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# INFINITESIMAL GENERATORS OF TIME CHANGED PROCESSES

BY HENRYK GZYL
*Universidad Central de Venezuela, Caracas*

We start with a standard Markov process $X$ and a continuous additive process $A$ of $X$ with fine support $\Phi$. We form the time changed process $X_\tau$, and we compute its weak infinitesimal generator in terms of the weak infinitesimal generator of the process $X$ and of the Lévy system of $(X_\tau, \tau)$. We give some examples.

**1. Introduction and preliminaries.** Let $X = (\Omega, \mathbf{F}, \mathbf{F}_t, X_t, \theta_t, P^x)_{x \in E}$ be a standard Markov process with state space $(E_\Delta, \mathbf{E}_\Delta)$ where $E$ is a locally compact, second countable metric space. As usual we denote by $\mathbf{E}^*$ the universal completion of $E$, $\Delta$ the cemetary point and $E_\Delta = E \cup \{\Delta\}$. (The reader should look in [1], [4] or [10] for any concept we use without explicit definition.) Let $A = (A_t)_{t>0}$ be a continuous additive functional which we assume to be perfect; i.e., we assume that $A_0 = 0, A_t \in \mathbf{F}_t$ that almost surely $t \to A_t$ is continuous and that almost surely $A_{t+s} = A_t + A_s \circ \theta_t \forall t, s \geqslant 0$. We shall assume that $E^x A_t < \infty$ for $x \in E$ and $t \geqslant 0$. We define $\tau_t = \inf\{u : A_u > t\}$, the usual right continuous inverse of $A_t$.

Denote by $\Phi$ the set$\{x : P^x(\tau_0 = 0) = 1\}$. $\Phi$ is called the (fine) support of $A$. If we consider $\Phi$ with the induced topology, then its Borel sets $\mathbf{\Phi}$ can be obtained as $\mathbf{\Phi} = \mathbf{E}|_\Phi$ and also $\mathbf{\Phi}^* = \mathbf{E}^*|_\Phi$, $\mathbf{\Phi}_\Delta^* = \mathbf{E}^*|_{\Phi_\Delta}$.

From $X$ and $\tau$ one constructs the process $X_\tau = (\Omega, \mathbf{F}, \mathbf{F}_t, X_{\tau_t}, \theta_{\tau_t}, P^x)$, which is a strong Markov process on $(\Phi_\Delta, \mathbf{\Phi}_\Delta^*)$ with transition semigroup $Q_t f(x) = I_\Phi(x) E^x f(X_{\tau_t})$, defined on $b\mathbf{\Phi}^*$ and extended to $b\mathbf{\Phi}_\Delta^*$ in the usual way.

Below we recall the notion of weak infinitesimal generator of a Markov process and in Section 2 we explore the relationship between the weak infinitesimal generator $G$ of $X$ and the weak infinitesimal generator $\hat{G}$ of $X_\tau$. We shall remark that our results are actually valid for random time substitutions not necessarily associated to additive functionals. That is, let $(\sigma_t)_{t>0}$ be an increasing, right continuous family of stopping times such that $X_\sigma$ is at least a Hunt process and a Lévy system can be found for the joint process $(X_\sigma, \sigma)$; then if care is taken as to what the state space of $X_{\sigma_t}$ is, our proof still provides us with a method of computing the weak infinitesimal generator of $X_\sigma = (\Omega, \mathbf{F}, \mathbf{F}_{\sigma_t}, X_{\sigma_t}, \theta_{\sigma_t}, P^x)$.

The result we present here extends a previous result by Dynkin which we treat in Example 1. This same situation has also been treated in an abstract setting (see [8] and references quoted therein). In some of the examples below we consider the situation described in the previous paragraph. Also, in [6], Karoui and Reinhard compute the generator of the process obtained from $X$ by time changing it with respect to $\sigma_t = \tau_{A_t}$. They use an approach entirely different from ours.

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Let us now continue with an outline of the properties of the process $(X_t, \tau)$ that we are going to need below. In order to have a nice behavior for the left limits of $X_t$, i.e., for $X_t$ to be quasileft-continuous we bring in the following result (privately transmitted to me by Bernard Maisonneuve).

LEMMA 1.1. *The process* $(\tau_t)_{t>0}$ *is quasileft-continuous with respect to* $\{F_{\tau_t}\}_{t>0}$.

COMMENT. As a consequence of this lemma $X_\tau$ is quasileft-continuous on $[0, \hat{\zeta})$, $\hat{\zeta} = A_\zeta$.

PROOF. Let $T_n$ be an increasing sequence of stopping times of $\{\mathbf{F}_{\tau_t}\}$, with limit $T$. Put $D_t = \tau(A_t)$, then it is easy to see that $\tau_{T_n-}$, $\tau_T$ are stopping times of $\mathbf{F}_{D_t}$ and $\tau_{T_n-}$ increases to $\tau_{T-}$.

Observe that $\tau_t = D_{\tau_t-}$ and apply the quasileft-continuity of the process $D_t$ (see for example the remark on page 81 of [6] or in Maisonneuve's "Ensembles regenerativs", *Asterisque* **15**, page 27, 1974) with respect to $\mathbf{F}_{D_t}$ to obtain that $\tau_{T_n} \nearrow \tau_T$ a.s.

Now it is easy to verify that $(X_\tau, \tau)$ is a Markov additive process according to [2]. Let us proceed to make some regularity assumptions on the continuous part of $\tau$ and on the Lévy system of $(X_\tau, \tau)$.

ASSUMPTION A.1. The continuous part $\tau^c$ of $\tau$ satisfies

$$\tau_t^c = \int_0^t a(X_{\tau_s})\, ds \tag{1.2}$$

where $a$ is finely continuous for the process $X_\tau$.

ASSUMPTION A.2. The Lévy system of $(X_\tau, \tau)$ (see [2]) is such that for any bounded previsible (relative to $\mathbf{F}_{\tau_t}$) process $Z$, and for any bounded $f \in \mathbf{\Phi} \otimes \mathbf{\Phi}_\Delta \otimes B(\overline{\mathbb{R}}_+)$ the following holds

$$E^x \Sigma_{s>0} Z_s f(X_{\tau_s-}^-, X_{\tau_s}, \Delta\tau_s) I_{\{X_{\tau_s-}^- \neq X_{\tau_s} \text{ or } \tau_{s-} \neq \tau_s\}} \tag{1.3}$$
$$= E^x \int_0^\infty Z_s\, ds \int_{\Phi_\Delta \times \overline{\mathbb{R}}_+} L(X_{\tau_s}, dy, du) f(X_{\tau_s}, y, u).$$

Furthermore, for any $g \in b(\mathbf{\Phi}_\Delta \otimes B(\overline{\mathbb{R}}_+))$ and $x \in \Phi$,

$$t \to \int_{\Phi_\Delta \times \overline{\mathbb{R}}_+} L(X_{\tau_s}, dy, du) g(y, u) \text{ is a.s. } [P^x] \text{ right continuous.}$$

In (1.3) $X_t^- = X_{t-}$. We should remark that the real assumptions in A1 and A2 are not the forms given by (1.2) and (1.3), which can be obtained by appropriate time changing with respect to a strictly increasing continuous additive functional. The real assumptions lie in the regularity conditions which make the proof of 2.1 easy, and in order to study them, one would have to take a closer look at Lévy systems of time changed processes [5], exit systems [9], and the conditional structure of $\zeta$ given $X_\zeta$ [11], as well as to the original process $X$ itself.

Let us now introduce the definitions of weak infinitesimal generator of $X$ and $X_\tau$, following [4], as we are going to use them in the next section.

For the process $X$ define

$$\mathbf{L} = \{f \in b\mathbf{E}_\Delta : P_t f(x) \to f(x) \forall x \in E \text{ as } t \downarrow 0\}$$

and then define $\mathbf{D} = U^\alpha \mathbf{L}$, which turns out to be independent of $\alpha$, for $\alpha > 0$. One can then prove that $\mathbf{D} = \{f \in b\mathbf{E}_\Delta : (1/t)(P_t f - f)$ converges boundedly in $t$, to $g \in \mathbf{L}\}$.

For $f \in \mathbf{D}$, the limit of $(1/t)(P_t f(x) - f(x))$ as $t \downarrow 0$ is denoted by $Gf(x)$, and the mapping $G : \mathbf{D} \to \mathbf{L}$ satisfies $U^\alpha(\alpha - G)g = g \forall g \in \mathbf{D}$ and $(\alpha - G)(U^\alpha f = f \forall f \in \mathbf{L}$.

Since in general $Q_t : b\Phi^* \to b\Phi^*$, we define

$$\hat{\mathbf{L}} = \{f \in b\Phi^*_\Delta : Q_t f(x) \to f(x) x \in \Phi \text{ as } t \downarrow 0\}.$$

If we write $W^\alpha f(x) = E^x \int_0^\infty e^{-\alpha t} f(X_{\tau_t})\, dt$ for $f \in b\Phi^*$, then $W^\alpha$ the resolvent associated with $Q_t$, and as before we put $\hat{\mathbf{D}} = W^\alpha \hat{\mathbf{L}}$. It is easy to verify now that $\hat{\mathbf{D}} = \{f \in b\Phi^*_\Delta : (1/t)(Q_t f(x) - f(x))$ converges boundedly in $t$, to $g \in \hat{\mathbf{L}}\}$.

We are going to denote $\lim_{t \downarrow 0}(1/t)(Q_t f - f)$, for $f \in \hat{\mathbf{D}}$, by $\hat{G}f(x)$, and, as before, one verifies that the mapping $\hat{G} : \hat{\mathbf{D}} \to \hat{\mathbf{L}}$ satisfies $W^\alpha(\alpha - \hat{G})g = g$ and $(\alpha - G)W^\alpha f = f$ for $g \in \hat{\mathbf{D}}$ and $f \in \hat{\mathbf{L}}$, respectively.

In the course of the proof of 2.1 we will have to make use of the following result about exit systems. Let $M = \{(t, \omega) : A_{t+\varepsilon}(\omega) - A_{t-\varepsilon}(\omega) > 0\}$, then $\vec{M} = \{(\tau_s - (\omega), \omega) : \tau_{s-}(\omega) \neq \tau_s(\omega), s \in \mathbb{R}_+\}$, is the set of positive left end points of intervals contiguous to $M$. It is proved in [9] that there exists a pair $(\tilde{P}, K)$ with $K$ a continuous additive functional with support $\Phi$, $\tilde{P}$ a kernel from $\mathbf{F}^*$ to $\mathbf{E}^*$ such that (among other things) for every bounded positive well-measurable (relative to $\mathbf{F}_t$) process $Z$ and every bounded $\mathbf{F}^*$-measurable function $H$, the following identity holds.

$$(1.4) \quad E^x \Sigma_{s \in \vec{M}} Z_s I_\Phi(X_s) H \circ \theta_s = E^x \Sigma_{s>0} Z_{\tau_s -} I_\Phi(X_{\tau_s -}) H \circ \theta_{\tau_s -} I_{(0,\infty)}(\Delta\tau_s)$$
$$= E^x \int_0^\infty Z_s \tilde{E}^{X_s}(H)\, dK_s.$$

It is also mentioned in [9] (with reference for the proof) that there exists a sequence of stopping times $T_n$ such that

$$(1.5) \qquad E^x \Sigma_{s \in \vec{M}} Z_s I_{\Phi^c}(X_s) H \circ \theta_s = \Sigma_n E^x\{Z_{T_n} E^{X_{T_n}}[H]\}.$$

Actually, since the points of the set $\{s \in \vec{M}, X_s \notin \Phi\}$ are isolated from the right, one can take $T_1 = \inf\{s \in \vec{M} : X_s \notin \Phi\}$ and $T_n = T_1 + T_{n-1} \circ \theta_T = T_{n-1} + T_1 \circ \theta_{T_{n-1}}$. Also $T_n > T_{n-1}$ if $T_{n-1} < \infty$. In the course of the proof of Theorem 2.1 we will have to compute a limit of the form $\lim_{t \downarrow 0}(1/t)E^x\{E^{X(T_1)}(f(X_{\tau_0}) - f(x_0));\ A_{T_1} \leqslant t\}$ for $x \in \Phi$.

Let $Z$ be a bounded previsible process and $B_t^g = g(X_T)I_{[A_T, \infty)}(t)$, then $E^x(Z_{A_T} g(X_T);\ A_T \leqslant t] = E^x \int_0^\infty Z_s\, dB_s^g$. According to [11], there exists a right continuous, adapted, increasing process $C_t^g$ such that for each $\mu$, $C^g$ is the dual previsible projection of $B_t^g$ relative to $P^\mu$. We shall denote by $C_t$ the dual projection of $B_t^1$, and remark that there exists a bounded positive kernel $B$ such that $C_t^g = \int_0^t Bg(X_s)\, dC_s$ (see appendix to [9]). At this point we state our

ASSUMPTION A.3. We shall assume that $t \to Bg(X_t)$ is a.s. right continuous and that $C_t \equiv t$ whenever $\cup_n [T_n] \neq \phi$.

With this assumption we can prove the following

PROPOSITION 1.6. *For* $g \in bE^*$, $x \in \Phi$

$$\lim_{t \downarrow 0} \frac{1}{t} E^x\left[ g(X_{T_1}); A_{T_1} \leqslant t \right] = Bg(x)$$

$$\lim_{t \downarrow 0} \frac{1}{t} E^x\left[ g(X_{T_n}); A_{T_n} \leqslant t \right] = 0 \forall n > 1.$$

PROOF. The first assertion follows from (A.3) and the second from the strong Markov property and dominated convergence theorem.

In the next section we are going to see how to compute $\lim_{t\downarrow 0}(1/t)(Q_t f(x) - f(x))$ for $f \in \mathbf{D}$ and $x \in \Phi$ in terms of $Gf$, and the kernels $K(x, A) = L(x, A - \{x\}, (0, \infty])$ and $B$. Then with one extra assumption on $A$ we will be able to extend $f \in \hat{\mathbf{D}}$ to $f \in \mathbf{D}$ and then show how to compute $\hat{G}f$ in terms of $Gf$ and the kernel $K$ applied to $f$.

The assumption we shall need is

ASSUMPTION A.4. Assume that $\lim_{t\downarrow 0} (1/t)E^x A_t = b(x)$ exists, is a bounded function on $\Phi$ and vanishes off $\Phi$.

We shall also mention that we shall be adhering to the "lifetime formalism," i.e., functions defined on $E(\Phi)$ will be considered extended to $E_\Delta(\Phi_\Delta)$ and vanishing on $\Delta$.

## 2. The weak infinitesimal generator of $X_\tau$.

Let us begin this section with

THEOREM 2.1. *Let* $f \in \mathbf{D}$ *be such that* a.s. $t \to Gf(X_t)$ *is right continuous (for example,* $f \in U^\alpha \mathbf{D}$ *which is dense in* $\mathbf{D}$) *and assume that* (A.1), (A.2), *and* (A.3) *hold. Then for* $x \in \Phi$

$$(2.2) \quad \lim_{t\downarrow 0} \frac{1}{t}(Q_t f(x) - f(x)) = a(x)Gf(x) + \int_\Phi K(x, dy)(f(y) - f(x)) + B(\delta f)(x)$$

*where* $\delta f(x) = E^x f(X_{\tau_0}) - f(x)$ *and* $K(x, A) = L(x, A - \{x\}, (0, \infty])$ *for* $A \in \Phi^*$.

PROOF. The basic idea comes from the fact that for $f \in \mathbf{D}$, $M_t = f(X_t) - f(X_0) - \int_0^t Gf(X_s)\,ds$ is a locally square integrable martingale. Therefore $M_{\tau_t}$ is only a locally square integrable martingale relative to $\mathcal{F}_{\tau_t}$. According to Theorem 1 in Kazamaki [7], there exists a continuous time change $\sigma_t$ (relative to $\mathcal{F}_{\tau_t}$), increasing from 0 to $\infty$, such that $M_{\tau(\sigma_t)}$ is a square integrable martingale and it vanishes at 0 for every $P^x$ with $x \in \Phi$. With this comment it is easy to see that nothing is lost if we assume that $M_{\tau_t}$ is square integrable for the following argument.

An application of the change of variables formula [10] and our assumption on $\tau_t^c$ yields

(2.3)

$$\int_0^{\tau_t} Gf(X_s)\,ds = \int_0^t Gf(X_{\tau_{s-}})\,d\tau_s + \Sigma_{s<t}\{\int_{\tau_{s-}}^{\tau_s} Gf(X_u)\,du - Gf(X_{\tau_{s-}})\Delta\tau_s\}$$
$$= \int_0^t Gf(X_{\tau_s})a(X_{\tau_s})\,ds + \Sigma_{s<t}\int_{\tau_{s-}}^{\tau_s} Gf(X_u)\,du.$$

Note now that the set $\{(\tau_{s-}(\omega), \omega) : \tau_{s-}(\omega) \neq \tau_s(\omega),\ s \in R_+\}$ is the set consisting of the positive left end points to the intervals contiguous to the perfect, homogeneous, random set $M = \{(t, \omega) : A_{t+\varepsilon} - A_{t-\varepsilon} > 0\ \forall \varepsilon > 0\}$. It is proved in [3] that $L_n^\varepsilon + \varepsilon$ and $R_n^\varepsilon$ are stopping times relative to $\{\mathbf{F}_t\}$ and it is easy to see that

$$\Sigma_{s<t} M_{\tau_s} - M_{\tau_{s-}} = \lim_{\varepsilon\to 0}\Sigma_n M_{R_n^\varepsilon\wedge\tau_t} - M_{(L_n^\varepsilon+\varepsilon)\wedge\tau_t}$$

and since $M_{\tau_t}$ is a $P^x$ martingale for $x \in \Phi$

$$E^x\Sigma_{s<t} M_{\tau_s} - M_{\tau_{s-}} = \lim_{\varepsilon\downarrow 0}\Sigma_n E^x\big(M_{R_n^\varepsilon\wedge\tau_t} - M_{(L_n^\varepsilon+\varepsilon)\wedge\tau_t}\big) = 0$$

because of the optional sampling theorem. Therefore,

$$E^x\Sigma_{s<t}\int_{\tau_{s-}}^{\tau_s} Gf(X_u)\,du = E^x\Sigma_{s<t}\{f(X_{\tau_s}) - f(X_{\tau_{s-}})\}I_{(0,\infty]}(\Delta\tau_s).$$

Also, taking $Z_s = |f(X_{s-}) - f(X_s)|I_{\{X_{s-}\neq X_s\}}$ and $H \equiv 1$ and substituting it in (1.4) it is easy to verify that the processes

$$\Sigma_{s<t}\{(f(X_{\tau_s}) - f(X_{\tau_{s-}}^-)\}I_\Phi(X_{\tau_{s-}})I_{(0,\infty]}(\Delta\tau_s)$$

and

$$\Sigma_{s\leq t}\{f(X_{\tau_s}) - f(X_{\tau_{s-}})\}I_\Phi(X_{\tau_{s-}})I_{(0,\infty]}(\Delta\tau_s)$$

are indistinguishable, and therefore from (1.3)

(2.4)
$$E^x\Sigma_{s\leq t}\{f(X_{\tau_s}) - f(X_{\tau_{s-}})\}I_\Phi(X_{\tau_{s-}})I_{(0,\infty]}(\Delta\tau_s)$$
$$= E^x\int_0^t ds\int_\Phi K(X_{\tau_s}, dy)(f(y) - f(X_{\tau_s})).$$

Consider now the term

$$E^x\Sigma_{s<t}\{f(X_{\tau_s}) - f(X_{\tau_{s-}})\}I_{\Phi^c}(X_{\tau_{s-}})I_{(0,\infty]}(\Delta\tau_s).$$

From the comments preceding (A.3) we see that this term can be written as

$$\Sigma_n E^x\{E^{X(T_n)}[f(X_{\tau_0}) - f(X_0)];\ T_n \leq \tau_t\}$$
$$= \Sigma_n E^x\{E^{X(T_n)}[f(X_{\tau_0}) - f(X_0)];\ A_{T_n} \leq t\};$$

i.e.,

(2.5)
$$E^x\Sigma_{s<t}\{f(X_{\tau_s}) - f(X_{\tau_{s-}})\}I_{\Phi^c}(X_{\tau_{s-}})I_{(0,\infty]}(\Delta\tau_s))\}$$
$$= \Sigma_n E^x\{\delta f(X_{T_n});\ A_{T_n} \leq t\}$$

Now, from (2.3), (2.4), (2.5) and our assumptions (A.1), (A.2), and (A.3) it follows that for $x \in \Phi$

$$\lim_{t \downarrow 0} \frac{1}{t}(Qf(x) - f(x)) = a(x)Gf(x) + \int_{\Phi} K(x, dy)(f(y) - f(x)) + B\delta f(x).$$

COMMENT. When $\Phi$ is projective, in particular when $F$ is finely perfect and closed, then $B \equiv 0$ and

$$\lim_{t \downarrow 0} \frac{1}{t}(Q_t f(x) - f(x)) = a(x)Gf(x) + \int_{\Phi} K(x, dy)(f(y) - f(x)).$$

THEOREM 2.6. *Let $f \in \hat{\mathbf{D}}$ be such that* a.s. $t \to \hat{G}f(X_{\tau_t})$ *is right continuous and assume that* (A.4) *holds. Then $\bar{f}(x) = E^x f(X_{\tau_0})$ is an extension of $f$ to $E$ such that*

$$\text{(2.7)} \qquad \lim_{t \downarrow 0} \frac{1}{t}(P_t \bar{f}(x) - \bar{f}(x)) = b(x)\hat{G}f(x) \qquad x \in \Phi$$

$$= 0 \qquad x \notin \Phi$$

and also

$$\text{(2.8)} \qquad \hat{G}f(x) = a(x)G\bar{f}(x) + \int_{\Phi} K(x, dy)(f(y) - f(x)).$$

PROOF. Certainly $E^x\{f(X_{\tau_0})\} = \bar{f}(x)$ is an extension of $f$ to $E$. Also from the optional sampling theorem follows that $P_t \bar{f}(x) - \bar{f}(x) = E^x \int_0^{A_t} \hat{G}f(X_{\tau_s})$ from which (2.7) follows if we investigate (A.4). Since

$$\hat{G}f(x) = \lim_{t \downarrow 0} \frac{1}{t}(Q_t \bar{f}(x) - \bar{f}(x) = \lim_{t \downarrow 0} \frac{1}{t}(Q_t f(x) - f(x)),$$

we obtain (2.8) from Theorem 2.1 and the fact that $E^x\{\bar{f}(X_{\tau_0}) - \bar{f}(X_0)\} = 0$.

COMMENTS. We shall see in (2.6) that if $x \in \mathring{\Phi}$, then

$$\lim_{t \downarrow 0} \frac{1}{t}(Q_t f(x) - f(x)) = a(x)Gf(x).$$

A similar calculation will show that $b(x) = 1/a(x)$ and therefore $\hat{G}f(x) = a(x)G\bar{f}(x)$.

Also from (2.8) it follows that if $1 > a(x)b(x)$ then

$$\hat{G}f(x) = \frac{1}{1 - a(x)b(x)} \int_{\Phi} K(x, dy)(f(y) - f(x)) \qquad \text{for } x \in \Phi - \mathring{\Phi}.$$

To end this section we mention that in the case where $A$ is adapted to $\sigma(X_s : s < t)$, $\Phi$ happens to be a Borel set and $X_\tau$ is a standard process with state space $(\Phi_\Delta, \mathbf{\Phi}_\Delta)$, if $\Phi$ is closed. Let us now recall Theorem I.2.3 of [4] as

THEOREM 2.9. *Every stochastically continuous transition semigroup on the topological state space $(E, \mathbf{O}, \mathbf{E})$ — $\mathbf{O}$ being the topology on $E$, is uniquely determined by its infinitesimal generator.*

Therefore, since $X$ is a right continuous normal process, its transition semigroup is uniquely determined by G. Also, since $\Phi$ is finely closed and $\mathbf{\Phi}$ is the $\sigma$-algebra on $\Phi$ generated by the induced topology, it follows that the semigroup $(Q_t)$ is stochastically continuous and uniquely determined by its weak infinitesimal generator $\hat{G}$, according to Theorem 2.9.

FINAL COMMENTS.

(i) Notice that for $f_1$, $f_2$ satisfying the conditions of Theorem 2.1,

$$\lim_{t\downarrow 0}\frac{1}{t}(Q_t f_i(x) - f_i(x)) \quad \text{and} \quad \int K(x, dy)(f_i(y) - f_i(x)), i = 1, 2,$$

depend only on the values of $f_i$ on $\Phi$; therefore,

$$a(x)Gf_1(x) + B(\delta f_1)(x) = a(x)Gf_2(x) + B(\delta f_2)(x).$$

This implies that if we extend $f \in \hat{\mathbf{D}}$ to $f' \in \mathbf{D}$ in a way different to the one mentioned in Theorem 2.6, we will obtain

$$\begin{aligned}\hat{G}f(x) &= a(x)G\bar{f}(x) + \int_\Phi K(x, dy)(f(y) - f(x)) \\ &= a(x)Gf'(x) + \int_\Phi K(x, dy)(f(y) - f(x)) + B(\delta f')(x).\end{aligned}$$

(ii) If we define $\mathbf{D}$ (and $\hat{\mathbf{D}}$) to be the class of $b\mathbf{E}$ (and $b\mathbf{\Phi}^*$) measurable functions for which there exists $g \in b\mathbf{E}$ (or $\mathbf{\Phi}^*$) such that

$$f(X_t) - f(X_0) - \int_0^t g(X_s)\, ds$$

is a $P^x$-local martingale for $x \in E (x \in \Phi)$, then we can drop the regularity assumptions in (A.1), (A.2), and (A.3).

## 3. Examples.

(3.1). Assume that $h(x)$ is a strictly positive, finely continuous function and put $A_t = \int_0^t h(X_s)\, ds$. Since in this case $t = \tau_{A_t} = A_{\tau_t}$, and, therefore, $t = \int_0^{\tau_t} h(X_{\tau_{A_s}})\, d\tau_{A_s} = \int_0^t h(X_{\tau_s})\, d\tau_s$ from which it follows that $d\tau_s = ds/h(X_{\tau_s}) \equiv a(X_{\tau_s})\, ds$. Also, since in this case $\tau_{s-} = \tau_s$ for every $s \geqslant 0$, it follows from Theorem 2.1 that

$$\hat{G}f(x) = a(x)Gf(x)$$

which is X.10.24 in [4].

(3.2). Let us assume that $A$ is such that $\tau$ is a pure jump process, i.e., $\tau^c \equiv 0$. In this case (2.2) yields that for $f \in \mathbf{D}$ and $x \in \Phi$

$$\hat{G}f(x) = \int_{\Phi_\Delta} K(x, dy)\{f(y) - f(x)\}$$

and in this case, when $f = I_{\{x_0\}}$ for $x_0 \in \Phi$ happens to be in $\mathbf{D}$, then if we put $\hat{T}_0 = \inf\{u : X_{\tau_u} \neq x_0\}$ it is easy to conclude that $P^{x_0}(T_0 > t)$ has a density with respect to Lebesgue measure given by $-E^{x_0}K(X_{\tau_t}, \{x_0\}^c)$. In this case the process $X_\tau$ is a pure jump process.

(3.3). Let us now consider a variation on the theme of Example 2 of [6]. We start with a standard Markov process on $(E, \mathbf{E})$ and a one-dimensional Lévy process $\tau = (W, \mathbf{G}, Q)$ with positive increments. By forming the appropriate products we can consider $X$ and $\tau$ independent relative to $(\Omega \times W, \mathbf{F} \otimes \mathbf{G}, P^x \otimes Q)$. We shall write $\tau_t = ct + \tau_t'$, where $\tau_t'$ is the discontinuous part of $\tau_t$, and assume that for every bounded $F \in \mathcal{B}(\mathbb{R}_+) \times \mathcal{B}(\overline{\mathbb{R}}_+)$ the following holds ($E$ stands for expectation with respect to $dQ$)

$$E\Sigma_{s<t} F(\tau_{s-}, \Delta\tau_s) I_{\{\tau_{s-} \neq \tau_s\}} = E\int_0^t ds \int_0^\infty F(\tau_s, u) n(du)$$

where $n(du)$ is the Lévy measure of $\tau$.

If we denote by $\tilde{P}^x$ the measure $P^x \otimes Q$, it is easy to see that

$$\begin{aligned}
\tilde{E}^x \Sigma_{s<t} \int_{\tau_{s-}}^{\tau_s} Gf(X_u)\, du &= E^x \int_0^t ds \int_0^\infty \left(\int_{\tau_s}^{\tau_s+u} Gf(X_v)\, dv\right) n(du) \\
&= E^x \int_0^\tau ds \int_0^\infty \int_0^\infty \left(\int_r^{r+u} Gf(X_v)\, dv\right) n(du) Q(\tau_s \in dr) \\
&= \int_0^t ds \int_0^\infty \int_0^\infty E^x \left\{\int_0^u Gf(X_v) \circ \theta_r\, dv\right\} du\; Q(\tau_s \in dr) \\
&= \int_0^t ds \int_0^\infty \int_0^\infty \{P_{u+r} f(x) - P_r f(x)\} n(du) Q(\tau_s \in dr) \\
&= \int_0^t ds \int_0^\infty E\left\{P_{\tau_s+u} f(x) - P_{\tau_s} f(x)\right\} n(du).
\end{aligned}$$

When we divide the first and last terms of this chain by $t$ and we let $t \downarrow 0$ we obtain that this case

$$\hat{G}f(x) = cGf(x) + \int_0^\infty \{P_u f(x) - f(x)\} n(du) \tag{3.4}$$

which extends the results in [6] for functions $f$ not vanishing in the neighborhood of $x$.

(3.5). With the same notations as in Sections 1 and 2, we shall verify that any $f \in \hat{\mathbf{D}}$ is in $D_e(A)$, the extended domain of $A$ for $X_{\tau(A_t)}$. See Chapter IX in [7] for all the facts that we do not comment on here. By definition, $f \in D_e(A)$ if there exists a bounded, universally measurable $g$, denoted by $Df$ in [7], such that

$$f(X_{\tau(A_t)}) - f(X_{\tau_0}) - \int_0^t g(X_{\tau(A_s)})\, dA_s$$

is a martingale with respect to $(\Omega, \mathbf{F}_{\tau(A_t)}, P^x)$ for $x \in \Phi$.

To prove that $f \in \hat{\mathbf{D}}$ implies that $f \in D_e(A)$ notice that $f(X_{\tau_t}) - f(X_{\tau_0}) - \int_0^t \hat{G}f(X_{\tau_s})\, ds = N_t$ is a martingale relative to $(\Omega, \mathbf{F}_{\tau_t}, P^x)$ for $x \in \Phi$. From the optional sampling theorem it follows that $f(X_{\tau(A_t)}) - f(X_{\tau_0}) - \int_0^{A_t} \hat{G}f(X_{\tau_s})\, ds = N_{A_t}$ is a martingale relative to $\mathbf{F}_{\tau(A_t)}$. Now, it is very easy to verify that $\int_0^{A_t} \hat{G}f(X_{\tau_s})\, ds = \int_0^t Gf(X_{\tau(A_s)})\, dA_s$, and the desired result follows from putting $g = Df = \hat{G}f$. This result and the result contained in Proposition 47, Chapter IX of [7], provide us with another representation for $Df$.

(3.6). Let us now consider a refinement of the result of Theorem 2.1. For this, let $x$ be an interior point of $\Phi$, and let $U$ be a neighborhood of $x$ such that $U \subset \Phi$ and let us put $T = \inf\{t > 0 : X_t \in \Phi - U\}$. From the right continuity of the trajectories of $X$ it follows that $P^x(T > 0) = 1$, and since $X_T \in \Phi$ it follows that $P^x(A_T > 0) = 1$.

It is proved in [11] that $\tau_t^c = \int_0^{\tau_t} I_\Phi(X_s)\, ds$, from which it follows that $\tau_t^c = \tau_t$ on $\tau_t \leqslant T$ a.s. $P^x$. It is easy to see now that

$$\Sigma_{s<t}(f(X_{\tau_s}) - f(X_{\tau_{s-}}^-))I_{(0,\infty]}(\Delta\tau_s) = \Sigma_{A_t<s<t}(f(X_{\tau_s}) - f(X_{\tau_{s-}}^-))I_{(0,\infty]}(\Delta\tau_s)$$

and therefore, taking expectations, dividing by $t$ and letting $t\downarrow 0$, we obtain

$$lim_{t\downarrow 0}(1/t)E^x\Sigma_{s<t}(f(X_{\tau_s}) - f(X_{\tau_{s-}}^-))I_{(0,\infty]}(\Delta\tau_s)$$
$$= \{\int_{\Phi_\Delta} K(x, dy)(f(y) - f(x))\}P^x(A_T = 0).$$

From this we can conclude that for $x \in \mathring{\Phi}$ ($\mathring{\Phi}$ denotes the interior of $\Phi$)

$$\hat{G}f(x) = a(x)Gf(x).$$

From this and (2.2) we can conclude that

$$\hat{G}f(x) = a(x)Gf(x) \qquad x \in \mathring{\Phi} \tag{3.7}$$
$$= a(x)Gf(x) + \int_{\Phi_\Delta} K(x, dy)(f(y) - f(x)) \qquad x \in \Phi - \mathring{\Phi}.$$

(3.8). Let us now start with two standard Markov processes $X^{(i)} = (\Omega^{(i)}, \mathbf{F}^{(i)}, \mathbf{F}_t^{(i)}, \theta^{(i)}, X_t^{(i)}, P_i^{x_i})$, with locally compact second countable state spaces $(E_i, \mathbf{E}_i)$, $i = 1, 2$. Assume for simplicity that both processes have infinite lifetime and form the product process $X = (X^{(1)}, X^{(2)}) = (\Omega, \mathbf{F}, \mathbf{F}_t, \theta_t, X_t, P^{(x_1,x_2)})$ where $\Omega = \Omega^{(1)} \times \Omega^{(2)}, P^{(x_1,x_2)} = P^{x_1} \otimes P^{x_2}$ for $(x_1, x_2) \in E_1 \times E_2, X_t = (X_t^{(1)}, X_t^{(2)}), \theta_t = (\theta_t^{(1)}, \theta_t^{(2)})$, $\mathbf{F}$ being the usual completion of $\sigma(X_t^{(1)} : t \geqslant 0) \times \sigma(X_t^{(2)} : t \geqslant 0)$, etc. One can define three semigroups on $b\mathbf{E}$, namely

$$P_t^{(1)}f(x, y) = E^{x,y}f(X_0^{(1)}, X_t^{(2)}) = E_2^y f(x, X_t^{(2)}),$$
$$P_t^{(2)}f(x, y) = E^{x,y}f(X_t^{(1)}, X_0^{(2)}) = E_1^x f(X_t^{(1)}X_0^{(2)})$$

and their product

$$P_t f(x, y) = P_t^{(1)}P_t^{(2)}f(x, y) = P_t^{(2)}P_t^{(1)}f(x, y) = E^{(x,y)}f(X_t^{(1)}, X_t^{(2)}).$$

If we denote by $G_i$ the generator of the process $X^i$, it is easy to see that the generators of $P_t^1$, $P_t^2$ and $P_t$ are given by $G_1 \otimes I$, $I \otimes G_2$ and $G = G_1 \otimes I + I \otimes G_2$,where certainly the last identity is valid at least on $\mathbf{D}(G_1 \otimes I) \cap \mathbf{D}(I \otimes G_2)$. Instead of $G_1 \otimes I$ and $I \otimes G_2$ we shall be writing $G_1$ and $G_2$, respectively.

Suppose now that we are given a continuous additive functional $A$ of $X^{(1)}$ whose support is a closed set $\Phi \subset E_1$. Then the time changed process $X_\tau = (X_\tau^{(1)}, X_\tau^{(2)})$ has the set $(\Phi \times E_2)_\Delta$ for state space and $\hat{\zeta} = A_\infty$ as lifetime.

Again we shall denote the time changed semigroup by $(Q_t)$, its generator by $\hat{G}$, the Lévy kernel of $(X_\tau^{(1)}, \tau)$ by $N(x, d\xi, du)$ and assume that A1 and A2 hold true. As in the proof of 2.1 we have to compute the limit of

$$\frac{1}{t}E^{(x,y)}\int_0^t a(X_{\tau_s})Gf(X_{\tau_s})\,ds \text{ and } \frac{1}{t}E^{(x,y)}\Sigma_{s<t}(f(X_{\tau_s}) - f(X_{\tau_{s-}}^-))I_{(0,\infty]}(\Delta\tau_s)$$

as $t\downarrow 0$ for $(x, y) \in \Phi \times E_2$.

It is easy to see that

$$\frac{1}{t}E^{(x,y)}\int_0^t a(X_{\tau_s})Gf(X_{\tau_s})\,ds \to a(x)(G_1f(x, y) + G_2f(x, y)).$$

In order to compute the other limit notice that we have set up things so that $(X_t^1)$ and $(X_t^2)$, and therefore $(\tau_t)$ and $(X_t^2)$, be independent processes; therefore, if we put $P_T^{(2)}(X_T^{(1)}, y) = E_2^y f(X_T^{(1)}, X_T^{(2)})$ for any $(\mathbf{F}_t^{(1)})$-stopping time $T$, it is easy to see that

$$E^{(x,y)}\Sigma_{s<t}(f(X_{\tau_s}) - f(X_{\tau_{s-}}^-))I_{(0,\infty](\Delta\tau_s)}$$

$$= E^x\Sigma_{s<t}(P_{\tau_s}^{(2)}(X_{\tau_s}^{(1)}, y) - P_{\tau_{s-}}^{(2)}(X_{\tau_{s-}}^{(1)-}, y))I_{(0,\infty]}(\Delta\tau_s)$$

$$= E^x\int_0^t ds\int_{\Phi_\Delta\times\bar{\mathbf{R}}_+} N(X_{\tau_s}^1, d\xi, du)\{P_{\tau_s+u}^{(2)}(\xi, y) - P_{\tau_s}^{(2)}(X_{\tau_s}, y)\},$$

and dividing by $t$ and letting $t\downarrow 0$ we obtain that

$$lim_{t\downarrow 0}\frac{1}{t}E^{(x,y)}\Sigma_{s<t}(f(X_{\tau_s}) - f(X_{\tau_{s-}}^-))I_{(0,\infty]}(\Delta\tau_s)$$

$$= \int_{\Phi_\Delta\times\bar{\mathbf{R}}_+} N(x, d\xi, du)\{P_u^{(2)}f(\xi, y) - f(x, y)\}.$$

From all this it follows that

$$\hat{G}f(x, y) = a(x)(G_1f(x, y) + G_2f(x, y)) + \int_{\Phi_\Delta\times\bar{\mathbf{R}}_+} N(x, d\xi, du)\{P_u^{(2)}f(\xi, y) - f(x, y)\}. \tag{3.9}$$

This result and (3.7) allows us to compare the behavior of $X$ and $X_\tau$ in various particular cases. For example, if $a > 0$ on $\mathring{\Phi}$ and $a \equiv 0$ on $\partial\Phi$, then within $\mathring{\Phi}$, $X$ and $X_\tau$ have the same behavior, and the second term describes the combined behavior of $(X_\tau^{(1)}, X_\tau^{(2)})$ on $\partial\Phi \times E_2$.

From 3.9 and Example 3.1 it follows that if $a > 0$ on $\Phi$, then one further time change with respect to the continuous additive functional of $X_\tau^1$ given by $B_t = \int_0^t a(X_{\tau_s}^1)\,ds$ provides us with another process $Y_t = X_{\tau_{\sigma_t}}$, $\sigma_1 = \inf\{u : B_u > t\}$, with the same hitting distributions as $X_\tau$, and with generator

$$\tilde{G}f(x, y) = Gf(x, y) + \int_{\Phi_\Delta\times\bar{\mathbf{R}}_+} \bar{N}(x, d\xi, du)\{P_u^{(2)}f(\xi, y) - f(x, y)\} \tag{3.10}$$

where $\bar{N}(x, d\xi, du) = N(x, d\xi, du)/a(x)$.

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A. P. 52120
CARACAS 1050-A
VENEZUELA