The Annals of Probability
1980, Vol. 8, No. 3, 431–440

CONDITIONS FOR ATTAINING $\bar{d}$ BY A MARKOVIAN JOINING[1]

BY MARTIN H. ELLIS[2]

Northeastern University

Necessary (and sometimes sufficient) conditions are given for a multi-step Markovian joining of two processes to attain the $\bar{d}$-distance between them. The conditions are applied to Markovian joinings of two-state Markov processes.

Introduction. The $\bar{d}$-distance is a natural measure of the difference between two discrete-time stationary random processes whose state space is finite or denumerable. Throughout this paper all processes considered are assumed to be of this type. The $\bar{d}$-distance between two processes may be thought of as the fraction of the places which must be altered to turn a generic string for one of the processes into a generic string for the other (see below, or [4]). A process is a joining of two processes if both processes are embedded in it. It can be shown [4] that for any pair of ergodic processes there is an ergodic joining which attains the $\bar{d}$-distance between them (in the following sense: any generic string for this process yields generic strings for each of the processes and the proportion of places where two such strings disagree equals the $\bar{d}$-distance between the two processes). See [4] for further discussion of $\bar{d}$.

Although the $\bar{d}$-distance is a very natural concept, its definition gives no "formula" which given two ergodic processes tells the $\bar{d}$-distance between them. Nor does it tell what kind of joinings can attain the $\bar{d}$-distance between two particular processes.

If the two processes being compared are Markov processes or embedded in Markov processes it is natural to ask whether the $\bar{d}$-distance between them can be attained by a Markovian joining (i.e., a joining which is a Markov process with respect to the "joint states," described below) and if so to determine if a particular Markovian joining attains $\bar{d}$.

The aim of this paper will be to give conditions (Theorem 1) which a Markovian joining of two processes must satisfy if the joining attains the $\bar{d}$-distance between them. The conditions found will be shown sufficient to guarantee the attainment of $\bar{d}$ if the two processes have "relatively prime periods" (Theorem 2). In all cases, if a Markovian joining fails to satisfy the conditions then there is another Markovian joining which matches the two processes more closely (Theorem 3). Using the

---

Received February 1978; revised February 1979.

[1]Research partially supported by National Science Foundation Grant #020-0736A

[2]Martin Herbert Ellis died February 16, 1980 at the age of 30.

AMS 1970 *subject classifications.* Primary 28A65, 60J10.

Key words and phrases. Stationary random process, discrete time process, Markov process, Markov chain, difference between two processes, $\bar{d}$-distance, joint process, Markovian joining, attaining the $\bar{d}$-distance.

conditions developed it will be shown (Theorem 4) that the $\bar{d}$-distance between two two-state Markov processes is never attained by a Markovian joining unless the Markovian joining attains the partition distance between them. The paper concludes with some observations.

First, however, the concepts will be carefully defined and some notation introduced.

Notation. A process can be represented as a $1 - 1$ bimeasurable measure-preserving transformation T on a probability space together with a finite or countable measurable partition P of the space. Given two processes $\mathfrak{T}_1 = (T_1, P_1)$ and $\mathfrak{T}_2 = (T_2, P_2)$, $\mathfrak{T} = (T, P)$ is a joining (of $\mathfrak{T}_1$ and $\mathfrak{T}_2$) if $P = P_1 \times P_2$ and $\mathfrak{T}$'s marginals are $\mathfrak{T}_1$ and $\mathfrak{T}_2$. For each joining $\mathfrak{T} = (T, P)$ of $\mathfrak{T}_1 = (T_1, P_1)$ and $\mathfrak{T}_2 = (T_2, P_2)$ let $d_{\mathfrak{T}}$ be the measure of the set of points in $\mathfrak{T}$'s measure space whose P_1 name and P_2 name differ; $d_{\mathfrak{T}}$ is the distance (between $\mathfrak{T}_1$ and $\mathfrak{T}_2$) attained by $\mathfrak{T}$. The $\bar{d}$-distance between $\mathfrak{T}_1$ and $\mathfrak{T}_2$, denoted $\bar{d}(\mathfrak{T}_1, \mathfrak{T}_2)$, is the infimum of $\{d_{\mathfrak{T}}$: $\mathfrak{T}$ is a joining of $\mathfrak{T}_1$ and $\mathfrak{T}_2\}$.

For every $n \in N^+$ and every P-n-string G (i.e., every $G \in P^n$) let $h(g)$ denote the proportion of the n terms in G whose P_1 and P_2 name differ, and for $\{Z_i\}_0^\infty$ an infinite sequence of states in P let $h(\{Z_i\}_0^\infty) = \liminf\{h(G) : G$ is a finite initial segment of $\{Z_i\}_0^\infty\}$. Then for each ergodic joining $\mathfrak{T}$ of $\mathfrak{T}_1$ and $\mathfrak{T}_2$, $d_{\mathfrak{T}} = h(\{Z_i\}_0^\infty\}$, where $\{Z_i\}_0^\infty$ is any generic sequence for $\mathfrak{T}$. As mentioned in the introduction, it can be shown that $\bar{d}(\mathfrak{T}_1, \mathfrak{T}_2) = \inf\{h(\{Z_i\}_0^\infty)$: the first and second marginals of $\{Z_i\}_0^\infty$ are generic sequences for $\mathfrak{T}_1$ and $\mathfrak{T}_2$ respectively$\}$, and that the infimum is attained by a sequence $\{Z_i\}_0^\infty$ which is generic for an ergodic joining of $\mathfrak{T}_1$ and $\mathfrak{T}_2$.

Call a joining $\mathfrak{T} = (T, P)$ an m-step joining if T is an m-step Markov process with respect to P. The primary result in this paper is Theorem 1, which gives necessary conditions which an m-step joining of $\mathfrak{T}_1$ and $\mathfrak{T}_2$ must satisfy if $d_{\mathfrak{T}} = \bar{d}(\mathfrak{T}_1, \mathfrak{T}_2)$.

Throughout this paper $\mathfrak{T} = (T, P)$ will denote an ergodic m-step joining of $\mathfrak{T}_1 = (T_1, P_1)$ and $\mathfrak{T}_2 = (T_2, P_2)$, and μ will denote $\mathfrak{T}$'s measure.

Let E and F be P-m-strings. For each $n \in \mathbb{N}^+$ with $\mu(E \cap T^{-m-n}F) > 0$, let $\rho(\cdot, E, F, n)$ denote the measure on P-n-strings satisfying

$$\rho(G, E, F, n) = \frac{\mu(E \cap T^{-m}G \cap T^{-m-n}F)}{\mu(E \cap T^{-m-n}F)}$$

for every P-n-string G; if E, F and n are understood, denote $\rho(\cdot, E, F, n)$ by ρ. Let

$$d(\rho) = \Sigma h(G)\rho(G)$$

the sum being taken over all P-n-strings G. Let ρ_1 and ρ_2 denote the respective marginals of ρ, and let

$$\bar{d}(\rho_1, \rho_2) = \inf\{d(\nu): \nu \text{ is a measure on } P\text{-}n\text{-strings whose marginals are } \rho_1 \text{ and } \rho_2 \text{ respectively}\}.$$

By compactness, the infimum is attained. Note that $d(\rho) > \bar{d}(\rho_1, \rho_2)$.

THEOREM 1. *If* $d_{\mathfrak{T}} = \bar{d}(\mathfrak{T}_1, \mathfrak{T}_2)$, *then for all* P-m-strings E *and* F *and for all* $n \in \mathbb{N}^+$ *with* $\mu(E \cap T^{-m-n}F) > 0$, $d(\rho) = \bar{d}(\rho_1, \rho_2)$.

PROOF. Suppose the hypotheses of the theorem hold but the conclusion doesn't. Choose P-m-strings E and F and $n \in \mathbb{N}^+$ for which $\mu(E \cap T^{-m-n}F) > 0$ and $d(\rho) > \bar{d}(\rho_1, \rho_2)$. Let $\{Z_i\}_0^\infty$ be a generic sequence for $\mathfrak{T}$. Let i_0 be the smallest natural number for which

$$E = \langle Z_{i_0}, \cdots, Z_{i_0} + m - 1\rangle \quad \text{and}$$
$$F = \langle Z_{i_0} + m + n, \cdots, Z_{i_0} + 2m + n - 1\rangle$$

and for $r \in \mathbb{N}$ let i_{r+1} be the smallest natural number satisfying

$$i_{r+1} \geqslant i_r + m + n, E = \langle Z_{i_{r+1}}, \cdots, Z_{i_{r+1}} + m - 1\rangle \text{and}$$
$$F = \langle Z_{i_{r+1}} + m + n, \cdots, Z_{i_{r+1}} + 2m + n - 1\rangle.$$

Then

$$\lim_{r\to\infty} \frac{i_r}{r} \leqslant \frac{m + n}{\mu(E \cap T^{-m-n}F)}. \tag{1}$$

Let ν denote a measure on P-n-strings which attains $\bar{d}(\rho_1, \rho_2)$. Independently choose a P-n-string G_r for each $r \in \mathbb{N}$ according to the measure ν assigns to P-n-strings. Modify $\{Z_i\}_0^\infty$ to obtain $\{\hat{Z}_i\}_0^\infty$ as follows.

$$\hat{Z}_i = \text{the } k\text{th term of } G_r \qquad \text{if } i = i_r + m + k \quad \text{for } 0 \leqslant k \leqslant n - 1$$
$$= Z_i \qquad \text{otherwise.}$$

Let $\{\hat{X}_i\}$ and $\{\hat{Y}_i\}$ be the marginals of $\{\hat{Z}_i\}$. Since $\mathfrak{T}$ is an m-step Markov process, the P-n-strings of $\{Z_i\}$ being replaced are independent and distributed according to the measure ρ assigns to P-n-strings, hence the marginals of the P-n-strings being replaced are independent and distributed according to the measure ρ_1 (respectively ρ_2) assigns to P_1-n-strings (respectively P_2-n-strings). The n-strings that are replacing them, however, are also independent and distributed according to the measure ρ_1 (respectively ρ_2) assigns to P_1-n-strings (respectively P_2-n-strings). Hence $\{\hat{X}_i\}$ and $\{\hat{Y}_i\}$ are generic sequences for $\mathfrak{T}_1$ and $\mathfrak{T}_2$ respectively.

Furthermore, by (1) the terms of $\{Z_i\}$ being replaced have density at least $n\mu(E \cap T^{-m-n}F)(m + n)^{-1}$, and for the terms being replaced the probability that the P_1 name and P_2 name differ will decrease from $d(\rho)$ to $\bar{d}(\rho_1, \rho_2)$. Hence

$$\begin{aligned}\bar{d}(\mathfrak{T}_1, \mathfrak{T}_2) &\leqslant h(\{\hat{Z}_i\}_0^\infty) \\ &\leqslant h(\{Z_i\}_0^\infty) - n\mu(E \cap T^{-m-n}F)(n + m)^{-1}\big(d(\rho) - \bar{d}(\rho_1, \rho_2)\big) \\ &= d_{\mathfrak{T}} - n\mu(E \cap T^{-m-n}F)(n + m)^{-1}\big(d(\rho) - \bar{d}(\rho_1, \rho_2)\big).\end{aligned}$$

Contradiction. ▯

Theorem 1 is true not only for the $\bar{d}$-distance function, but for any function on the joint atoms. That is, if $f : P \to \mathbb{R}$, for each P-n-string $G = \langle g_1, \cdots, g_n \rangle$ let

$$h_f(G) = n^{-1}\Sigma_{i=1}^{n} f(g_i)$$

and if ρ is a measure on P-n-strings let

$$d_f(\rho) = \Sigma h_f(G)\rho(G)$$

the sum being taken over all P-n-strings G. If $\bar{d}_f(\rho_1, \rho_2)$ and $\bar{d}_f(\mathfrak{T}_1, \mathfrak{T}_2)$ are analogously defined and $\bar{d}_f(\mathfrak{T}_1, \mathfrak{T}_2) < \infty$, then the analogous version of Theorem 1 is proved by the preceding proof. Note that f^*, the function which assigns a value of zero to the atoms of P with the same P_1 and P_2 names and a value of one to all other atoms in P, yields the usual $\bar{d}$-distance.

Since $\mathfrak{T}$ is a multi-step Markov process, either $\mathfrak{T}$ is aperiodic (has period one) or $\mathfrak{T}$ has period greater than one. Let t denote the period of $\mathfrak{T}$. Then $\mathfrak{T}$ is isomorphic to the direct product of a Bernoulli shift with a rotation on t elements; furthermore, since $\mathfrak{T}_1$ (respectively $\mathfrak{T}_2$) is a factor of $\mathfrak{T}$, it is isomorphic to the direct product of a Bernoulli shift with a rotation on t_1 (respectively t_2) elements, and t_1 and t_2 divide t ([1], [5]).

The converse of Theorem 1 is true if t_1 and t_2 are relatively prime:

THEOREM 2. *If $\mathfrak{T}_1$ and $\mathfrak{T}_2$ have relatively prime periods and $d_{\mathfrak{T}} \neq \bar{d}(\mathfrak{T}_1, \mathfrak{T}_2)$ then there are P-m-strings E and F and an $n \in \mathbb{N}^+$ for which $\mu(E \cap T^{-m-n}F) > 0$ and $d(\rho) > \bar{d}(\rho_1, \rho_2)$.*

PROOF. The proof divides into three cases.

(i) If $t = 1$, then P is weak Bernoulli for $\mathfrak{T}$. Then for all P-m-strings E and F with $\mu(E) > 0$ and $\mu(F) > 0$, $\rho(\cdot, E, F, n)$ is defined for all sufficiently large $n \in \mathbb{N}^+$, and

$$\lim_{n\to\infty} d(\rho(\cdot, E, F, n)) = d_{\mathfrak{T}}$$

whereas

$$\lim_{n\to\infty} \bar{d}(\rho_1(\cdot, E, F, n), \rho_2(\cdot, E, F, n)) = \bar{d}(\mathfrak{T}_1, \mathfrak{T}_2).$$

(ii) If $t > 1$ and at least one of $\mathfrak{T}_1$, $\mathfrak{T}_2$ is aperiodic, say $t_1 = 1$, let $w = mt$,

$$\hat{\mathfrak{T}}_1 = (T_1^w, \vee_{i=0}^{w-1} T_1^i P_1)$$
$$\hat{\mathfrak{T}}_2 = (T_2^w, \vee_{i=0}^{w-1} T_2^i P_2)$$
$$\hat{\mathfrak{T}} = (T^w, \vee^{w-1} T^i P)$$

and let f be the function on P-w-strings $G = \langle g_1, \cdots, g_w \rangle$ satisfying

$$f(G) = w^{-1}\Sigma_{i=0}^{w-1} f^*(g_i)$$

for every P-w-string G.

Then $\hat{\mathfrak{T}}_1$ is ergodic, $\hat{\mathfrak{T}}_2$ consists of t_2 ergodic components (if $t_2 = 1$ then $\hat{\mathfrak{T}}_2$ is ergodic), and $\hat{\mathfrak{T}}$ is a nonergodic one-step joining of $\hat{\mathfrak{T}}_1$ and $\hat{\mathfrak{T}}_2$ consisting of t ergodic components each of measure t^{-1}; each of these t components (with its measure normalized to one) is a mixing one-step joining of $\hat{\mathfrak{T}}_1$ with a component of $\hat{\mathfrak{T}}_2$. The $\bar{d}_f$-distance between $\hat{\mathfrak{T}}_1$ and each component of $\hat{\mathfrak{T}}_2$ equals $\bar{d}_f(\hat{\mathfrak{T}}_1, \hat{\mathfrak{T}}_2)$, and $\bar{d}_f(\hat{\mathfrak{T}}_1, \hat{\mathfrak{T}}_2) = \bar{d}(\mathfrak{T}_1, \mathfrak{T}_2)$. If $d_{\mathfrak{T}} \neq \bar{d}(\mathfrak{T}_1, \mathfrak{T}_2)$, then $\hat{\mathfrak{T}}$ will not optimally match (at least) one of the components of $\hat{\mathfrak{T}}_2$ with $\hat{\mathfrak{T}}_1$. Since each of the components of $\hat{\mathfrak{T}}$ is an aperiodic one-step joining, part (i) of this proof is applicable to the nonoptimally matched component(s) of $\hat{\mathfrak{T}}$, hence we can conclude that there are P-w-strings E' and F' (corresponding to $\bigvee_{i=0}^{w-1} T^i P$-1-strings in a nonoptimal-matching component of $\hat{\mathfrak{T}}$) and an $n \in \mathbb{N}^+$ for which $d(\rho(\cdot, E', F', n)) > \bar{d}(\rho_1(\cdot, E', F', n), \rho_2(\cdot, E', F', n))$. Let E be the last m terms of E' and F be the first m terms of F', and n be as just described; then the conclusion of the theorem holds.

(iii) If both $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are periodic and t_1 and t_2 are relatively prime, consider

$$\mathfrak{T}_1' = \left(T_1^{t_1}, \bigvee_{i=0}^{t_1-1} T_1^i P_1\right)$$

$$\mathfrak{T}_2' = \left(T_2^{t_1}, \bigvee_{i=0}^{t_1-1} T_2^i P_2\right)$$

and

$$\mathfrak{T}' = \left(T^{t_1}, \bigvee_{i=0}^{t_1-1} T^i P\right).$$

Then $\mathfrak{T}_1'$ consists of t_1 components each of which is mixing, $\mathfrak{T}_2'$ is an ergodic periodic (period t_2) process, and $\mathfrak{T}'$ is a nonergodic multi-step joining of $\mathfrak{T}_1'$ and $\mathfrak{T}_2'$ consisting of t_1 ergodic components each of measure t_1^{-1}; each of these t_1 components (with its measure normalized to one) is a multi-step joining of $\mathfrak{T}_2'$ with a component of $\mathfrak{T}_1'$. If $d_{\mathfrak{T}} \neq \bar{d}(\mathfrak{T}_1, \mathfrak{T}_2)$, then as in (ii) $\mathfrak{T}'$ does not optimally match $\mathfrak{T}_2'$ with one of the components of $\mathfrak{T}_1'$. Since each component of $\mathfrak{T}_1'$ is mixing, hence aperiodic, part (ii) of this proof applies to the nonoptimally matched components of $\mathfrak{T}'$, whence the conclusion of the theorem follows. □

If $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are periodic and their periods are not relatively prime, the converse of Theorem 1 may not hold, as the following example shows.

EXAMPLE. Let $\mathfrak{T}_1$ and $\mathfrak{T}_2$ both equal the Markov process with states $\{1, 2, 3\}$ and transition matrix

$$\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 \end{bmatrix}$$

Thus $\bar{d}(\mathfrak{T}_1, \mathfrak{T}_2) = 0$. Let $\mathfrak{T}$ be the one-step joining of $\mathfrak{T}_1$ and $\mathfrak{T}_2$ with states $\{(1, 3), (2, 3), (3, 1), (3, 2)\}$ (i.e., the other five pairs have measure zero) and transition matrix

$$\begin{bmatrix} 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{bmatrix}$$

Then $d_{\mathfrak{T}} = 1$, yet for all 1-strings E and F for all $n \in \mathbb{N}^+$ with $\mu(E \cap T^{-1-n}F) > 0$, $d(\rho) = \bar{d}(\rho_1, \rho_2) = 1$.

The modified process given in the proof of Theorem 1 (the process for which $\{\hat{Z}_i\}_0^\infty$ is a generic sequence) may fail to be a q-step joining for any $q \in \mathbb{N}$. The problem stems from the fact that $Z_{i_{r+1}}$ may not be the first term after Z_{i_r} which initiates E followed by a gap of length n followed by F. Yet the stipulation that $i_{r+1} > i_r + m + n$ cannot be omitted, for if it is omitted the P-n-strings being replaced may not be separated by at least m terms, hence may not be independent of one another, whence the modification may not yield a sequence whose marginals are generic sequences for $\mathfrak{T}_1$ and $\mathfrak{T}_2$.

However, E can be lengthened by concatenating a string to its left to obtain a P-$(t + n + m - 1)$-string $\hat{E} = \langle e_1, e_2, \cdots, e_{t+n+m-1}\rangle$ for which $\langle e_1, \cdots, e_t\rangle$ does not equal $\langle e_{i+1}, \cdots, e_{i+t}\rangle$ for $1 \leq i < m + n$. Then the construction given in the proof of Theorem 1 when applied to $\rho(\cdot, \hat{E}, F, n)$ will yield a $(t + 2n + 2m - 2)$-step joining.

THEOREM 3. *If there are P-m-strings E and F and an $n \in \mathbb{N}^+$ such that $d(\rho) > \bar{d}(\rho_1, \rho_2)$, then there is a multi-step joining $\hat{\mathfrak{T}}$ of $\mathfrak{T}_1$ and $\mathfrak{T}_2$ with $d_{\hat{\mathfrak{T}}} < d_{\mathfrak{T}}$.*

PROOF. Given P-strings G_1 and G_2 let $G_1 \circ G_2$ denote G_1 concatenated with G_2 and let $G_1^{(r)}$ denote rG_1's concatenated together.

Let G_1 be a P-k-string with $k \geq m$ such that $\mu(G_1 \circ G_1) > 0$, let G_2 be a P-k-string unequal to G_1 with $\mu(G_1 \circ G_2) > 0$, and let G_3 be a P-l-string with $l \geq n - 2$ such that $\mu(G_2 \circ G_3 \circ E) > 0$. Choose $r \in \mathbb{N}^+$ so that $(r - 1)k \geq m + n - 1$, and let $\hat{E} = G_1^{(r)} \circ G_2 \circ G_3 \circ E$. Then $\mu(\hat{E}) > 0$, and $\hat{E}$ has the properties described before the statement of this theorem, so $\hat{\mathfrak{T}}$, the process obtained by applying the construction given in the proof of Theorem 1 to $\rho(\cdot, \hat{E}, F, n)$, is an $(rk + k + l + 2m + n - 1)$-step joining of $\mathfrak{T}_1$ and $\mathfrak{T}_2$ attaining a smaller distance than $\mathfrak{T}$ does. □

Theorems 2 and 3 imply the following:

COROLLARY 1. *If $\mathfrak{T}_1$ and $\mathfrak{T}_2$ can be joined by a multi-step joining and if $\mathfrak{T}_1$ and $\mathfrak{T}_2$ have relatively prime periods, then either there is a multi-step joining attaining $\bar{d}(\mathfrak{T}_1, \mathfrak{T}_2)$ or there is no best multi-step joining.*

COROLLARY 2. *If $\mathfrak{T}_1$ and $\mathfrak{T}_2$ can be joined by a mixing multi-step joining then either there is a mixing multi-step joining attaining $\bar{d}(\mathfrak{T}_1, \mathfrak{T}_2)$ or there is no best mixing multi-step joining.*

PROOF. If $\mathfrak{T}_1$ and $\mathfrak{T}_2$ can be joined by a mixing joining, they must be mixing, hence aperiodic. Thus, if $\mathfrak{T}$ is a mixing multi-step joining and $d_{\mathfrak{T}} \neq \bar{d}(\mathfrak{T}_1, \mathfrak{T}_2)$, Theorem 2 implies the hypothesis of Theorem 3 holds, and the proof of Theorem 3 yields a mixing multi-step joining $\hat{\mathfrak{T}}$ of $\mathfrak{T}_1$ and $\mathfrak{T}_2$ with $d_{\hat{\mathfrak{T}}} < d_{\mathfrak{T}}$. □

In particular, if $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are multi-step Markov processes then their direct product is a multi-step joining, so Corollary 1 applies; if in addition $\mathfrak{T}_1$ and $\mathfrak{T}_2$ are aperiodic then the direct product is mixing and Corollary 2 applies.

The usefulness of Theorem 1 comes in demonstrating that $\mathfrak{T}$ not attain the $\bar{d}$-distance, for by Theorem 1 it suffices to find P-m-strings E and F and $n \in \mathbb{N}^+$ for which $\mu(E \cap T^{-m-n}F) > 0$ and $d(\rho) > \bar{d}(\rho_1, \rho_2)$. Theorem 1 will be used in this manner to prove Theorem 4. First, however, some notation is needed.

Let $p(\mathfrak{T}_1, \mathfrak{T}_2)$ denote the partition distance between $\mathfrak{T}_1$ and $\mathfrak{T}_2$. That is, $p(\mathfrak{T}_1, \mathfrak{T}_2) = 1 - \Sigma_i \min\{r_i, s_i\}$, where r_i is the measure of the ith atom of P_1, s_i the measure of the ith atom of P_2. Clearly $p(\mathfrak{T}_1, \mathfrak{T}_2) \leqslant \bar{d}(\mathfrak{T}_1, \mathfrak{T}_2)$.

Let $M(\mathfrak{T}_1, \mathfrak{T}_2) = \inf\{d_{\mathfrak{T}}\colon \mathfrak{T}$ is a one-step joining of $\mathfrak{T}_1$ and $\mathfrak{T}_2\}$. A compactness argument will show that if the set of distances is nonempty, then the infimum is attained. Clearly $\bar{d}(\mathfrak{T}_1, \mathfrak{T}_2) \leqslant M(\mathfrak{T}_1, \mathfrak{T}_2)$.

Let (κ, λ) denote the one-step two-state Markov process with transition matrix

$$\begin{pmatrix} 1-\kappa & \kappa \\ \lambda & 1-\lambda \end{pmatrix}.$$

Then (κ, λ) has positive entropy if and only if $0 < \min\{\kappa, \lambda\} < 1$; here the measure of the first state is $\lambda(\kappa + \lambda)^{-1}$ and the measure of the second state is $\kappa(\kappa + \lambda)^{-1}$.

THEOREM 4. *If* (α, β) *and* (γ, δ) *have positive entropy, and* $M((\alpha, \beta), (\gamma, \delta)) > p((\alpha, \beta), (\gamma, \delta))$, *then* $M((\alpha, \beta), (\gamma, \delta)) > \bar{d}((\alpha, \beta), (\gamma, \delta))$.

PROOF. Given the pair $((\alpha, \beta), (\gamma, \delta))$, let A denote the first state of (α, β), B denote the second state of (α, β), C denote the first state of (γ, δ), D denote the second state of (γ, δ), and if $\mathfrak{T}$ is a joining for the pair $((\alpha, \beta), (\gamma, \delta))$, order the four states of $\mathfrak{T}$ lexicographically (thus $(A, C) = 1$, $(A, D) = 2$, $(B, C) = 3$, and $(B, D) = 4$); let μ denote $\mathfrak{T}$'s measure.

Given (α, β) and (γ, δ), the pairs $((\alpha, \beta), (\gamma, \delta))$, $((\beta, \alpha), (\delta, \gamma))$, $((\gamma, \delta), (\alpha, \beta))$, and $((\delta, \gamma), (\beta, \alpha))$ all have the same value for M, for $\bar{d}$, and for p. Using this symmetry it can and henceforth will be assumed that $((\alpha, \beta), (\gamma, \delta))$ is in one of the following forms

(i) $((\alpha, \beta), (\alpha - u, \beta + u))$ $\quad u \geqslant 0$
(ii) $((\alpha, \beta), (\alpha - u, \beta + w))$ $\quad w > u \geqslant 0$
(iii) $((\alpha, \beta), (\alpha + u, \beta + w))$ $\quad w, u > 0$ and $\mu(C) - \mu(A) \geqslant 0$.

Note that $p((\alpha, \beta), (\gamma, \delta)) = \mu(C) - \mu(A)$.

In [2] it is shown that when $M((\alpha, \beta), (\gamma, \delta)) > p((\alpha, \beta), (\gamma, \delta))$, the one-step joining of (α, β) and (γ, δ) with transition matrix

$$\begin{pmatrix} \min\{1-\alpha, 1-\gamma\} & \max\{0, \gamma-\alpha\} & \max\{0, \alpha-\gamma\} & \min\{\alpha, \gamma\} \\ \min\{\delta, 1-\alpha\} & \max\{0, 1-\alpha-\delta\} & \max\{0, \alpha+\delta-1\} & \min\{\alpha, 1-\delta\} \\ \min\{\beta, 1-\gamma\} & \max\{0, \beta+\gamma-1\} & \max\{0, 1-\beta-\gamma\} & \min\{\gamma, 1-\beta\} \\ \beta & 0 & \delta-\beta & 1-\delta \end{pmatrix}$$

attains $M((\alpha, \beta), (\gamma, \delta))$ with the exception of one class of pairs; each pair in this exceptional class has a one-step joining attaining M whose transition matrix has e_{42}, e_{21}, e_{43}, and e_{31} all positive. For $((\alpha, \beta), (\gamma, \delta))$ with $M((\alpha, \beta), (\gamma, \delta)) > p((\alpha, \beta), (\gamma, \delta))$, let $\mathfrak{T}^*((\alpha, \beta), (\gamma, \delta))$ denote the one-step joining of (α, β) and (γ, δ) attaining $M((\alpha, \beta), (\gamma, \delta))$ described above; when $((\alpha, \beta), (\gamma, \delta))$ is understood, denote $\mathfrak{T}^*((\alpha, \beta), (\gamma, \delta))$ by $\mathfrak{T}^*$. Let $\mathfrak{T}^* = (T, P)$ (hence $P = \{1, 2, 3, 4\} = \{(A, C), (A, D), (B, C), (B, D)\}$).

For every $((\alpha, \beta), (\gamma, \delta))$ with $M((\alpha, \beta), (\gamma, \delta)) > p((\alpha, \beta), (\gamma, \delta))$ P-1-strings E and F and an $n \in \mathbb{N}^+$ will now be found for which $\mu(E \cap T^{-1-n}F) > 0$ and $d(p) > \bar{d}(\rho_1, \rho_2)$. Then by Theorem 1 it may be concluded that $d_{\mathfrak{T}^*} > \bar{d}((\alpha, \beta), (\gamma, \delta))$; since for any other one-step joining $\mathfrak{T}'$ of (α, β) and (γ, δ), $d_{\mathfrak{T}'} \geqslant d_{\mathfrak{T}^*}$, the theorem follows.

For $((\alpha, \beta), (\gamma, \delta))$ in the exceptional class, $\mathfrak{T}^*$ assigns $\langle 4, 2, 1\rangle$ and $\langle 4, 3, 1\rangle$ positive measure. Thus, for $E = \langle 4\rangle$, $F = \langle 1\rangle$, $n = 1$, $d(\rho) > \bar{d}(\rho_1, \rho_2)$: ρ assigns positive measure to $\langle 2\rangle = \langle (A, D)\rangle$ and to $\langle 3\rangle = \langle (B, C)\rangle$, hence ρ can be improved upon by pairing $\langle A\rangle$ with $\langle C\rangle$ and $\langle B\rangle$ with $\langle D\rangle$ (until the measure of $\langle 2\rangle$ or $\langle 3\rangle$ is reduced to zero).

If $((\alpha, \beta), (\gamma, \delta))$ is in form (i), or if $\beta + \gamma \leqslant 1$ and $((\alpha, \beta), (\gamma, \delta))$ is in form (ii), there is a one-step joining of (α, β) and (γ, δ) attaining $p((\alpha, \beta), (\gamma, \delta))$.

If $((\alpha, \beta), (\gamma, \delta))$ is in form (ii), $\beta + \gamma > 1$, and $\delta = 1$ (whence $\gamma < 1$), then $\mathfrak{T}^*$ assigns $\langle 3, 2, 3, 2, 3, 4\rangle$ and $\langle 3, 1, 4, 3, 1, 4\rangle$ positive measure. Thus, for $E=\langle 3\rangle$, $F = \langle 4\rangle$, $n=4$, $d(\rho)>\bar{d}(\rho_1, \rho_2)$: ρ assigns positive measure to $\langle 2, 3, 2, 3\rangle = \langle (A, D), (B, C), (A, D), (B, C)\rangle$ and to $\langle 1, 4, 3, 1\rangle = \langle (A, C), (B, D), (B, C), (A, C)\rangle$, hence ρ can be improved upon by pairing $\langle A, B, A, B\rangle$ with $\langle C, D, C, C\rangle$ and $\langle A, B, B, A\rangle$ with $\langle D, C, D, C\rangle$ (until the measure of $\langle 2, 3, 2, 3\rangle$ or $\langle 1, 4, 3, 1\rangle$ is reduced to zero).

If $((\alpha, \beta), (\gamma, \delta))$ is in form (ii), $\beta + \gamma > 1$, and $\delta < 1$, or if it is form (iii) and $\gamma = 1$ (whence $\delta < 1$), then $\mathfrak{T}^*$ assigns $\langle 4, 3, 2, 4\rangle$ and $\langle 4, 4, 3, 4\rangle$ positive measure. Thus, for $E = \langle 4\rangle$, $F = \langle 4\rangle$, $n = 2$, $d(\rho) > \bar{d}(\rho_1, \rho_2)$ (the proof is like those above).

If $((\alpha, \beta), (\gamma, \delta))$ is in form (iii) and $\delta = 1$ (whence $\gamma < 1$), then $\mathfrak{T}^*$ assigns $\langle 1, 2, 3, 1, 4\rangle$ and $\langle 1, 1, 2, 3, 4\rangle$ positive measure. Thus, for $E = \langle 1\rangle$, $F = \langle 4\rangle$, $n = 3$, $d(\rho) > \bar{d}(\rho_1, \rho_2)$ (the proof is like those above).

Finally, if $((\alpha, \beta), (\gamma, \delta))$ is in form (iii), $\gamma < 1$, and $\delta < 1$, then $\mathfrak{T}^*$ assigns $\langle 1, 2, 4, 4\rangle$, $\langle 1, 1, 2, 4\rangle$, and $\langle 1, 4, 3, 4\rangle$ positive measure. Thus, for $E = \langle 1\rangle$, $F = \langle 4\rangle$, $n = 2$, $d(\rho) > \bar{d}(\rho_1, \rho_2)$: ρ assigns positive measure to $\langle 2, 4\rangle = \langle (A, D), (B, D)\rangle$, $\langle 1, 2\rangle = \langle (A, C), (A, D)\rangle$ and $\langle 4, 3\rangle = \langle (B, D), (B, C)\rangle$, hence ρ can be improved upon by pairing $\langle A, B\rangle$ with $\langle C, D\rangle$, $\langle A, A\rangle$ with $\langle D, C\rangle$, and $\langle B, B\rangle$ with $\langle D, D\rangle$ (until the measure of $\langle 2, 3\rangle$ or $\langle 1, 2\rangle$ or $\langle 4, 3\rangle$ is reduced to zero). □

Theorem 4 together with Corollary 2 to Theorem 3 imply that if (α, β) and (γ, δ) have positive entropy and $M((\alpha, \beta), (\gamma, \delta)) > p((\alpha, \beta), (\gamma, \delta))$ then there is a mixing multi-step joining of (α, β) and (γ, δ) attaining a smaller distance than

$M((\alpha, \beta), (\gamma, \delta))$. If either (α, β) or (γ, δ) has zero entropy, it is not hard to show that $M((\alpha, \beta), (\gamma, \delta)) = \bar{d}((\alpha, \beta), (\gamma, \delta))$.

In [3] it is shown that there are (α, β) and (γ, δ) with positive entropy for which $\bar{d}((\alpha, \beta), (\gamma, \delta)) = p((\alpha, \beta), (\gamma, \delta))$ and $M((\alpha, \beta), (\gamma, \delta)) > \bar{d}((\alpha, \beta), (\gamma, \delta))$.

The following example shows that Theorem 4 doesn't extend to one-step three-state Markov processes.

EXAMPLE. Let $\mathfrak{T}_1$ and $\mathfrak{T}_2$ be the one-step three state-Markov processes with $P_1 = P_2 = \{1, 2, 3\}$ and respective transition matrices

$$\begin{bmatrix} 1-\alpha & 0 & \alpha \\ 0 & 1-\alpha & \alpha \\ \beta & \beta & 1-2\beta \end{bmatrix} \text{ and } \begin{bmatrix} 0 & 1-\alpha & \alpha \\ 1-\alpha & 0 & \alpha \\ \beta & \beta & 1-2\beta \end{bmatrix}$$

where $0 < \alpha < 1$ and $0 < \beta \leqslant \frac{1}{2}$. Let $\mathfrak{T}$ be the one-step joining of $\mathfrak{T}_1$ and $\mathfrak{T}_2$ which assigns positive measure to states (1, 1), (1, 2), (2, 1), (2, 2), and (3, 3), and whose transition matrix with respect to these states is

$$\begin{bmatrix} 0 & 1-\alpha & 0 & 0 & \alpha \\ 1-\alpha & 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & 1-\alpha & \alpha \\ 0 & 0 & 1-\alpha & 0 & \alpha \\ \beta & 0 & 0 & \beta & 1-2\beta \end{bmatrix}.$$

It is not hard to show that $d_{\mathfrak{T}} = \bar{d}(\mathfrak{T}_1, \mathfrak{T}_2) = 2\beta(1-\alpha)/(\alpha+2\beta)(2-\alpha)$, yet $\mathfrak{T}$ doesn't attain the partition distance between $\mathfrak{T}_1$ and $\mathfrak{T}_2$, since $p(\mathfrak{T}_1, \mathfrak{T}_2) = 0$.

Question. Is there a positive integer k and one-step k-state Markov processes $\mathfrak{T}_1$ and $\mathfrak{T}_2$ whose transition matrices have no zeros for which $M(\mathfrak{T}_1, \mathfrak{T}_2) = \bar{d}(\mathfrak{T}_1, \mathfrak{T}_2)$ and $\bar{d}(\mathfrak{T}_1, \mathfrak{T}_2) \neq p(\mathfrak{T}_1, \mathfrak{T}_2)$?

If the answer to the question is no, it may point to the proper extension of Theorem 4.

Note that the attainment or nonattainment of $\bar{d}$ between two processes with relatively prime periods by a one-step joining $\mathfrak{T}$ is completely determined by the set of nonzero entries in $\mathfrak{T}$'s transition matrix. If $\mathfrak{T}$ doesn't attain $\bar{d}$, by Theorem 2 there are states e and f, $n \in \mathbb{N}^+$, and a collection of $(n+2)$-strings with first term e and last term f which has positive measure and whose strings can be separated into their marginal $(n+2)$-strings and rematched more efficiently. Any other one-step joining whose set of nonzero entries in its transition matrix include all those of $\mathfrak{T}$ must assign those $(n+2)$-strings positive measure, hence by Theorem 1 cannot attain $\bar{d}$. Conversely, if $\mathfrak{T}$ is a one-step joining which attains $\bar{d}$, Theorems 1 and 2 imply that any other one-step joining whose set of nonzero entries is contained in $\mathfrak{T}$'s must attain $\bar{d}$.

The preceding observations apply to m-step joining (T, P) of two processes with relatively prime periods when the joinings are viewed as one-step Markov processes $(T, V_0^{m-1}T^iP)$.

Finally, if $\mathcal{T}_1$ and $\mathcal{T}_2$ do not have relatively prime periods, let c denote the greatest common divisor of their periods. Then there are c different kinds of ergodic joinings of $\mathcal{T}_1$ and $\mathcal{T}_2$ corresponding to the c possible "phase shifts" between their periods. The converse of Theorem 1 will not necessarily hold here, although $d(\rho) = \bar{d}(\rho_1, \rho_2)$ for all ρ implies $\mathcal{T}$ is a best possible joining *among joinings with the same phase shift*, for, as the example following Theorem 2 shows, there may be no joining with a particular phase shift attaining $\bar{d}$. Nonetheless, as shown above, it is true that the attainment or nonattainment by $\mathcal{T}$ of the smallest distance attainable by any joining with the same phase shift as $\mathcal{T}$ is completely determined by the set of nonzero entries in $\mathcal{T}$'s transition matrix (when $\mathcal{T}$ is viewed as a one-step Markov process $(T, \bigvee_{i=0}^{m-1} T^i P)$).

REFERENCES

[1] ADLER, R. L., SHIELDS, P. and SMORODINSKY, M. (1972). Irreducible Markov shifts. *Ann. Math. Statist.* **43** 1027–1029.

[2] ELLIS, M. H. (1978). Distances between two-state Markov processes attainable by Markov joinings. *Trans. Amer. Math. Soc.* **241** 129–153.

[3] ELLIS, M. H. (1980). On Kamae's conjecture concerning the $\bar{d}$-distance between two-state Markov processes. *Ann. Probability* **8** 372–376.

[4] ORNSTEIN, D. S. (1974). *Ergodic Theory, Randomness, and Dynamical Systems*. Yale University Press, New Haven.

[5] THOUVENOT, J. P. (1975). Une classe de systèmes pour lesquels la conjecture de Pinsker est vraie. *Israel J. Math.* **21** 208–214.

DEPARTMENT OF MATHEMATICS
NORTHEASTERN UNIVERSITY
360 HUNTINGTON AVENUE
BOSTON, MASSACHUSETTS 02115