$		n	j_n	$n + j_n$
0	57	57	21	41	61	42	25	67		63	11	74
1	56	57	22	40	62	43	24	67		64	10	74
2	55	57	23	39	62	44	24	68		65	10	75
3	54	57	24	38	62	45	23	68		66	9	75
4	53	57	25	38	63	46	22	68		67	8	75
5	53	58	26	37	63	47	22	69		68	8	76
6	52	58	27	36	63	48	21	69		69	7	76
7	51	58	28	36	64	49	20	69		70	7	77
8	50	58	29	35	64	50	20	70		71	6	77
9	50	59	30	34	64	51	19	70		72	5	77
10	49	59	31	33	64	52	18	70		73	5	78
11	48	59	32	32	64	53	18	71		74	4	78
12	47	59	33	32	65	54	17	71		75	4	79
13	47	60	34	31	65	55	16	71		76	3	79
14	46	60	35	30	65	56	16	72		77	3	80
15	45	60	36	29	65	57	15	72		78	2	80
16	44	60	37	29	66	58	14	72		79	2	81
17	44	61	38	28	66	59	14	73		80	1	81
18	43	61	39	27	66	60	13	73	$n^* =$	81	0	81
19	42	61	40	27	67	61	12	73				
20	41	61	41	26	67	62	12	74				

$V_g = A_0 = 56.05631$

and for $\forall n < N$

$$(2.26) \qquad j_n \leqslant j_{n+1} + 1$$

(2.25) follows since $A_{n-1} \geqslant A_n$ by Lemma 2.2 and Theorem 2.2 for $n < n^*$, and because $A_{n^*-1} = V(n^* - 1, n^* - 1) \geqslant f(n^*, n^*) = A_n$ by Lemma 2.1B. (2.26) may be proved by contradiction. Suppose that

$$(2.27) \qquad \exists k \geqslant 2 \quad \text{such that } j_n = j_{n+1} + k.$$

By (2.15) we know that $(n, n + j_n - 1)$ is a continuation state (C.S.). But $(n, n + j_n - 1)$ a C.S.$\Rightarrow (n + 1, n + j_n - 1)$ a C.S. by Lemma 2.3C.$\Rightarrow (n + 1, n + j_{n+1} + 1 + (k - 2))$ a C.S. by (2.27) $\Rightarrow (n + 1, n + 1 + j_{n+1})$ a C.S. by Lemma 2.3B, a contradiction!

3. Asymptotic results. In this section we consider again the payoff function g of Example 2.1. Our objective will be to estimate the value V_g when N is large.

Thus, suppose that the payoff sequence is defined for $n = 0, \cdots, N$ by

$$(3.1) \qquad g(n, m_n) = m_n - cn,$$

where $c > 0$ is given. It is clear that if we fix c and let $N \to \infty$, then $V_g \sim N$. thus we are led to approximate V_g when both N and c are large. Formally, we let $N \to \infty$ and $c \to \infty$ in such a manner that

$$(3.2) \qquad c/N \to \alpha,$$

where $\alpha \geqslant 0$ is fixed. We will obtain the limiting value of V_g/N and then argue that the myopic strategy has an expected payoff which is asymptotic with V_g.

THEOREM 3.1. *Let g be defined by* (3.1); *then where limit is understood in the sense of* (3.2), *we have*

$$(3.3) \qquad \lim V_g/N = (1 - 2\alpha^{\frac{1}{2}})^+ .$$

For proof we will need to appeal to

LEMMA 3.1. *Let g be defined by* (3.1) *and $\{j_n\}_0^N$ by* (2.14); *then for $\forall n < N$*

$$(3.4) \qquad j_{n+1} \leqslant j_n \leqslant j_{n+1} + 1.$$

PROOF OF THE LEMMA. Consider an urn containing N objects, and for the moment let $V(n, m)$ defined by (1.2) be denoted by $V^N(n, m)$ to make the dependence on N explicit. Clearly, for $N = 1, 2, \cdots$ and $n \leqslant m \leqslant N$

$$(3.5) \qquad V^N(n, m) \leqslant V^{N+1}(n, m).$$

Moreover, for given $n = 0, 1, \cdots, N - 1$, it is easily seen that

$$(3.6) \qquad V^N(n + 1, n + 1) = V^{N-1}(n, n) + 1 - c$$

by simply imagining that the objects numbered $n + 2, \cdots, N$ remaining in the urn after state $(n + 1, n + 1)$ is reached are relabeled $n + 1, \cdots, N - 1$, respectively. Thus, recalling (for an urn with N balls) that $A_n = V(n, n)$ we have from

(3.5) and (3.6) for $n = 0, \cdots, N-1$

$$A_{n+1} \leq A_n + 1 - c. \tag{3.7}$$

Moreover, by (2.14)

$$A_{n+1} > g(n+1, n+j_n+1) = n + j_n + 1 - c(n+1) \tag{3.8}$$

and

$$A_n \leq g(n, n+j_n) = n + j_n - cn. \tag{3.9}$$

Combining (3.7), (3.8), and (3.9) gives $j_n > j_{n+1} - 1$, which is equivalent to the left-hand inequality in (3.4). The right-hand inequality is simply a restatement of (2.26).

We remark that the relationship (3.4) does not hold for all $f \in \mathcal{G}$; indeed, we have been able to invent specifications of f for which j_n and j_{n+1} differ by as much as $N - 3$.

Returning now to the proof of the theorem, observe that with g defined by (3.1) we have from (2.10)

$$n^*/N \to (1-2\alpha)^+ \tag{3.10}$$

under the limiting operation (3.2); indeed for $\alpha > \frac{1}{2}$, $n^* \to 0$, so that $V_g \to 0$. The proof that $V_g/N \to 0$ when $\alpha = \frac{1}{2}$ depends on the manner in which $c/N \to \alpha$, and we omit the details.

The interesting case is $\alpha < \frac{1}{2}$. From (3.10) $n^* \to \infty$ for $\alpha < \frac{1}{2}$, so that $V_g = A_0$ is defined by (2.13). That is

$$V_g = A_0 = \frac{1}{N}\{j_1 A_1 + B_1(j_1)\} \tag{3.11}$$

where now by (2.12)

$$\begin{aligned} B_n(k) &= \Sigma_{j=n+k}^{N}(j - cn) \\ &= \frac{N(N+1)}{2} - \frac{(n+k-1)(n+k)}{2} - c(N-n-k+1), \end{aligned}$$

so that in particular

$$B_1(j_1) = \frac{N(N+1)}{2} - \frac{j_1(1+j_1)}{2} - c(N-j_1). \tag{3.12}$$

By (2.14) $A_1 \leq g(1, 1+j_1) = 1 + j_1 - c$, so that from (3.11) and (3.12) we have

$$A_0 \leq \frac{1}{N}\left\{\frac{j_1(1+j_1)}{2} + \frac{N(N+1)}{2} - cN\right\}.$$

Accordingly, since by (3.4) and (2.14) $j_1 \leq j_0 < 1 + A_0$ we have

$$A_0 < \frac{1}{N}\left\{\frac{(A_0+1)(A_0+2)}{2} + \frac{N(N+1)}{2} - cN\right\}. \tag{3.13}$$

Moreover, a similar argument using (3.11), (2.14) and the right-hand inequality of (3.4) yields

$$A_0 > \frac{1}{N}\left\{\frac{(A_0-1)(A_0-2)}{2} + \frac{N(N+1)}{2} - cN\right\}. \tag{3.14}$$

Call $\bar{\Delta} = \limsup A_0/N$ and $\underline{\Delta} = \liminf A_0/N$; then from (3.13) and (3.14), $\bar{\Delta}^2 + 1 - 2\alpha \leq 2\bar{\Delta}$ and $2\underline{\Delta}^2 + 1 - 2\alpha \geq 2\underline{\Delta}$, yielding $\bar{\Delta} \leq 1 - 2\alpha^{\frac{1}{2}}$ and $\underline{\Delta} \geq 1 - 2\alpha^{\frac{1}{2}}$. Theorem 3.1 is now proved.

In Table 2 below we present the values of V_g/N for several choices of N and $\alpha \equiv c/N$. The entry labeled $N = \infty$ corresponds to the quantity defined in (3.3).

TABLE 2.
Values of V_g/N for several choices of N and $\alpha \equiv c/N$.

α / N	0.05	0.10	0.20	0.40
10	0.7693054	0.6318649	0.4367301	0.1611111
20	0.7263049	0.5918319	0.4020225	0.1333360
40	0.7049675	0.5722625	0.3846906	0.1194465
50	0.7006958	0.5683633	0.3812674	0.1166686
100	0.6922140	0.5605631	0.3743904	0.1111131
250	0.6871415	0.5558941	0.3702808	0.1077874
500	0.6854550	0.5543393	0.3689129	0.1066821
1000	0.6846138	0.5535229	0.3682299	0.1061324
∞*	0.6837723	0.5527865	0.3675445	0.1055729

*quantity defined in (3.3).

THEOREM 3.2. *Let g be defined by* (3.1), *let σ_g be the myopic stopping time* (1.7), *and call the value of σ_g*

$$V_g' = Eg(\sigma_g, m_{\sigma_g});$$

then

$$\lim V_g'/N = \left(1 - 2\alpha^{\frac{1}{2}}\right)^+$$

where the limiting operation continues to be defined by (3.2).

PROOF. Again we consider only the interesting case $\alpha < \frac{1}{2}$. From (1.7) and Example 2.1

$$\sigma_g = \inf\{n \geq 0: m_n \geq a_n^N\},$$

where for $n = 1, \cdots, N$

$$a_n^N = N + \tfrac{1}{2} - \left[2(N-n) + \tfrac{1}{4}\right]^{\frac{1}{2}}.$$

We have then

$$V_g' = E(m_{\sigma_g} - \sigma_g) = \Sigma_{n=1}^{\infty} E(m_n - cn | \sigma_g = n) P(\sigma_g = n)$$

$$= \left(\frac{N + a_1^N}{2} - c\right)\left(\frac{N - a_1^N + 1}{N}\right) + \Sigma_{n=2}^{N}\left(\frac{N - a_n^N}{2} - cn\right)\left(\frac{N - a_n^N + 1}{N - n + 1}\right)$$

$$\times \Pi_{i=2}^{n}\left(\frac{a_{i-1}^N - (i-1)}{N - i + 2}\right) = \Sigma_{n=1}^{N} b_n^N, \text{ say.}$$

Now, for each fixed n

$$\lim_{N\to\infty} \frac{b_n^N}{N} = 2\alpha^{\frac{1}{2}}\left(1 - (\alpha/2)^{\frac{1}{2}} - n\alpha\right)\left(1 - 2\alpha^{\frac{1}{2}}\right)^{n-1},$$

and moreover, the sequence $\{b_n^N/N\}_1^N$ can be shown to be uniformly bounded below. Thus from Fatou's lemma

$$\begin{aligned}\liminf V_g'/N &= \liminf \Sigma_{n=1}^N b_n^N/N \geq \Sigma_{n=1}^\infty \liminf b_n^N/N \\ &= \Sigma_{n=1}^\infty 2\alpha^{\frac{1}{2}}\left(1 - (\alpha/2)^{\frac{1}{2}} - n\alpha\right)\left(1 - 2\alpha^{\frac{1}{2}}\right)^{n-1} \\ &= 1 - 2\alpha^{\frac{1}{2}}.\end{aligned}$$

Thus, $\liminf V_g'/N \geq 1 - 2\alpha^{\frac{1}{2}}$, and by Theorem 3.1

$$\liminf V_g'/N \leq \limsup V_g'/N \leq \limsup V_g/N = 1 - 2\alpha^{\frac{1}{2}},$$

completing the proof.

REMARKS. 1. Consider again Example 2.2; that is, suppose the payoff sequence is given by

$$h(n, m_n) = \gamma^n m_n, \qquad n = 0, \cdots, N; 0 < \gamma < 1;$$

then

$$\lim V_h/N = \frac{1 - (1 - \gamma^2)^{\frac{1}{2}}}{\gamma}, \tag{3.15}$$

where the limiting operation is again defined by (3.2).

2. Not surprisingly, the asymptotic results (3.3) and (3.15) both coincide with those results which would be obtained if the urn were sampled with replacement, under which assumption we are in the monotone case. For example, if sampling is with replacement, and h defines the payoff sequence, then it is optimal to stop at time

$$t = \inf\{n \geq 0: m_n \geq \beta\}$$

where

$$\beta = N/\gamma + \tfrac{1}{2} - \left(\left(N/\gamma + \tfrac{1}{2}\right)^2 - N(N+1)\right)^{\frac{1}{2}},$$

and the value is

$$\tilde{V}_g = Eg(t, m_t) = \frac{\gamma(N+\beta)}{2}\,\frac{(N - \beta + 1)}{N - \gamma\beta + \gamma}.$$

It then follows that as $N \to \infty$

$$\lim \tilde{V}_g/N = \frac{1 - (1 - \gamma^2)^{\frac{1}{2}}}{\gamma}.$$

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