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REPRESENTATIONS OF MARKOV PROCESSES AS MULTIPARAMETER TIME CHANGES[1]

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Let $Y_1, Y_2, \cdots$ be independent Markov processes. Solutions of equations of the form $Z_i(t) = Y_i(\int_0^t \beta_i(Z(s))\,ds)$, where $\beta_i(z) \geq 0$, are considered. In particular it is shown that, under certain conditions, the solution of this "random time change problem" is equivalent to the solution of a corresponding martingale problem.

These results give representations of a large class of diffusion processes as solutions of $X(t) = X(0) + \Sigma_{i=1}^N \alpha_i W_i(\int_0^t \beta_i(X(s))\,ds)$ where $\alpha_i \in \mathbb{R}^d$ and the W_i are independent Brownian motions. A converse to a theorem of Knight on multiple time changes of continuous martingales is given, as well as a proof (along the lines of Holley and Stroock) of Liggett's existence and uniqueness theorems for infinite particle systems.

1. Introduction. It is well known (Volkonski (1958), Lamperti (1967)) that if $Y(t)$ is a Feller process with infinitesimal operator A and $\tau(t)$ is defined as follows

$$\tau(t) = \inf\{r : \int_0^r \beta(Y(s))^{-1}\,ds \geq t\}, \tag{1.1}$$

where $\beta(x)$ is continuous and satisfies $0 < \beta_1 \leq \beta(x) \leq \beta_2 < \infty$ for some β_1, β_2, then,

$$Z(t) = Y(\tau(t)) = Y\left(\int_0^t \beta(Z(s))\,ds\right) \tag{1.2}$$

is a Feller process with infinitesimal operator βA. (The conditions on β can be relaxed at a cost of losing the complete description of the generator of $Z(t)$.) For example, it is possible to represent diffusions with generators of the form $Af = \beta f''$ as random time changes of Brownian motion.

In this paper we consider representations of Markov processes involving different, dependent, random time changes of several independent Markov processes. In particular we show that a wide class of diffusion processes (those with uniformly elliptic generators) can be represented as solutions of equations of the form

$$X(t) = X(0) + \Sigma_{i=1}^N \alpha_i W_i\left(\int_0^t \beta_i(X(s))\,ds\right), \tag{1.3}$$

where the W_i are independent scalar Brownian motions, and the α_i are elements of $\mathbb{R}^d$. (In general N will be much larger than d.) Note that $X(t)$ can be written as an affine transformation, $X(t) = X(0) + MZ(t)$, of the N-dimensional process $Z(t)$ with components

$$Z_i(t) = W_i\left(\int_0^t \beta_i(X(s))\,ds\right) = W_i\left(\int_0^t \beta_i(X(0) + MZ(s))\,ds\right). \tag{1.4}$$

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This observation leads us to consider systems of equations of the form

$$Z_i(t) = Y_i\left(\int_0^t \beta_i(Z(s))\,ds\right), \qquad 1 \leq i \leq N, \tag{1.5}$$

rather than equations of more complicated forms such as (1.3). Throughout, the $Y_i(t)$ will be assumed to be independent, right continuous, Markov processes.

Multiple random time changes of this type appear first in the work of Helms (1974) and in a somewhat different form in Holley and Stroock (1976). Representations of the form (1.3) were used in Kurtz (1978) to obtain strong approximation theorems for Markov chains. Our approach follows closely that of Helms.

With reference to (1.5) define

$$\tau_i(t) = \int_0^t \beta_i(Z(s))\,ds, \tag{1.6}$$

and for $u \in [0, \infty)^N$ define

$$\mathcal{F}_u = \sigma(Y_i(s_i) : s_i \leq u_i, \quad i = 1, 2, \cdots, N). \tag{1.7}$$

Intuitively, $\tau(t) = (\tau_1(t), \tau_2(t), \cdots)$ is a $\mathcal{F}_u$ stopping time (as it always is for $N = 1$) in the sense that $\{\tau(t) \leq u\} \in \mathcal{F}_u$ for all u. (Here $v \leq u$ denotes $v_i \leq u_i, i = 1, 2, \cdots, N$.) We will, in fact, require $\tau(t)$ to be a stopping time as an additional condition on our solutions. This allows us to use the optional sampling theorem (Kurtz (1980)) to show that solutions of (1.5) are solutions of a corresponding martingale problem. We also give a converse.

In Section 2, we collect various lemmas concerning Markov processes and the martingale problem. In Section 3, we show that the solution of (1.5) is, in a certain sense, equivalent to the solution of a corresponding martingale problem. In Section 4, we discuss the representation (1.3), in Section 5, we discuss the problem of solving (1.5), and in Section 6, we show the relationship between our results and a theorem of Knight (1970) on multiple time changes of continuous martingales. Sections 4, 5 and 6 are independent of each other.

Throughout E (or E_k) will denote a complete separable metric space which is locally compact or a product of locally compact spaces. By E^Δ we will denote some compactification of E; usually E^Δ will be the one point compactification (if E is locally compact) or a countable product of one point compactifications (if E is a product of locally compact spaces). $B(E)$ will denote the Banach space of bounded Borel measurable function with the sup norm; $C(E)$ will denote the subspace of bounded continuous functions; $C_\Delta(E)$ will denote the subspace whose elements are restrictions to E of functions in $C(E^\Delta)$. Note that if $E_i, i = 1, 2, \cdots$ are locally compact spaces, E_i^Δ their one point compactifications, $E = \Pi E_i$ and $E^\Delta = \Pi E_i^\Delta$, then the linear combinations of functions of the form $f(z) = \Pi_{i \in I} f_i(z_i)$, where I is finite and $f_i \in C_\Delta(E_i)$, form a dense subspace of $C_\Delta(E)$.

By a Feller process, we mean a Markov process with locally compact state space E whose semigroup satisfies $T(t)1 = 1$, and is strongly continuous on $\hat{C}(E)$, the space of continuous functions vanishing at infinity (in the one point compactification), and hence on $C_\Delta(E)$ (E^Δ being the one point compactification). $D_E[0, \infty)$ ($D_{E^\Delta}[0, \infty)$) will denote the space of right continuous functions having left limits with values in $E(E^{\Delta})$ given the Skorohod topology. $\mathfrak{M}(E)$ ($\mathfrak{M}(E^{\Delta})$) will denote the σ-algebra of subsets of $D_E[0,\infty)$ ($D_{E^{\Delta}}[0,\infty)$) generated by the coordinate random variables.

2. The martingale problem. The notion of the martingale problem was introduced by Stroock and Varadhan (1969) as a means of characterizing the Markov process associated with a given generator. In this section we develop the results on the martingale problem we will need in the later sections. Most of these results appear in the literature for specific classes of processes (see, for example, Stroock and Varadhan (1969) and Holley and Stroock (1976)).

Let $D \subset C_{\Delta}(E)$ be a collection of functions such that $1 \in D$ and D_S (the linear span of D) is dense in $C_{\Delta}(E)$; let $A : D \to B(E)$ satisfy $A1 = 0$. By a *solution of the martingale problem* (A, D), we mean a measurable stochastic process $X(t)$ defined on a probability space $(\Omega, \mathcal{F}, P)$ such that for every $f \in D$

$$f(X(t)) - \int_0^t Af(X(s))\,ds \tag{2.1}$$

is a martingale with respect to the increasing family $\mathcal{F}_t = \sigma(X(s) : s \leqslant t)$. If $\mathcal{G}_t \subset \mathcal{F}$ is an increasing family of σ-algebras such that $\mathcal{G}_t \supset \mathcal{F}_t$ and (2.1) is a $\mathcal{G}_t$-martingale for every $f \in D$, we will say X is a solution of the martingale problem (A, D) with respect to $\mathcal{G}_t$.

Typically A will extend to a linear operator on D_S (possibly to a "multivalued" operator) and it is immediate (linear combinations of martingales are martingales) that any solution of the martingale problem for (A, D) is a solution for (A, D_S). It will be convenient, however, to allow sets D that are not linear.

LEMMA 2.2. *Under the above hypotheses on* D *and* A, *a solution* X *of the martingale problem* (A, D) *has a modification with sample paths in* $D_{E^{\Delta}}[0,\infty)$.

PROOF. For $f \in D$ extend f to E^{Δ} by continuity and set $Af = 0$ on $E^{\Delta} - E$. Then X remains a solution of the martingale problem for the extended operator. Since D_S is dense in $C_{\Delta}(E)$ there exist $f_i \in D, i = 1, 2, \cdots$ that separate points of E^{Δ} (again extending the f_i by continuity).

Since (2.1) is a martingale for all $f \in D$ and $\int_0^t Af(X(s))\,ds$ is continuous, it follows (see Breiman (1968) Theorem 14.7) that with probability one $f(X(s))$ has right and left limits through the rationals at every t (i.e., $\lim_{\mathbb{Q} \ni s \to t+} f(X(s))$ and $\lim_{\mathbb{Q} \ni s \to t-} f(X(s))$ exist for all $t \geqslant 0$). By countability we in fact have

$$P\{\lim_{\mathbb{Q} \ni s \to t+} f_i(X(s)) \text{ and } \lim_{\mathbb{Q} \ni s \to t-} f_i(X(s)) \text{ exist for all } t \geqslant 0, i = 1, 2, \cdots\} = 1. \tag{2.3}$$

Since the f_i are continuous and separate points in E^{Δ} and E^{Δ} is compact, it follows that with probability one

$$\lim_{\mathbb{Q} \ni s \to t+} X(s) \equiv Y(t) \tag{2.4}$$

exists for every t and $Y(t)$ has sample paths in $D_{E^{\Delta}}[0,\infty)$ by Breiman (1968)

Proposition 14.4. Since

$$(2.5)\quad \lim_{\mathbb{Q}\ni s\to t+} E(f(X(s))|\mathcal{F}_t) = f(X(t)) + \lim_{\mathbb{Q}\ni s\to t+} E(\textstyle\int_t^s Af(u)\,du|\mathcal{F}_t) = f(X(t))$$

it follows that $E(f(Y(t))|\mathcal{F}_t) = f(X(t))$ for all $f \in D$ and hence for all $f \in C(E^\Delta)$. Therefore,

$$(2.6)\quad E\big((f(Y(t)) - f(X(t)))^2|\mathcal{F}_t\big) = E(f^2(Y(t))|\mathcal{F}_t) - 2f(X(t))E(f(Y(t))|\mathcal{F}_t) + f^2(X(t)) = 0$$

and $Y(t) = X(t)$ a.s. □

In the light of Lemma 2.2 we will always assume that solutions of the martingale problem (A, D) have sample paths in $D_{E^\Delta}[0, \infty)$ (if necessary extending f and Af as above).

Let $D^+ = \{f \in D : \inf_x f(x) > 0\}$. If $f \in D$ implies $f + c \in D$ all $c > 0$ and $A(f + c) = Af$, any solution of the martingale problem (A, D^+) is a solution of the martingale problem (A, D).

An alternative characterization of solutions of the martingale problem is then given by the fact that for $f \in D^+$ and $Hf \equiv Af/f$, (2.1) is a martingale if and only if

$$(2.7)\quad f(X(t))\exp\{-\textstyle\int_0^t Hf(X(s))\,ds\}$$

is a martingale. See Theorem (1.1) of Holley and Stroock (1976) or, more generally, this equivalence is a consequence of the integration by parts formula for semimartingales (see Meyer (1976), page 303). For example, denoting (2.1) by M_1 and (2.7) by M_2, we have

$$M_2(t) = \textstyle\int_0^t \exp\{-\int_0^u Hf(X(s))\,ds\}\,dM_1(u).$$

LEMMA 2.8. *Let X be a solution of the martingale problem (A, D) with respect to $\mathcal{G}_t$. Suppose that X is unique in the sense that any other solution Y with $Y(0) = X(0)$ in distribution induces the same distribution on $\mathcal{M}(E^\Delta)$, as X. Then for any $\mathcal{G}_t$ stopping time $\tau < \infty$ a.s., we have*

$$(2.9)\quad E(f(X(\tau + t))|\mathcal{G}_\tau) = E(f(X(\tau + t))|X(\tau))$$

for all bounded measurable f. In particular, X is a Markov process.

PROOF. In order to verify (2.9) for a stopping time τ we must show that for every $F \in \mathcal{G}_\tau$

$$(2.10)\quad \textstyle\int_F f(X(\tau + t))\,dP = \int_F E(f(X(\tau + t))|X(\tau))\,dP.$$

For an arbitrary $F \in \mathcal{G}_\tau$ we will construct a new solution of the martingale problem Y satisfying $Y(0) = X(0)$ in distribution and use Y to verify (2.10) (actually the equality of the Laplace transforms of the terms in (2.10)).

For $\alpha < \infty$, let $D_{E^\Delta}[0, \alpha)$ denote the space of right continuous E^Δ-valued functions on $[0, \alpha)$ that have left limits in $(0, \alpha]$. Let $\Omega_1 = D_{E^\Delta}[0, \infty) \times [0, \infty)$ and identify Ω_1 with the product space

$$\Omega_2 \times \Omega_3 = \{[\omega_1, \alpha] : \omega_1 \in D_{E^\Delta}[0,\alpha), \alpha \geqslant 0\} \times D_{E^\Delta}[0,\infty) \tag{2.11}$$

using the one to one correspondence $(\omega, \alpha) \leftrightarrow ([\omega_1, \alpha], \omega_2)$ defined by

$$\begin{aligned} \omega(t) &= \omega_1(t), & t < \alpha \\ &= \omega_2(t-\alpha), & t \geqslant \alpha \end{aligned} \tag{2.12}$$

(let $D_{E^\Delta}[0,0)$ be a set containing a single element).

Define the random variables

$$\begin{aligned} \tau_0 &= \alpha, \\ Y(t) &= \omega(t), \\ Y_1(t) &= \omega_1(t) & t < \alpha \\ &= \omega_1(\alpha -) & t \geqslant \alpha \\ Y_2(t) &= \omega_2(t). \end{aligned} \tag{2.13}$$

Note that by the correspondence in (2.12) these random variables are defined on both Ω_1 and $\Omega_2 \times \Omega_3$. We leave it to the reader to show

$$\begin{aligned} \mathcal{F}^{(1)} &\equiv \sigma(\tau_0, Y(t) : t \geqslant 0) = \sigma(\tau_0, Y_1(t) : t \geqslant 0) \times \sigma(Y_2(t) : t \geqslant 0) \\ &\equiv \mathcal{F}^{(2)} \times \mathcal{F}^{(3)}. \end{aligned} \tag{2.14}$$

For $A \in \mathcal{F}^{(2)}$ and $B \in \mathcal{F}^{(3)}$ define

$$\begin{aligned} Q(A \times B) = E\big(&\chi_A(X(\cdot), \tau) \\ &\big[\chi_F E(\chi_B(X(\tau + \cdot)) | X(\tau)) + \chi_{F^c} \chi_B(X(\tau + \cdot))\big]\big). \end{aligned} \tag{2.15}$$

A theorem of Morando (1969) implies that Q extends to a probability measure on $\mathcal{F}^{(2)} \times \mathcal{F}^{(3)} = \mathcal{F}^{(1)}$. Observe that on the probability space $(\Omega_1, \mathcal{F}^{(1)}, Q)$, $(Y(t \wedge \tau_0), \tau_0)$ has the same distribution as $(X(t \wedge \tau), \tau)$. In particular $Y(0) = X(0)$ in distribution.

We will now show that Y is a solution of the martingale problem (A, D). Without loss of generality we can assume $f \in D$ implies $f + c \in D$ all $c > 0$, and $A(f + c) = Af$. For $f \in D^+$ we must show that

$$Z_f(t) = f(Y(t)) \exp\{-\textstyle\int_0^t Hf(Y(s))\, ds\} \tag{2.16}$$

is a martingale with respect to $\mathcal{F}_t = \sigma(Y(s) : s \leqslant t)$. The optional sampling theorem implies

$$f(X(t \wedge \tau)) \exp\{-\textstyle\int_0^{t \wedge \tau} Hf(X(s))\, ds\} \tag{2.17}$$

is a $\mathcal{G}_{t \wedge \tau}$ martingale and that

$$f(X(\tau + t)) \exp\{-\textstyle\int_0^{\tau + t} Hf(X(s))\, ds\} \tag{2.18}$$

is a $\mathcal{G}_{\tau + t}$ martingale. Since $(Y(t \wedge \tau_0), \tau_0)$ has the same distribution as $(X(t \wedge \tau), \tau)$, $Z_f(t \wedge \tau_0)$ is a martingale with respect to $\mathcal{F}_t^{\tau_0} = \sigma(Y(s \wedge \tau_0), \chi_{\{\tau_0 < s\}} : s \leqslant t)$. Set

$\mathcal{H}_t = \mathcal{F}^{(2)} \times \sigma(Y(\tau_0 + s): s \leqslant t)$. If $A \in \mathcal{F}^{(2)}, B \in \sigma(Y(\tau_0 + s): s \leqslant t)$, then by (2.15)

$$(2.19)\quad E_Q(\chi_A \chi_B Z_f(\tau_0 + t + u))$$
$$= E\big(\chi_A(X(\cdot), \tau)\exp\{-\textstyle\int_0^\tau Hf(X(s))\,ds\}$$
$$\times\big[\chi_F E\big(\chi_B(X(\tau + \cdot))f(X(\tau + t + u))\exp\{-\textstyle\int_\tau^{\tau+t+u} Hf(X(s))\,ds\}\,|\,X(\tau)\big)$$
$$+ \chi_{F^c}\chi_B(X(\tau + \cdot))f(X(\tau + t + u))\exp\{-\textstyle\int_\tau^{\tau+t+u} Hf(X(s))\,ds\}\big]\big).$$

Recalling that $F \in \mathcal{G}_\tau \subset \mathcal{G}_{\tau+t}$ and noting that $\chi_B(X(\tau + \cdot))$ is $\mathcal{G}_{\tau+t}$ measurable, the fact that (2.18) is a $\mathcal{G}_{\tau+t}$-martingale implies the right side of (2.19) is equal to

$$E\big(\chi_A(X(\cdot), \tau)\exp\{-\textstyle\int_0^\tau Hf(X(s))\,ds\}$$
$$(2.20)\quad \times\big[\chi_F E\big(\chi_B(X(\tau + \cdot))f(X(\tau + t))\exp\{-\textstyle\int_\tau^{\tau+t} Hf(X(s))\,ds\}\,|\,X(\tau)\big)$$
$$+ \chi_{F^c}\chi_B(X(\tau + \cdot))f(X(\tau + t))\exp\{-\textstyle\int_\tau^{\tau+t} Hf(X(s))\,ds\}\big]\big)$$
$$= E_Q(\chi_A \chi_B Z_f(\tau_0 + t)),$$

and hence $Z_f(\tau_0 + t)$ is an $\mathcal{H}_t$-martingale. Since $(t - \tau_0)\vee 0$ is an $\mathcal{H}_t$-stopping time the optional sampling theorem implies $Z_f(t \vee \tau_0)$ is an $\mathcal{H}_{(t-\tau_0)\vee 0}$-martingale. Let $A \in \mathcal{F}_t$. Then $A \cap \{\tau_0 \leqslant t\} \in \mathcal{H}_{(t-\tau_0)\vee 0}$ and $A \cap \{\tau_0 > t\} \in \mathcal{F}_t^{\tau_0} \subset \mathcal{H}_{(t-\tau_0)\vee 0}$, consequently

$$\textstyle\int_A Z_f(t + u)\,dQ = \int_{A\cap\{\tau_0 \leqslant t\}} Z_f((t + u)\vee\tau_0)\,dQ$$
$$+ \textstyle\int_{A\cap\{\tau_0 > t\}}\big[Z_f((t + u)\vee\tau_0) + Z_f((t + u)\wedge\tau_0) - Z_f(\tau_0)\big]\,dQ$$
$$(2.21)\quad = \textstyle\int_{A\cap\{\tau_0 \leqslant t\}} Z_f(t\vee\tau_0)\,dQ$$
$$+ \textstyle\int_{A\cap\{\tau_0 > t\}}\big[Z_f(t\vee\tau_0) + Z_f(t\wedge\tau_0) - Z_f(\tau_0)\big]\,dQ$$
$$= \textstyle\int_A Z_f(t)\,dQ$$

which shows $Z_f(t)$ is an $\mathcal{F}_t$-martingale.

By the uniqueness assumption we have

$$E\big(\textstyle\int_0^\infty e^{-\lambda t} f(X(t))\,dt\big) = E_Q\big(\int_0^\infty e^{-\lambda t} f(Y(t))\,dt\big)$$
$$= E_Q\big(\textstyle\int_0^{\tau_0} e^{-\lambda t} f(Y(t))\,dt\big)$$
$$(2.22)\quad + E_Q\big(e^{-\lambda\tau_0}\textstyle\int_0^\infty e^{-\lambda t} f(Y(\tau_0 + t))\,dt\big)$$
$$= E\big(\textstyle\int_0^\tau e^{-\lambda t} f(X(t))\,dt\big)$$
$$+ E\big(e^{-\lambda\tau}\big[\chi_F E\big(\textstyle\int_0^\infty e^{-\lambda t} f(X(\tau + t))\,dt\,|\,X(\tau)\big)$$
$$+ \chi_{F^c}\textstyle\int_0^\infty e^{-\lambda t} f(X(\tau + t))\,dt\big]\big)$$

and we conclude

$$(2.23)\quad E\left(e^{-\lambda\tau}\chi_F E\left(\int_0^\infty e^{-\lambda t} f(X(\tau+t))\,dt \,|\, X(\tau)\right)\right) = E\left(e^{-\lambda\tau}\chi_F \int_0^\infty e^{-\lambda t} f(X(\tau+t))\,dt\right).$$

Since F is an arbitrary element of $\mathcal{G}_\tau$ and $e^{-\lambda\tau}$ is $\mathcal{G}_\tau$ measurable we have

$$(2.24)\quad E\left(\int_0^\infty e^{-\lambda t} f(X(\tau+t))\,dt \,|\, X(\tau)\right) = E\left(\int_0^\infty e^{-\lambda t} f(X(\tau+t))\,dt \,|\, \mathcal{G}_\tau\right).$$

For continuous f, the uniqueness of the Laplace transform implies $E(f(X(\tau + t))|\mathcal{G}_\tau) = E(f(X(\tau + t))|X(\tau))$ which in turn implies (2.9) for all bounded measurable f. □

REMARK. If uniqueness holds for all solutions, then (2.24) follows immediately from the fact that $E(\chi_F\chi_B(X(\tau + \cdot)))/P(F)$ and $E(\chi_F E(\chi_B(X(\tau + \cdot)) |X(\tau)))/P(F)$ are measures on $D_{E^\Delta}[0,\infty)$ which give solutions of the martingale problem with the same initial distribution.

LEMMA 2.25. *Let X be a solution of the martingale problem (A, D) with respect to $\mathcal{G}_t$. Let $h(x,u)$ be bounded and continuous on $E \times \mathbb{R}^m$ with $h(\cdot,u) \in D$ for all u. Assume that $Ah(x,u)$ is bounded in x and u and is continuous in u for each x. Let $\tau(t)$ be a continuous $\mathbb{R}^m$-valued process adapted to $\mathcal{G}_t$ and define $g(x,t) = h(x,\tau(t))$. Suppose that $g(x,t)$ is absolutely continuous in t for each (x,ω) and that $g' \equiv (\partial/\partial t)g$ is bounded in (x,t,ω) and continuous in x for almost every t. Then*

$$(2.26)\qquad g(X(t),t) - \int_0^t \left(Ag(X(s),s) + g'(X(s),s)\right) ds$$

is a $\mathcal{G}_t$-martingale. If $\inf_{x,u} h(x,u) > 0$, then

$$(2.27)\quad g(X(t),t)\exp\left\{-\int_0^t \left(Ag(X(s),s) + g'(X(s),s)\right)/g(X(s),s)\,ds\right\}$$

is a $\mathcal{G}_t$-martingale.

PROOF. Since X is a solution of the martingale problem, for each u

$$(2.28)\qquad h(X(t),u) - \int_{t_k}^t Ah(X(s),u)\,ds$$

is a $\mathcal{G}_t$-martingale for $t \geqslant t_k$. Approximating $\tau(t_k)$ by $\mathcal{G}_{t_k}$ measurable random variables assuming finitely many values and using the continuity in u of h and Ah it follows that

$$(2.29)\qquad g(X(t),t_k) - \int_{t_k}^t Ag(X(s),t_k)\,ds$$

is a $\mathcal{G}_t$-martingale for $t \geqslant t_k$. Consequently for any partition $0 = t_0 < t_1 < t_2 \cdots$, we can define a $\mathcal{G}_t$-martingale $M(t)$ by setting

$$(2.30)\qquad \begin{aligned} M(t) = {} & g(X(t),t_k) - \int_{t_k}^t Ag(X(s),t_k)\,ds \\ & - \sum_{l=0}^{k-1}\left(g(X(t_{l+1}),t_{l+1}) - g(X(t_{l+1}),t_l)\right) \\ & - \sum_{l=0}^{k-1}\int_{t_l}^{t_{l+1}} Ag(X(s),t_l)\,ds \end{aligned}$$

for $t_k \leqslant t < t_{k+1}$. As $\max_k|t_{k+1} - t_k| \to 0$, $M(t)$ converges to (2.26) a.s. and in L_1 by the continuity in t of $g(x,t)$ and the continuity in x of $g'(x,t)$. Hence (2.26) is a $\mathcal{G}_t$-martingale. The proof that (2.27) is a martingale is the same as for (2.7). □

LEMMA 2.31. *Let $X(t)$ be a Feller process with locally compact state space E and infinitesimal operator A. Let $\Psi(x,t)$ and $(\partial/\partial t)\Psi(x,t) \equiv \Psi'(x,t)$ be bounded and continuous with compact support in $E \times [0,\infty)$, and let $f \in \mathfrak{D}(A)$. Then*

$$g(x,t) = E_x\big(\exp\{-\textstyle\int_0^t \Psi(X(s), t-s)\,ds\} f(X(t))\big) \tag{2.32}$$

is in $\mathfrak{D}(A)$ for each t,

$$\lim_{\varepsilon\to 0} \varepsilon^{-1}(g(x,t+\varepsilon) - g(x,t)) \equiv g_t(x,t) \tag{2.33}$$

exists uniformly in x for each t,

$$g(x,0) = f(x), \tag{2.34}$$

and

$$g_t(x,t) = Ag(x,t) - \Psi(x,t)g(x,t). \tag{2.35}$$

PROOF. A simple calculation gives

$$\begin{aligned} g_t(x,t) &= E_x\big(\exp\{-\textstyle\int_0^t \Psi(X(s),t-s)\,ds\} Af(X(t))\big) \\ &\quad - E_x\big((\Psi(X(t),0) + \textstyle\int_0^t \Psi'(X(s),t-s)\,ds) \\ &\quad \times \exp\{-\textstyle\int_0^t \Psi(X(s),t-s)\,ds\} f(X(t))\big). \end{aligned} \tag{2.36}$$

To compute Ag, note that the Markov property gives

$$\begin{aligned} &\varepsilon^{-1}\big[E_x(g(X(\varepsilon),t)) - g(x,t)\big] \\ &= \varepsilon^{-1}\big[E_x\big(\exp\{-\textstyle\int_\varepsilon^{t+\varepsilon} \Psi(X(s),t+\varepsilon-s)\,ds\} f(X(t+\varepsilon))\big) - g(x,t)\big] \\ &= \varepsilon^{-1} E_x\big((1 - \exp\{-\textstyle\int_0^\varepsilon \Psi(X(s),t+\varepsilon-s)\,ds\}) \\ &\quad \times \exp\{-\textstyle\int_\varepsilon^{t+\varepsilon} \Psi(X(s),t+\varepsilon-s)\,ds\} \\ &\quad \times f(X(t+\varepsilon))\big) + \varepsilon^{-1}\big[g(x,t+\varepsilon) - g(x,t)\big]. \end{aligned} \tag{2.37}$$

Letting $\varepsilon \to 0$, we obtain

$$Ag(x,t) = \Psi(x,t)g(x,t) + g_t(x,t). \tag{2.38}$$

The uniform continuity of Ψ and Ψ' ensure the uniformity of the various limits. □

LEMMA 2.39. *Let E be locally compact. Let $\varphi_i(x), i = 1, 2, \cdots, m$, be positive and continuous and $\varphi_i(x) = 1$ for x outside some compact set. Let $0 = t_0 < t_1 < \cdots < t_m < T$. Then there exist $\Psi_n(x,t)$ satisfying the conditions of Lemma 2.31 such that*

$$Q_n \equiv \exp\{-\textstyle\int_0^T \Psi_n(X(s), T-s)\,ds\} \leqslant \Pi_{i=0}^m \sup_x |\varphi_i(x)| \tag{2.40}$$

and

$$\lim_{n\to\infty} Q_n = \Pi_{i=0}^m \varphi_i(X(t_i)). \tag{2.41}$$

PROOF. Let $\rho \geqslant 0$ be continuously differentiable with compact support in $(0,\infty)$ and $\int_0^\infty \rho(s)\,ds = 1$ $(\rho = 0$ on $(-\infty, 0])$. Take

$$\Psi_n(x,t) = -\Sigma_{i=0}^m n\rho((T-t-t_i)n)\ln \varphi_i(x). \tag{2.42}$$

□

LEMMA 2.43. *Let A be the infinitesimal operator for a Feller process with locally compact state space E, and let D be a core for A. Then the martingale problem (A, D) has a unique solution for each initial distribution*, i.e., *if X and Y are solutions of the martingale problem (A, D) and $X(0)$ is equal to $Y(0)$ in distribution, then X and Y have the same distribution on $D_E[0, \infty)$.*

REMARK. This lemma is essentially the same as Theorem (4.2) in Holley and Stroock (1976). The proof we give illustrates techniques we will need later.

PROOF. Since D is a core, any solution of the martingale problem (A, D) is also a solution of the martingale problem $(A, \mathfrak{D}(A))$. It is sufficient to let X be the Feller process and Y another solution of the martingale problem. Let $g(x, t)$ be as in Lemma (2.31) with $f \equiv 1$ and define

$$h(x,t) = g(x, T - t \wedge T). \tag{2.44}$$

Then by Lemma 2.25

$$h(Y(t),t)\exp\{-\textstyle\int_0^t(Ah(Y(s),s) + \partial_s h(Y(s),s))/h(Y(s),s)\,ds\} \tag{2.45}$$

is a martingale. Therefore, since $h(Y(T), T) = 1$, and (2.35) implies the integrand is $\Psi(Y(s), T-s)$ for $0 \leqslant s \leqslant T$

$$\begin{aligned} E\big(\exp\{-\textstyle\int_0^T\Psi(Y(s),T-s)\,ds\}\big) &= E(h(Y(0),0)) = E(g(X(0),T)) \\ &= E\big(\exp\{-\textstyle\int_0^T\Psi(X(s),T-s)\,ds\}\big). \end{aligned} \tag{2.46}$$

The result now follows by Lemma 2.39. □

3. **Multiple random time change.** For each $1 \leqslant i \leqslant N$ (N may be infinite, then take $1 \leqslant i < \infty$), let A_i be the infinitesimal operator for a semigroup on $C_\Delta(E_i)$ corresponding to a Feller process with locally compact state space E_i (E_i^Δ the one point compactification), and let β_i be a nonnegative Borel measurable function on $\Pi_{i=1}^N E_i$. A process $Z(t)$, with state space $\Pi_{i=1}^N E_i$ and defined on a probability space $(\Omega, \mathfrak{F}, P)$, solves the *random time change problem* $(A_i, \beta_i, 1 \leqslant i \leqslant N)$, if there exists a family of σ-algebras $\mathfrak{F}_u \subset \mathfrak{F}, u \in [0, \infty)^N$, that is increasing in the sense that $\mathfrak{F}_u \subset \mathfrak{F}_v$ if $u \leqslant v$ (i.e., $u_i \leqslant v_i$, $1 \leqslant i \leqslant N$), and right continuous processes Y_i with state space E_i such that

$$\Pi_{i=1}^n f_i(Y_i(u_i))\exp\{-\textstyle\int_0^{u_i} H_i f_i(Y_i(s))\,ds\} \tag{3.1}$$

is a martingale with respect to $\mathfrak{F}_u$ for $n \leqslant N, n < \infty, f_i \in \mathfrak{D}^+(A_i)$; ($H_i f_i = A_i f_i / f_i, \mathfrak{D}^+(A_i) = \{f \in \mathfrak{D}(A_i) : \inf_x f_i(x) > 0\}$;)

$$Z_i(t) = Y_i\big(\textstyle\int_0^t \beta_i(Z(s))\,ds\big); \tag{3.2}$$

and

$$\tau(t) = (\tau_1(t), \tau_2(t), \cdots), \tag{3.3}$$

where $\tau_i(t) = \int_0^t \beta_i(Z(s))\,ds$ is an $\mathfrak{F}_u$ stopping time for all $t \geqslant 0$ (i.e., $\{\tau(t) \leqslant u\} \in \mathfrak{F}_u$ for all u).

REMARK. Condition (3.1) implies $f_i(Y_i(t))\exp\{-\int_0^t H_i f_i(Y_i(s))\,ds\}$ is a martingale with respect to $\mathcal{H}_t^i = \sigma(Y_i(s_i) : 0 \leqslant s_i \leqslant t) \vee \sigma(Y_j(s_j) : 0 \leqslant s_j < \infty, j \neq i)$. By the uniqueness of the solution of the martingale problem $(A_i, \mathfrak{D}(A_i))$ it follows that Y_i is Markovian with respect to $\mathcal{H}_t^i$ and, in particular,

$$(3.4)\quad P\{Y_i(\cdot) \in B \mid \mathcal{H}_0^i\} = P\{Y_i(\cdot) \in B \mid Y_i(0), Y_j(\cdot) j \neq i\} = P\{Y_i(\cdot) \in B \mid Y_i(0)\}$$

for all $B \in \mathfrak{M}(E_i)$. In particular the Y_i are independent if the initial positions $Y_i(0)$ are independent.

For a stopping time in the sense used in (3.3), we define

$$(3.5)\qquad \mathcal{F}_\tau = \{A : A \cap \{\tau \leqslant u\} \in \mathcal{F}_u \text{ all } u\}$$

as in the one dimensional case. Let

$$(3.6)\qquad D = \{\Pi_{i=1}^n f_i : n \leqslant N, n < \infty, f_i \in \mathfrak{D}(A_i)\},$$

and note that

$$(3.7)\qquad D^+ = \{\Pi_{i=1}^n f_i : n \leqslant N, n < \infty, f_i \in \mathfrak{D}^+(A_i)\}.$$

For $\Pi_{i=1}^n f_i \in D$, define

$$(3.8)\qquad A\Pi_{i=1}^n f_i = \Sigma_{j=1}^n \beta_j A_j f_j \Pi_{i \neq j} f_i$$

(e.g., if $n = 2, Af_1 f_2 = \beta_1 f_2 A_1 f_1 + \beta_2 f_1 A_2 f_2$.) For $\Pi_{i=1}^n f_i \in D$ let $c > \max_i \|f_i\|$. Then

$$(3.9)\qquad \Pi_{i=1}^n f_i = \Pi_{i=1}^n([f_i + c] - c) = \Sigma_{I \subset \{1,2,\cdots,n\}}(-c)^{n-|I|}\Pi_{i \in I}[f_i + c],$$

where $|I|$ is the cardinality of I, and

$$(3.10)\qquad A\Pi_{i=1}^n f_i = \Sigma_{I \subset \{1,2\cdots n\}}(-c)^{n-|I|} A\Pi_{i \in I}[f_i + c],$$

(recall $A_i c = 0$). It follows that any solution of the martingale problem (A, D^+) is a solution for (A, D).

The following theorem states that the martingale problem (A, D) (or (A, D^+)) and the random time change problem $(A_i, \beta_i, 1 \leqslant i \leqslant N)$ are equivalent.

THEOREM 3.11. *Let A_i and β_i be as above, and assume* $\sup_z \beta_i(z) < \infty$.

(a) *If $Z(t)$ is a solution of the random time change problem $(A_i, \beta_i, 1 \leqslant i \leqslant N)$, then it is also a solution of the martingale problem (A, D) with respect to $\mathcal{G}_t = \mathcal{F}_{\tau(t)}$.*

(b) *Assume $\inf_z \beta_i(z) > 0$. If $Z(t)$ is a solution of the martingale problem (A, D), then it is also a solution of the random time change problem with*

$$(3.12)\qquad \tau_i(t) = \int_0^t \beta_i(Z(s))\,ds,$$

$$(3.13)\qquad Y_i(t) = Z_i(\gamma_i(t)) \text{ where } \gamma_i(t) = \inf\{r : \int_0^r \beta_i(Z(s))\,ds \geqslant t\},$$

$$(3.14)\qquad \mathcal{F}_u = \sigma(Y_i(s_i) : s_i \leqslant u_i, i = 1, 2, \cdots) \vee \cap_i \mathcal{G}_{\gamma_i(u_i)}.$$

REMARKS. (i) $\{\tau(t) \leqslant u\} = \cap_i\{\gamma_i(u_i) \geqslant t\} = \cap_n\{\vee_{i=1}^n \gamma_i(u_i) \geqslant t\} \in \cap_i \mathcal{G}_{\gamma_i(u_i)}$. Hence $\tau(t)$ is a $\mathcal{F}_u$ stopping time.

(ii) Theorem (5.5) and a portion of Lemma (5.10) of Holley and Stroock (1976) are essentially a special case of our theorem although their definition of "stopping vector" differs from ours (see Section 5).

(iii) The theorem of Knight (1970) is a more general statement of part (b) of our theorem in the case of finitely many Brownian motions (see Section 6).

(iv) The assumption in (b) that $\inf_z \beta_i(z) > 0$ is used only to ensure that $\int_0^\infty \beta_i(Z(s))\,ds = \infty$ and hence that Y_i is defined for all t. This assumption can be dropped at the (possible) expense of having to enlarge the sample space in order to be able to define $Y_i(t)$ for $t > \int_0^\infty \beta_i(Z(s))\,ds$. Knight (1970) carries out a similar argument. The assumption $\sup_z \beta_i(z) < \infty$ can also be relaxed in particular cases.

(v) $\mathcal{F}_u$ can be enlarged so that $\mathcal{F}_u = \cap_n \mathcal{F}_{u_n}$ for any decreasing sequence $\{u_n\}$ such that $\lim_{n\to\infty} u_n = u$. Then for a (real valued) $\mathcal{F}_{\tau(t)}$ stopping time $\gamma, \tau(\gamma)$ is a $\mathcal{F}_u$-stopping time (see Helms (1974), Lemma 6).

PROOF.

(a) Since

$$X(u) = \Pi_{i=1}^n f_i(Y_i(u_i))\exp\{-\int_0^{u_i} H_i f_i(Y_i(s_i))\,ds_i\} \tag{3.15}$$

is an $\mathcal{F}_u$-martingale, the optional sampling theorem (Theorem (2.15) of Kurtz (1980)) implies $X(\tau(t))$ is a martingale with respect to $\mathcal{G}_t \equiv \mathcal{F}_{\tau(t)}$. A change of variable $s_i = \tau_i(s)$ in each integral gives

$$X(\tau(t)) = \Pi_{i=1}^n f_i(Z_i(t))\exp\{-\int_0^t \beta_i(Z(s)) H_i f_i(Z_i(s))\,ds\}. \tag{3.16}$$

Since $X(\tau(t))$ is a martingale if and only if

$$\Pi_{i=1}^n f_i(Z_i(t)) - \int_0^t A(\Pi_{i=1}^n f_i)(Z(s))\,ds \tag{3.17}$$

is a martingale, part (a) follows.

(b) Note that (3.2) and (3.3) follow from the definitions and Remark (i) above. It remains to show that $X(u)$ given by (3.15) is an $\mathcal{F}_u$-martingale. We need the following lemma.

LEMMA 3.18. *Let $M_i(t), 1 \leq i \leq n < \infty$ be right continuous processes such that $\Pi_{i\in I} M_i(t)$ is a $\mathcal{G}_t$-martingale for all subsets $I \subset \{1, 2, \cdots, n\}$. Let $\tau_0, \tau_1, \cdots, \tau_n$ be stopping times, $\tau = \bigvee_i \tau_i$, $\overline{M}_i = \sup_{t<\tau}|M_i(t)|$ and assume $E(\Pi_{i\in I}\overline{M}_i) < \infty$ for all $I \subset \{1, 2, \cdots, n\}$. Then*

$$E(\Pi_{i=1}^n M_i(\tau_i)|\mathcal{G}_{\tau_0}) = \Pi_{i=1}^n M_i(\tau_i \wedge \tau_0). \tag{3.19}$$

PROOF. Since $M(\tau_1) = M(\tau_1 \vee \tau_0) - M(\tau_0) + M(\tau_1 \wedge \tau_0)$, (3.19) follows from the optional sampling theorem in the case $n = 1$. Proceeding by induction on n, assume (3.19) holds for $n = m - 1$ with $(\tau_0, \tau_1, \cdots, \tau_{m-1})$ replaced by any collection of stopping times $(\tau_0', \tau_1', \cdots, \tau_{m-1}')$ with $\vee_i \tau_i' \leq \tau$. Then

$$\begin{aligned} E(\Pi_{i=1}^m M_i(\tau_i)|\mathcal{G}_{\tau_0}) &= E(M_1(\tau_1)E(\Pi_{i=2}^m M_i(\tau_i)|\mathcal{G}_{\tau_0\vee\tau_1})|\mathcal{G}_{\tau_0}) \\ &= E(\Pi_{i=1}^m M_i(\tau_i \wedge (\tau_1 \vee \tau_0))|\mathcal{G}_{\tau_0}), \end{aligned} \tag{3.20}$$

where the second equality follows from the induction hypothesis. Let $\gamma = \wedge_i(\tau_i \vee \tau_0) = (\wedge_i \tau_i) \vee \tau_0$. Repeating the procedure in (3.20) for each $\mathcal{G}_{\tau_0 \vee \tau_k}$, we obtain

$$\begin{aligned} E\big(\Pi_{i=1}^m M_i(\tau_i) | \mathcal{G}_{\tau_0}\big) &= E\big(\Pi_{i=1}^m M_i(\tau_i \wedge \gamma) | \mathcal{G}_{\tau_0}\big) \\ &= E\big(\Pi_{i=1}^m M_i(\tau_i \wedge \tau_0) \chi_{\{\tau_0 = \gamma\}} | \mathcal{G}_{\tau_0}\big) \\ &\quad + E\big(\Pi_{i=1}^m M_i(\gamma) \chi_{\{\tau_0 < \gamma\}} | \mathcal{G}_{\tau_0}\big) \\ &= \Pi_{i=1}^m M_i(\tau_i \wedge \tau_0) \chi_{\{\tau_0 = \gamma\}} + E\big(\Pi_{i=1}^m M_i(\gamma) | \mathcal{G}_{\tau_0}\big) \chi_{\{\tau_0 < \gamma\}} \\ &= \Pi_{i=1}^m M_i(\tau_i \wedge \tau_0). \end{aligned} \tag{3.21}$$

Note that we have used the facts that $\{\tau_0 = \gamma\}, \{\tau_0 < \gamma\} \in \mathcal{G}_{\tau_0}$ and that $\Pi_{i=1}^m M_i$ is a martingale. □

Returning to the proof of Theorem 3.11, let $\Psi_i(x_i, t)$ satisfy the conditions of Lemma 2.19 and $f_i \in \mathfrak{D}^+(A_i)$. Define

$$\begin{aligned} h_i(x_i, t) &= g_i(x_i, u_i - t) & t \leqslant u_i \\ &= f_i(x_i) & t > u_i \end{aligned} \tag{3.22}$$

where $g_i(x_i, t)$ is the solution of

$$\frac{\partial}{\partial t} g_i(x_i, t) = A_i g_i(x_i, t) - \Psi_i(x_i, t) g_i(x_i, t) \tag{3.23}$$

with $g_i(x_i, 0) = f_i(x_i)$ (see Lemma 2.31). Define $Kh(x, t) \equiv (\partial/\partial t)h(x, t)/h(x, t)$ and let

$$\begin{aligned} M_i(t) = h_i(Z_i(t), \tau_i(t)) \exp\{ & -\textstyle\int_0^t \beta_i(Z(s))[Kh_i(Z_i(s), \tau_i(s)) \\ & + H_i h_i(Z_i(s), \tau_i(s))] \, ds\}. \end{aligned} \tag{3.24}$$

For $n \leqslant N, n < \infty$, Lemma 2.25 implies $M_1(t), \cdots, M_n(t)$ satisfy the conditions of Lemma 3.18. Let $u \leqslant v$. Then

$$E\big(\Pi_{i=1}^n M_i(\gamma_i(v_i)) | \mathcal{G}_{\bar{\gamma}_n(u)}\big) = \Pi_{i=1}^n M_i(\bar{\gamma}_n(u)) \tag{3.25}$$

where $\bar{\gamma}_n(u) = \wedge_{i=1}^n \gamma_i(u_i)$. In particular,

$$\begin{aligned} & E\big(\Pi_{i=1}^n f_i(Y_i(v_i)) \exp\{ -\textstyle\int_{u_i}^{v_i} H_i f_i(Y_i(s)) \, ds\} \exp\{ -\int_0^{u_i} \Psi_i(Y_i(s), u_i - s) \, ds\} | \mathcal{G}_{\bar{\gamma}_n(u)}\big) \\ & \quad = E\big(\Pi_{i=1}^n f_i(Y_i(u_i)) \exp\{ -\textstyle\int_0^{u_i} \Psi_i(Y_i(s), u_i - s) \, ds\} | \mathcal{G}_{\bar{\gamma}_n(u)}\big). \end{aligned} \tag{3.26}$$

From Lemma 2.39, we can conclude that

$$\begin{aligned} & E\big(\Pi_{i=1}^n f_i(Y_i(v_i)) \exp\{ -\textstyle\int_{u_i}^{v_i} H_i f_i(Y_i(s)) \, ds\} \Pi_{k=1}^{m_i} \varphi_{ik}(Y_i(t_{ik})) | \mathcal{G}_{\bar{\gamma}_n(u)}\big) \\ & \quad = E\big(\Pi_{i=1}^n f_i(Y_i(u_i)) \Pi_{k=1}^{m_i} \varphi_{ik}(Y_i(t_{ik})) | \mathcal{G}_{\bar{\gamma}_n(u)}\big) \end{aligned} \tag{3.27}$$

where φ_{ik} are as in Lemma 2.39 and $0 \leqslant t_{ik} \leqslant u_i$. It follows that $X(u)$ is an $\mathcal{F}_u$-martingale. □

REMARK. Since (3.12) is a martingale, the optional sampling theorem implies $E(X(\tau + u)|\mathfrak{F}_\tau) = X(\tau)$ for any $\mathfrak{F}_u$ stopping time τ. This in turn implies the multiparameter "strong Markov property" in Helms (1974, Theorem 4).

Existence of solutions of the martingale problem is quite easy to establish under the assumption that the $\beta_i(z)$ are continuous.

THEOREM 3.28. *Let A_i and β_i be as above, and suppose that the β_i are bounded and continuous. Then there exists a solution of the martingale problem (A, D).*

PROOF. Let ε_i satisfy $\sup_{z\in E}\Sigma_{i=1}^N \varepsilon_i\beta_i(z) < \infty$ and define

$$A_i^{(m)} = A_i\left(I - \frac{1}{m\varepsilon_i}A_i\right)^{-1} = m\varepsilon_i\left[\left(I - \frac{1}{m\varepsilon_i}A_i\right)^{-1} - I\right]. \tag{3.29}$$

Then $A_i^{(m)}$ is the Yosida approximation of A_i (see Yosida (1968), page 246) and

$$\lim_{m\to\infty} A_i^{(m)}f = A_i f \text{ for } f \in \mathfrak{D}(A_i). \tag{3.30}$$

Note that $A_i^{(m)}$ is defined on $B(E_i)$ and since $\|(I - (1/m\varepsilon_i)A_i)^{-1}\| \leqslant 1$ we have $\|A_i^{(m)}\| \leqslant 2m\varepsilon_i$. We can also consider $A_i^{(m)}$ as an operator on $B(\Pi_{j=1}^N E_j)$ ($A_i^{(m)}f(z)$ is defined for $f \in B(\Pi_{j=1}^N E_j)$ by fixing all components of z but the ith) and $\|A_i^{(m)}\| \leqslant 2m\varepsilon_i$ in this space as well. Define

$$A^{(m)}f = \Sigma_{i=1}^N \beta_i A_i^{(m)} f \tag{3.31}$$

and note that

$$|A^{(m)}f(z)| \leqslant \Sigma_{i=1}^N \beta_i(z)\|A_i^{(m)}f\| \leqslant \Sigma_{i=1}^N \beta_i(z) 2m\varepsilon_i \|f\|. \tag{3.32}$$

Consequently $\|A^{(m)}\| \leqslant \sup_z \Sigma 2m\varepsilon_i\beta_i(z) < \infty$.

For $\Pi_{i=1}^n f_i \in D$

$$A^{(m)}\Pi_{i=1}^n f_i = \Sigma_{j=1}^n \beta_j \Pi_{i\neq j} f_i A_j^{(m)} f_j. \tag{3.33}$$

Since $A^{(m)}$ is bounded it is the generator of a jump process $Z^{(m)}$ for which $\Sigma_{j=1}^N \beta_j m\varepsilon_j$ is the jump intensity, $\beta_i\varepsilon_i/(\Sigma_{j=1}^n \beta_j\varepsilon_i)$ the probability that the next jump is in the ith coordinate, and $m\varepsilon_i\int_0^\infty e^{-m\varepsilon_i t}P_i(t, z_i, \Gamma)\,dt$ the distribution of the jump ($P_i(t, z_i, \Gamma)$ denotes the transition function corresponding to A_i). By Theorem (3.11) $Z^{(m)}$ is a solution of the random time change problem

$$Z_i^{(m)}(t) = Y_i^{(m)}\left(\int_0^t \beta_i(Z^{(m)}(s))\,ds\right), \tag{3.34}$$

where $Y_i^{(m)}$ is a Feller process with generator $A_i^{(m)}$. If $Y_i^{(m)}(0)$ is independent of m, then (3.30) implies $Y_i^{(m)}$ converges weakly to Y_i (see Kurtz (1975) Theorem (4.29)). From this representation, the boundedness of β_i and the tightness of the $Y_i^{(m)}$, it follows that for every $\eta > 0$ and $T > 0$, there exist compact $K_i \subset E_i$ such that

$$\inf_m P\left\{Z_i^{(m)}(t) \in K_i, i = 1, 2, \cdots, t \leqslant T\right\} \geqslant 1 - \eta. \tag{3.35}$$

By (3.30),

$$\lim_{m\to\infty} A^{(m)}\Pi_{i=1}^n f_i = A\Pi_{i=1}^n f_i, \tag{3.36}$$

for all $\Pi_{i=1}^n f_i \in D$. Since the linear span of D is dense in $C_\Delta(E)$, it follows from

Lemma (1.23) of Kurtz (1978) that the sequence $Z^{(m)}$ is tight in $D_{E^\Delta}[0, \infty)$. From (3.35), we conclude that, in fact, $Z^{(m)}$ is tight in $D_E[0, \infty)$.

Let $Z^{(m_k)}$ be a subsequence converging weakly to a process Z. For $g = \prod_{i=1}^n f_i \in D, h_j \in C_\Delta(E)$ and $0 \leq t_1 < t_2 \cdots < t_l < t_{l+1}$, (3.36) implies

$$\begin{aligned} 0 &= \lim_{k\to\infty} E\Big(\Big[g\big(Z^{(m_k)}(t_{l+1})\big) - g\big(Z^{(m_k)}(t_l)\big) \\ &\quad - \textstyle\int_{t_l}^{t_{l+1}} A^{(m_k)} g\big(Z^{(m_k)}(s)\big)\,ds\Big]\prod_{j=1}^l h_j\big(Z^{(m_k)}(t_j)\big)\Big) \\ &= \lim_{k\to\infty} E\Big(\Big[g\big(Z^{(m_k)}(t_{l+1})\big) - g\big(Z^{(m_k)}(t_l)\big) \qquad (3.37)\\ &\quad - \textstyle\int_{t_l}^{t_{l+1}} A g\big(Z^{(m_k)}(s)\big)\,ds\Big]\prod_{j=1}^l h_j\big(Z^{(m_k)}(t_j)\big)\Big) \\ &= E\Big(\Big[g(Z(t_{l+1})) - g(Z(t_l)) - \textstyle\int_{t_l}^{t_{l+1}} Ag(Z(s))\,ds\Big]\prod_{j=1}^l h_j(Z(t_j))\Big) \end{aligned}$$

provided $P\{Z(t_j) = Z(t_j-)\} = 1, j = 1, 2, \cdots, l+1$ (see Billingsley (1968), page 124). By the right continuity of Z the right side of (3.37) is in fact zero for all choices of $t_1, t_2, \cdots, t_{l+1}$. This in turn implies $g(Z(t)) - \int_0^t Ag(Z(s))\,ds$ is a martingale with respect to $\mathcal{G}_t = \sigma(Z(s) : s \leq t)$. □

4. Representations of diffusion processes. In this section we consider diffusion processes in $\mathbb{R}^d$ with generators of the form

$$Au(x) = \tfrac{1}{2}\Sigma_{1 \leq i,j \leq d} c_{ij}(x)\partial_i\partial_j u(x) + \Sigma_{1 \leq j \leq d} b_j(x)\partial_j u(x). \qquad (4.1)$$

Stroock and Varadhan (1969) have given very general conditions under which the martingale problem corresponding to A (defined on an appropriate domain) has a unique solution and hence determines a Markov process. We will show that a large class of such processes can be represented as solutions of equations of the form

$$X(t) = X(0) + \Sigma_{k=1}^N \alpha_k W_k\big(\textstyle\int_0^t \beta_k(X(s))\big)\,ds \qquad (4.2)$$

where the W_i are independent scalar Brownian motions with variance and drift parameters σ_k^2 and $m_k, \alpha_k \in \mathbb{R}^d, \beta_k$ is nonnegative and measurable, and $N < \infty$. When the representation holds then

$$\big((c_{ij}(x))\big) \equiv C(x) = \Sigma_{k=1}^N \beta_k(x)\sigma_k^2\alpha_k\alpha_k^T \qquad (4.3)$$

(we think of α_k as a column vector and α_k^T is its transpose), and

$$b(x) \equiv \big(b_1(x), \cdots, b_j(x)\big)^T = \Sigma_{k=1}^N \beta_k(x) m_k \alpha_k. \qquad (4.4)$$

Representations of this form were used in Kurtz (1978) to obtain pathwise error estimates for diffusion approximations of Markov chains. In that case, the Markov chains satisfied

$$X(t) = X(0) + \Sigma_{k=1}^N \alpha_k Y_k\big(\textstyle\int_0^t \beta_k(X(s))\,ds\big), \qquad (4.5)$$

where the Y_k are Poisson processes and the diffusion approximation is obtained by replacing the Y_k by Brownian motions with the same mean and variance.

If

$$\sup_{i,x}|b_i(x)| < \infty \qquad (4.6)$$

and there exist $0 < \mu < \lambda < \infty$ such that

$$\mu|\xi|^2 \leqslant \Sigma_{1<i,j<d} c_{ij}(x)\xi_i\xi_j \leqslant \lambda|\xi|^2 \text{ for all } \xi, x \in \mathbb{R}^d \tag{4.7}$$

(i.e., A is uniformly elliptic) we will see that such representations always exist and that β_k and α_k can be selected so that $N < \infty, \inf_{k,x}\beta_k(x) > 0$ and $\sup_{k,x}\beta_k(x) < \infty$. This selection is based on the following lemma which is essentially a lemma of Motzkin and Wasow (1953).

LEMMA 4.8. *For $0 < \mu < \lambda < \infty$, let $S_d(\mu, \lambda)$ denote the collection of positive definite $(d \times d)$ matrices $((a_{ij}))$ satisfying*

$$\mu|\xi|^2 \leqslant \Sigma a_{ij}\xi_i\xi_j \leqslant \lambda|\xi|^2 \text{ for all } \xi \in \mathbb{R}^d. \tag{4.9}$$

Then there exist $\alpha_i \in \mathbb{R}^d, 1 \leqslant i \leqslant N < \infty$, such that every $A \in S_d(\mu, \lambda)$ can be represented as

$$A = \Sigma_{i=1}^N \beta_i(A)\alpha_i\alpha_i^T \tag{4.10}$$

where $\beta_i(A) > 0$. Furthermore the $\beta_i(A)$ can be taken to be C^∞ functions of the coefficients.

REMARK. The representation is in no way unique, and N may be arbitrarily large, the size depending on λ/μ.

PROOF. We begin by observing that it is sufficient to prove only that $\beta_i(A) \geqslant 0$. If all $A \in S_d(\mu/2, 2\lambda)$ can be represented

$$A = \Sigma_{i=1}^N \gamma_i(A)\alpha_i\alpha_i^T, \tag{4.11}$$

where $\gamma_i(A) \geqslant 0$, then there is an $\varepsilon > 0$ sufficiently small such that for any $A \in S_d(\mu, \lambda), A_0 \equiv A - \Sigma_{i=1}^N \varepsilon\alpha_i\alpha_i^T \in S_d(\mu/2, 2\lambda)$. Hence

$$A = \Sigma_{i=1}^N(\varepsilon + \gamma_i(A_0))\alpha_i\alpha_i^T \tag{4.12}$$

and we have (4.10) with $\beta_i(A) = \varepsilon + \gamma_i(A_0) > 0$.

Next observe that $S_d(\mu, \lambda)$ is a compact subset of the $m \equiv n(n + 1)/2$ dimensional vector space of symmetric matrices; in fact, it is a compact subset of the interior of the cone of positive definite matrices. Every $A \in S_d(\mu, \lambda)$ can be represented as

$$A = \Sigma_{i=1}^m \eta_i\eta_i^T, \tag{4.13}$$

where the $\eta_i\eta_i^T$ are linearly independent, hence span the space of symmetric matrices.

The sets

$$U(\eta_1, \cdots, \eta_m) = \{\Sigma_{i=1}^m c_i\eta_i\eta_i^T : c_i > 0\} \tag{4.14}$$

with the $\eta_i\eta_i^T$ linearly independent form an open cover of $S_d(\mu, \lambda)$. Let $U_k = U(\eta_1^{(k)}, \cdots, \eta_m^{(k)}), k = 1, 2, \cdots, K$ be a finite subcover. Each $A \in U_k$ has a unique representation

$$A = \Sigma c_i^{(k)}(A)\eta_i^{(k)}\eta_i^{(k)T}, \tag{4.15}$$

with $c_i^{(k)}(A) > 0$, and in fact the $c_i^{(k)}(A)$ are linear functions. Finally there exist nonnegative C^∞ functions $\lambda_k(A)$ such that $\lambda_k(A)$ has compact support in U_k and $\Sigma_k \lambda_k(A) = 1$ on $S_d(\mu, \lambda)$. Therefore, for every $A \in S_d(\mu, \lambda)$,

$$A = \Sigma \lambda_k(A) A = \Sigma \lambda_k(A) c_i^{(k)}(A) \eta_i^{(k)} \eta_i^{(k)T}, \tag{4.16}$$

which is the desired representation. □

From Lemma 4.8 we conclude the following theorem.

THEOREM 4.17. *Let* $C(x) \equiv ((c_{ij}(x)))$ *and* $b(x) = (b_1(x), \cdots, b_d(x))^T$ *satisfy* (4.6) *and* (4.7) *for some* $0 < \mu < \lambda < \infty$. *Then there exist* $\beta_k(x)$, α_k, σ_k^2 *and* m_k, $1 \leqslant k \leqslant N < \infty$ *satisfying* (4.3) *and* (4.4) *with* $\inf_{k,x} \beta_k(x) > 0$ *and* $\sup_{k,x} \beta_k(x) < \infty$. *The* $\beta_k(x)$ *can be taken to be* C^∞ *functions of* $C(x)$ *and* $b(x)$.

PROOF. By Lemma 4.8 we can write

$$C(x) = \Sigma_{i=1}^{N_1} \beta_i(x) \alpha_i \alpha_i^T. \tag{4.18}$$

Let e_j be the vector in $\mathbb{R}^d$ with jth component 1 and the other components 0. Let $\varphi(s)$ and $\Psi(s)$ be C^∞ functions on $\mathbb{R}$ such that $\inf_s \varphi(s) > 0$, $\inf_s \Psi(s) > 0$, and $\varphi(s) - \Psi(s) = s$. Then

$$b(x) = \Sigma_{j=1}^d \varphi(b_j(x)) e_j + \Sigma_{j=1}^d \Psi(b_j(x))(-e_j). \tag{4.19}$$

Let $\sigma_i^2 = 1$ and $m_i = 0$ for $1 \leqslant i \leqslant N_1$ and $\sigma_i^2 = 0$ and $m_i = 1$ for $N_1 + 1 \leqslant i \leqslant N \equiv N_1 + 2d$; $\beta_i(x) = \varphi(b_j(x))$ and $\alpha_i = e_j$ for $i = N_1 + j$, $1 \leqslant j \leqslant d$, and $\beta_i(x) = \Psi(b_j(x))$ and $\alpha_i = -e_j$ for $i = N_1 + d + j$, $1 \leqslant j \leqslant d$. □

Theorem 4.17 shows that to every operator A given by (4.1) with $C(x)$ and $b(x)$ satisfying (4.6) and (4.7) we can associate an equation of the form of (4.2). We now want to show that solving (4.2) is in a sense equivalent to solving a martingale problem for A. Letting $M: \mathbb{R}^N \to \mathbb{R}^d$ be given by

$$M = (\alpha_1, \alpha_2, \cdots, \alpha_N) = ((\alpha_{ik})) \tag{4.20}$$

solving (4.2) is equivalent to solving

$$Z_k(t) = W_k\left(\textstyle\int_0^t \beta_k(X(0) + MZ(s))\, ds\right) \tag{4.21}$$

and setting

$$X(t) = X(0) + MZ(t). \tag{4.22}$$

We are interested in the relationship between the martingale problem for A given by (4.1) (on an appropriate domain) and the random time change problem corresponding to (4.21). Specifically, let $C_\Delta(\mathbb{R}^d)$ ($C_\Delta(\mathbb{R})$) denote the space of continuous functions on $\mathbb{R}^d(\mathbb{R})$ having a limit at infinity in the one point compactification. Let $D = C_\Delta(\mathbb{R}^d) \cap C^2(\mathbb{R}^d)$, let A_k be the generator corresponding to W_k in $C_\Delta(\mathbb{R})$ ($A_k f = \frac{1}{2}\sigma_k^2 f'' + m_k f'$ for f such that f, f' and $f'' \in C_\Delta(\mathbb{R})$) and set $\tilde{\beta}_k(z) = \beta_k(X(0) + Mz)$ (we will assume that $X(0)$ is a fixed element of $\mathbb{R}^d$).

THEOREM 4.23. *Let* (A, D), A_k, α_k, β_k *and* $\tilde{\beta}_k$ *be as above; let* $N < \infty$ *and assume* β_k *is nonnegative, measurable and* $\sup_x \beta_k(x) \equiv \bar{\beta}_k < \infty$. *Suppose* (4.3) *and*

(4.4) *hold. If* $Z(t)$ *is a solution of the random time change problem* $(A_k, \tilde{\beta}_k, 1 \leqslant k \leqslant N)$ *then* $X(t) = X(0) + MZ(t)$ *is a solution of the martingale problem* (A, D).

PROOF. Let $D_1 = \{g \in C(\mathbb{R}^N) : g(z) = \Pi_{k=1}^N f_k(z_k), f_k \in \mathfrak{D}(A_k)\}$, and for $g \in D_1$, define

$$A'g = \Sigma_{k=1}^N \Pi_{l \neq k} f_l \tilde{\beta}_k A_k f_k. \tag{4.24}$$

Then by Theorem 3.11 part (a),

$$g(Z(t)) - \int_0^t A'g(Z(s))\, ds \tag{4.25}$$

is a martingale with respect to $\mathcal{F}_t = \sigma(Z(s) : s \leqslant t)$ for all $g \in D_1$. Note that $C_\Delta^2(\mathbb{R}) = \{f \in C_\Delta(\mathbb{R}) : f', f'' \in C_\Delta(\mathbb{R})\} \subset \mathfrak{D}(A_k)$ and if $g(z) = \Pi_{k=1}^N f_k(z_k), f_k \in C_\Delta^2(\mathbb{R})$ then

$$A'g = \Sigma_{k=1}^N \tfrac{1}{2} \tilde{\beta}_k \sigma_k^2 \partial_k^2 g + \Sigma_{k=1}^N \tilde{\beta}_k m_k \partial_k g. \tag{4.26}$$

Define $A'g$ by (4.26) for all $g \in C^2(\mathbb{R}^N)$ (the space of bounded, continuous functions with two bounded, continuous derivatives). We claim that (4.25) is a $\mathcal{F}_t$-martingale for all $g \in C^2(\mathbb{R}^N)$. Let $D_2 = \{g \in C(\mathbb{R}^N) : g(z) = \Sigma_{i=1}^m c_i \Pi_{k=1}^N f_k^i(z_k), f_k \in C_\Delta^2(\mathbb{R})\}$. Clearly (4.25) is a $\mathcal{F}_t$-martingale for all $g \in D_2$. If $g \in C^2(\mathbb{R}^N)$, then there exist $g_n \in D_2$ such that $\sup_n \|g_n\| < \infty$, $\sup_n \|\partial_k g_n\| < \infty$ and $\sup_n \|\partial_k \partial_l g_n\| < \infty$ for all $1 \leqslant k, l \leqslant N$ and

$$\begin{aligned} &\lim_{n\to\infty} \sup_{z\in\Gamma} |g_n(z) - g(z)| \\ &\quad = \lim_{n\to\infty} \sup_{z\in\Gamma} |\partial_k g_n(z) - \partial_k g(z)| \\ &\quad = \lim_{n\to\infty} \sup_{z\in\Gamma} |\partial_k \partial_l g_n(z) - \partial_k \partial_l g(z)| = 0 \end{aligned} \tag{4.27}$$

for all compact $\Gamma \subset \mathbb{R}^N$ and all $1 \leqslant k, l \leqslant N$. Noting that $\sup_{t \leqslant T} |Z(t)| < \infty$ a.s. the fact that $g_n(Z(t)) - \int_0^t A'g_n(Z(s))\, ds$ is a martingale with respect to $\mathcal{F}_t$ implies (4.25) is also.

Finally, for $f \in D$, define $g(z) = f(X(0) + Mz)$. Then $g \in C^2(\mathbb{R})$ and (4.25) is a $\mathcal{F}_t$-martingale, but (4.25) is just

$$f(X(t)) - \int_0^t Af(X(s))\, ds \tag{4.28}$$

and hence $X(t)$ is a solution of the martingale problem (A, D). ☐

The converse to Theorem 4.23 is more difficult primarily because the equation $X(t) = X(0) + MZ(t)$ does not in general uniquely determine $Z(t)$ in terms of $X(t)$. In fact, in order to obtain $Z(t)$ with the desired properties (i.e., a solution of the random time change problem), we will need to use a Brownian motion independent of $X(t)$. We will need the following lemmas.

LEMMA 4.29. *Suppose* $\sup_x |c_{ij}(x)| < \infty$ *and* $\sup_x |b_j(x)| < \infty$ *for all* $1 \leqslant i, j \leqslant d$ *and that* $X(t)$ *is a right continuous solution of the martingale problem* (A, D). *Then* $X(t)$ *is continuous, and*

$$\tilde{X}_j(t) \equiv X_j(t) - X_j(0) - \int_0^t b_j(X(s))\, ds \tag{4.30}$$

and

$$\tilde{X}_i(t)\tilde{X}_j(t) - \int_0^t c_{ij}(X(s))\,ds \tag{4.31}$$

are martingales.

REMARK. In the notation of Kunita and Watanabe (1967), $\langle \tilde{X}_i, \tilde{X}_j \rangle_t = \int_0^t c_{ij}(X(s))\,ds$.

PROOF. By approximating $|x|^n$ by functions in D, it is easy to show $E(|X(t)|^n) < \infty$ for all $n > 0, t \geqslant 0$, and by approximating $\Pi_{i=1}^d x_i^{n_i}$ by functions in D that

$$u(X(t)) - \int_0^t Au(X(s))\,ds \tag{4.32}$$

is a martingale for any polynomial u. In particular (4.30) and

$$\begin{aligned} M_{ij}(t) = {} & X_i(t)X_j(t) - \int_0^t X_i(s)b_j(X(s))\,ds \\ & - \int_0^t X_j(s)b_i(X(s))\,ds \\ & - \int_0^t c_{ij}(X(s))\,ds \end{aligned} \tag{4.33}$$

are martingales.

Note that (4.31) is

$$\begin{aligned} & M_{ij}(t) - \tilde{X}_i(t)X_j(0) - \tilde{X}_j(t)X_i(0) \\ & - \int_0^t\Big[\Big(X_i(t) - X_i(s) - \int_s^t b_i(X(u))\,du\Big)b_j(X(s)) \\ & + \Big(X_j(t) - X_j(s) - \int_s^t b_j(X(u))\,du\Big)b_i(X(s))\Big]\,ds. \end{aligned} \tag{4.34}$$

The first three terms in (4.34) are clearly martingales. We leave the verification that the last is to the reader.

Using the fact that (4.32) is a martingale for any polynomial it follows that

$$\begin{aligned} E\big((X_i(t) - X_i(u))^4\big) = {} & E\Big(\int_u^t\Big[6c_{ii}(X(s))(X_i(s) - X_i(u))^2 \\ & + 4b_i(X(s))(X_i(s) - X_i(u))^3\Big]\,ds\Big), \end{aligned} \tag{4.35}$$

and (4.35) in turn implies that for $T > 0$ there exists $c_T > 0$ such that

$$E\big((X_i(t) - X_i(u))^r\big) \leqslant c_T(t-u)^2 \tag{4.36}$$

for all $0 \leqslant u < t \leqslant T$. The continuity of X now follows from Billingsley (1968) Theorem 12.3. □

LEMMA 4.37. *Let $M = (\alpha_1, \alpha_2, \cdots, \alpha_N)$ have rank d and select $\gamma_i \in \mathbb{R}^{N-d}$ (if M has rank d then $N \geqslant d$) such that*

$$L = \begin{pmatrix} \alpha_1 & \alpha_2 & \cdots & \alpha_N \\ \gamma_1 & \gamma_2 & \cdots & \gamma_N \end{pmatrix} \tag{4.38}$$

is a nonsingular $N \times N$ matrix.

Let $c_i \geqslant 0, i = 1, 2, \cdots, N$, and define $C = \sum_{i=1}^N c_i\alpha_i\alpha_i^T$. Then there exist a $(N-d) \times d$ matrix σ and a $(N-d) \times (N-d)$ nonnegative definite matrix F such that

$$G = \begin{pmatrix} C & C\sigma^T \\ \sigma C & \sigma C\sigma^T + F \end{pmatrix} = \sum_{i=1}^N c_i\begin{pmatrix} \alpha_i\alpha_i^T & \alpha_i\gamma_i^T \\ \gamma_i\alpha_i^T & \gamma_i\gamma_i^T \end{pmatrix}. \tag{4.39}$$

PROOF. We must have

$$\sigma C = \Sigma_{i=1}^{N} c_i \gamma_i \alpha_i^T. \tag{4.40}$$

Since $c_i \alpha_i^T$ is in the range of C we may take

$$\sigma = \Sigma_{i=1}^{N} c_i \gamma_i \alpha_i^T C^{\#}, \tag{4.41}$$

where $C^{\#}$ is the generalized inverse of C (see Penrose (1955)). While $C^{\#}$ is not a continuous function of C, it is measurable. Then

$$F = \Sigma_{i=1}^{N} c_i \gamma_i \gamma_i^T - \sigma C \sigma^T, \tag{4.42}$$

and it remains to show that F is nonnegative definite.

To see this, let $y \in \mathbb{R}^{N-d}$ and $x = -\sigma^T y$. Then

$$(x^T, y^T) G\binom{x}{y} = y^T F y \geqslant 0. \tag{4.43}$$

REMARK. If $c_i > 0$, all i then F is positive definite.

THEOREM 4.44. *Let* (A, D), A_k, α_k, β_k *and* $\tilde{\beta}_k$ *be as above; let* $N < \infty$ *and assume* $\inf_x \beta_k(x) > 0$ *and* $\sup_x \beta_k(x) \equiv \bar{\beta}_k < \infty$. *Suppose* (4.3) *and* (4.4) *hold. Let* $X(t)$ *be a solution of the martingale problem* (A, D) *and let* $B(t)$ *be a standard* $(N-d)$*-dimensional Brownian motion independent of* $X(t)$ *and adapted to the same filtration* $\mathcal{G}_t$. *Then there is a solution* $Z(t)$ *of the random time change problem* $(A_k, \tilde{\beta}_k, 1 \leqslant k \leqslant N)$ *such that* $X(t) = X(0) + MZ(t)$.

PROOF. We assume that the α_k span $\mathbb{R}^d$. (If not, add additional terms to (4.2) in which $W_k \equiv 0$.) Let γ_i and L be as in Lemma 4.37 and let

$$G(x) = \Sigma_{k=1}^{N} \sigma_k^2 \beta_k(x) \begin{pmatrix} \alpha_k \alpha_k^T & \alpha_k \gamma_k^T \\ \gamma_k \alpha_k^T & \gamma_k \gamma_k^T \end{pmatrix} \tag{4.45}$$

and

$$a(x) = \Sigma_{k=1}^{N} m_k \beta_k(x) \gamma_k. \tag{4.46}$$

By Lemma 4.37 there exist $\sigma(x)$ and $F(x)$ such that

$$G(x) = \begin{pmatrix} C(x) & C(x)\sigma^T(x) \\ \sigma(x)C(x) & \sigma(x)C(x)\sigma^T(x) + F(x) \end{pmatrix}. \tag{4.47}$$

Set $K = L^{-1}$ and note that $KG(x)K^T$ is the diagonal matrix with elements $\sigma_k^2 \beta_k(x)$ and

$$K\begin{pmatrix} b(x) \\ a(x) \end{pmatrix} = (m_1 \beta_1(x) \cdots m_N \beta_N(x))^T. \tag{4.48}$$

As in Lemma 4.29, let $\tilde{X}(t) = X(t) - \int_0^t b(X(s))\,ds$. Define (see Kunita and Watanabe (1967))

$$\begin{aligned} Y(t) &= \textstyle\int_0^t \sigma(X(s))\,d\tilde{X}(s) + \int_0^t F^{\frac{1}{2}}(X(s))\,dB + \int_0^t a(X(s))\,ds \\ &\equiv \tilde{Y}(t) + \textstyle\int_0^t a(X(s))\,ds \end{aligned} \tag{4.49}$$

(take $F^{\frac{1}{2}}$ to be the nonnegative definite square root of F) and set

$$Z(t) = L^{-1}\begin{pmatrix} X(t) - X(0) \\ Y(t) \end{pmatrix}. \tag{4.50}$$

Clearly $X(t) = X(0) + MZ(t)$. By Theorem 3.11, to show that $Z(t)$ is a solution of the random time change problem $(A_k, \tilde{\beta}_k, 1 \leqslant k \leqslant N)$ it is sufficient to show that it is a solution of the martingale problem (A', D_1) (A' defined by (4.24)). Let $L^{-1} \equiv K = ((k_{ij}))$ and note that $X(t) - X(0) = \tilde{X}(t) + \int_0^t b(X(s))\,ds$.

From Lemma 4.29

$$\langle \tilde{X}_i, \tilde{X}_j \rangle_t = \int_0^t c_{ij}(X(s))\,ds \tag{4.51}$$

and Theorem 2.1 of Kunita and Watanabe (1967) can be used to compute

$$\langle \tilde{Y}_i, \tilde{X}_j \rangle_t = \int_0^t (\sigma(X(s))C(X(s)))_{ij}\,ds \tag{4.52}$$

and

$$\langle \tilde{Y}_i, \tilde{Y}_j \rangle_t = \int_0^t (\sigma(X(s))C(X(s))\sigma^T(X(s)))_{ij}\,ds + \int_0^t F_{ij}(X(s))\,ds. \tag{4.53}$$

For $f \in C^2(\mathbb{R}^d \times \mathbb{R}^{N-d})$, Itô's formula (Kunita and Watanabe (1967), Theorem 2.2) gives

$$\begin{aligned} f(X(t) - X(0), Y(t)) - f(0,0) &= \Sigma_{i=1}^d \int_0^t f_{x_i}(X(s) - X(0), Y(s))\,d\tilde{X}_i(s) \\ &+ \Sigma_{i=1}^{N-d} \int_0^t f_{y_i}(X(s) - X(0), Y(s))\,d\tilde{Y}_i(s) \\ &+ \tfrac{1}{2}\Sigma_{1 \leqslant i,j \leqslant d} \int_0^t f_{x_i x_j}(X(s) - X(0), Y(s))c_{ij}(X(s))\,ds \\ &+ \Sigma_{1 \leqslant i \leqslant N-d;\, 1 \leqslant j \leqslant d} \int_0^t f_{x_j y_i}(X(s) - X(0), Y(s)) \\ &\quad \times (\sigma(X(s))C(X(s)))_{ij}\,ds \\ &+ \tfrac{1}{2}\Sigma_{1 \leqslant i,j \leqslant N-d} \int_0^t f_{y_i y_j}(X(s) - X(0), Y(s)) \\ &\quad \times (\sigma(X(s))C(X(s))\sigma^T(X(s)))_{ij}\,ds \\ &+ \tfrac{1}{2}\Sigma_{1 \leqslant i,j \leqslant N-d} \int_0^t f_{y_i y_j}(X(s) - X(0), Y(s))F_{ij}(X(s))\,ds \\ &+ \Sigma_{i=1}^d f_{x_i}(X(s) - X(0), Y(s))b_i(X(s))\,ds \\ &+ \Sigma_{i=1}^{N-d} \int_0^t f_{y_i}(X(s) - X(0), Y(s))a_i(X(s))\,ds. \end{aligned} \tag{4.54}$$

Write $K = (K^{(1)}, K^{(2)})$ where $K^{(1)}$ is $N \times d$ and $K^{(2)}$ is $N \times (N - d)$. Let $g \in D_1$ and set $f(x,y) = g(K^{(1)}x + K^{(2)}y)$. Then $g(Z(t)) = f(X(t) - X(0), Y(t))$. The first two terms on the right of (4.54) form a martingale (denote it $M_g(t)$) and (4.54) becomes

$$(4.55)\quad g(Z(t)) - g(Z(0))$$
$$= M_g(t) + \tfrac{1}{2}\Sigma_{k,l}\textstyle\int_0^t g_{z_k z_l}(Z(s))\big(K^{(1)}C(X(s))K^{(1)T}\big)_{k,l}ds$$
$$+ \Sigma_{k,l}\textstyle\int_0^t g_{z_k z_l}(Z(s))\big(K^{(2)}\sigma(X(s))C(X(s))K^{(1)T}\big)_{k,l}ds$$
$$+ \tfrac{1}{2}\Sigma_{k,l}\textstyle\int_0^t g_{z_k z_l}(Z(s))\big(K^{(2)}\sigma(X(s))C(X(s))\sigma^T(X(s))K^{(2)T}\big)_{k,l}ds$$
$$+ \tfrac{1}{2}\Sigma_{k,l}\textstyle\int_0^t g_{z_k z_l}(Z(s))\big(K^{(2)}F(X(s))K^{(2)T}\big)_{k,l}ds$$
$$+ \Sigma_k\textstyle\int_0^t g_{z_k}(Z(s))\big(K^{(1)}b(X(s))\big)_k ds + \Sigma_k\int_0^t g_{z_k}(Z(s))\big(K^{(2)}a(X(s))\big)_k ds$$
$$= M_g(t) + \tfrac{1}{2}\Sigma_{k,l}\textstyle\int_0^t g_{z_k z_l}(Z(s))\big(KG(X(s))K^T\big)_{k,l}ds$$
$$+ \Sigma_k\textstyle\int_0^t g_{z_k}(Z(s))\big(K^{(1)}b(X(s)) + K^{(2)}a(X(s))\big)ds$$
$$= M_g(t) + \tfrac{1}{2}\Sigma_k\textstyle\int_0^t\sigma_k^2\beta_k(X(s))g_{z_k z_k}(Z(s))\,ds$$
$$+ \Sigma_k\textstyle\int_0^t m_k\beta_k(X(s))g_{z_k}(Z(s))\,ds$$
$$= M_g(t) + \textstyle\int_0^t A'g(Z(s))\,ds.$$

It follows that $Z(t)$ is a solution of the martingale problem (A', D_1) and hence of the random time change problem $(A_k, \tilde{\beta}_k, 1 \leq k \leq N)$. □

REMARK. Suppose that $m_i = 0$ for all i in (4.2) and that $\beta_k(x) \leq \bar{\beta}_k$. Then the optional sampling theorem in Kurtz (1980) implies

$$(4.56)\qquad X(t) - X(0) = E\big(\Sigma_{k=1}^N\alpha_k W_k(\bar{\beta}_k t)|\mathcal{G}_t\big).$$

This identity allows one to estimate moments of $X(t)$ in terms of moments of the Brownian motion $\Sigma_{k=1}^N\alpha_k W_k(\bar{\beta}_k t)$.

5. Existence of random time changes: Liggett's theorems. Theorem 3.28 gives conditions for the existence of solutions to the martingale problem corresponding to (3.6) and (3.8) and hence for existence in a weak sense of solutions of the random time change problem. In this section, we consider briefly approaches one might take in proving existence and uniqueness of random time changes in the strong sense, i.e., given the Y_i. At this point we have no good general theorem, but some of the observations we make here may be useful in developing a general theory.

Given independent Markov processes Y_i and nonnegative measurable functions β_i, we would like to be able to solve

$$(5.1)\qquad Z_i(t) = Y_i\big(\textstyle\int_0^t\beta_i(Z(s))\,ds\big),\qquad i = 1, 2, \cdots, N \leq \infty,$$

with the added condition that $\tau(t) = (\int_0^t\beta_1(Z(s))\,ds, \int_0^t\beta_2(Z(s))\,ds, \cdots)$ is a stopping time for some family $\mathcal{F}_u$ for which (3.1) holds. Typically one might take

$$(5.2)\qquad \mathcal{F}_u = \sigma(\mathfrak{N}) \vee \cap_{\varepsilon>0}\sigma\{Y_i(s_i) : s_i \leq u_i + \varepsilon, i = 1, 2, \cdots\}$$

where $\mathfrak{N}$ is the collection of sets of measure zero. (Assume that the underlying probability space is complete.) Note that this definition of $\mathcal{F}_u$ ensures that $\mathcal{F}_u = \cap_{\varepsilon>0}\mathcal{F}_{u+\varepsilon}(u + \varepsilon = \{u_i + \varepsilon\})$ which in turn implies that limits of stopping times are stopping times.

An iterative approach

$$Z_i^{(n+1)}(t) = Y_i\big(\textstyle\int_0^t\beta_i(Z^{(n)}(s))\,ds\big) \tag{5.3}$$

runs into trouble immediately, since there is no reason to expect the $\int_0^t\beta_i(Z^{(n)}(s))\,ds$ to be the components of a stopping time. On the other hand, if $N < \infty$, the system

$$Z_i^{(n)}(t) = Y_i\big(\big[n\textstyle\int_0^t\beta_i(Z^{(n)}(s))\,ds\big]/n\big) \tag{5.4}$$

has a unique solution and $\tau_i^{(n)}(t) = [n\int_0^t\beta_i(Z^{(n)}(s))]/n$ are the components of a stopping time. One might be able to show the existence of $\lim_{n\to\infty}Z_i^{(n)}(t)$.

Another approach is suggested by the equation

$$\frac{d}{dt}\tau_i(t) = \beta_i(Y_1(\tau_1(t)), Y_2(\tau_2(t)), \cdots). \tag{5.5}$$

From (5.5) we see that for each $\omega, \tau(t)$ must be the solution of an autonomous system of ordinary differential equations. Helms (1974) uses this approach to prove existence and uniqueness for finitely many Y_i having bounded generators (i.e., jump processes with expected holding times bounded away from zero).

Alternatively, we may think of $\tau(t)$ as taking values in some linear space and (5.5) determining a single ordinary differential equation

$$\dot{\tau}(t) = F(\tau(t)) \tag{5.6}$$

in that space. Of course we are interested in solutions that take values in $\mathfrak{T}$, the collection of all stopping times. This is not unreasonable since for $\tau \in \mathfrak{T}$ and $\varepsilon \geq 0$

$$\begin{aligned}\{\tau + \varepsilon F(\tau) \leq u\} &= \{\tau + \varepsilon F(\tau) \leq u\} \cap \{\tau \leq u\} \\ &= \{\tau \leq u - \varepsilon F(\tau)\chi_{\{\tau<u\}}\} \in \mathcal{F}_u\end{aligned} \tag{5.7}$$

($F(\tau)\chi_{\{\tau<u\}}$ is $\mathcal{F}_u$ measurable) and hence $\tau + \varepsilon F(\tau)$ is a stopping time. The problem would be essentially solved if we could find a norm under which F was Lipschitz continuous. To illustrate that possibility, we prove Liggett's existence and uniqueness theorems for infinite particle systems (Liggett (1972)). Existence and uniqueness of the time change were considered by Holley and Stroock (1976) for spin flip models, using other methods.

We first prove two theorems concerning time changes for infinitely many Poisson processes, and then show how existence and uniqueness can be obtained under Liggett's conditions. (Of course, Liggett proves more in that he characterizes a core for the infinitesimal generators.)

THEOREM 5.8. *Let $Y_i, i = 1, 2, \cdots$, be independent Poisson processes, let $\mathcal{F}_u$ satisfy $\mathcal{F}_u = \cap_{\varepsilon>0}\mathcal{F}_{u+\varepsilon}$ and* (3.1). *Let $\beta_i(z) \geq 0$ be continuous functions on $E \equiv \{0, 1, 2, \cdots\}^\infty$, and let*

$$a_{ik} = \sup_{z\in E}|\beta_i(z + e_k) - \beta_i(z)| \tag{5.9}$$

(where $e_k \in E$ is the element whose kth *component is* 1 *and other components are zero) and*

$$b_i = \sup_{z^1, z^2 \in E} |\beta_i(z^1) - \beta_i(z^2)|. \tag{5.10}$$

Let $\| \ \|$ *denote the* l_p*-norm for some* $1 \leqslant p \leqslant \infty$ *and let* $\| \ \|^*$ *denote the* l_q*-norm where* $1/p + 1/q = 1$. *If there exist* $c_i > 0, i = 1, 2, \cdots$ *and* $M < \infty$ *such that*

$$\| c_i b_i \|_i < M \tag{5.11}$$

and

$$\| \ \| c_i a_{ik} / c_k \|_k^* \|_i < M \tag{5.12}$$

or

$$\| \ \| c_i a_{ik} / c_k \|_i \|_k^* < M \tag{5.13}$$

(where $\| m_{ik} \|_i$ *denotes* $\| \ \|$ *applied to the sequence* $m_{ik}, i = 1, 2, \cdots$ *for fixed* k*) then there exists an almost surely unique solution of*

$$Z_i(t) = Y_i\left(\textstyle\int_0^t \beta_i(Z(s))\, ds \right) \tag{5.14}$$

with $\tau(t) = (\int_0^t \beta_i(Z(s))\, ds) \in \mathcal{T}$ *all* $t \geqslant 0$.

REMARK. The author would like to thank Larry Gray and the referee for pointing out the possibility of eliminating a uniform boundedness hypothesis on the β_i used in an earlier version of this result.

Theorem 5.8 applies to spin flip systems (the spins being given by $(-1)^{Z_i}$). To obtain Liggett's results for models involving speed change and exclusion we need a somewhat different result. In the application Z_{ij}, below, is the number of transitions of particles from state i to state j.

THEOREM 5.15. *Let* $Y_{ij}, i, j = 1, 2, \cdots$ *be independent Poisson processes, let* $\mathcal{F}_u, u = \{u_{ij}\}$, *satisfy* $\mathcal{F}_u = \cap_{\varepsilon > 0} \mathcal{F}_{u+\varepsilon}$ *and the analog of* (3.1). *Let* $\beta_{ij}(z) \geqslant 0$ *be continuous functions on* $E \equiv \Pi_{k,l=1}^{\infty} \{0, 1, 2, \cdots\}$ *and suppose there exist* a_{ijk}, b_{ijk}, c_{ij}, *and* $M < \infty$ *such that*

$$\sup_{z \in E} |\beta_{ij}(z + e_{kl}) - \beta_{ij}(z)| \leqslant a_{ijk} + b_{ijl}, \tag{5.16}$$

$$c_{ij} = \sup_{z^1, z^2 \in E} |\beta_{ij}(z^1) - \beta_{ij}(z^2)|, \tag{5.17}$$

$$\sup_i \Sigma_j c_{ij} + \sup_j \Sigma_i c_{ij} < M, \tag{5.18}$$

$$\sup_i \Sigma_{j,k} a_{ijk} + \sup_j \Sigma_{i,k} a_{ijk} < M, \tag{5.19}$$

and

$$\sup_i \Sigma_{j,l} b_{ijl} + \sup_j \Sigma_{i,l} b_{ijl} < M. \tag{5.20}$$

Then there exists an almost surely unique solution of

$$Z_{ij}(t) = Y_{ij}\left(\textstyle\int_0^t \beta_{ij}(Z(s))\, ds \right), \tag{5.21}$$

with $\tau(t) = (\int_0^t \beta_{ij}(Z(s))\, ds)$ *a* $\mathcal{F}_u$*-stopping time for all* $t \geqslant 0$.

The proofs of the above theorems depend on the following lemma.

LEMMA 5.22. *Let Y_i, β_i, a_{ik} and $\mathcal{F}_u$ be as in Theorem 5.8, and set $F_i(\tau) = \beta_i(Y_1(\tau_1), Y_2(\tau_2), \cdots)$. Then for $\mathcal{F}_u$-stopping times $\tau^{(1)}$ and $\tau^{(2)}$*

$$E(|F_i(\tau^{(1)}) - F_i(\tau^{(2)})|) \leq \Sigma_k a_{ik} E(|\tau_k^{(1)} - \tau_k^{(2)}|). \tag{5.23}$$

PROOF. Note that

$$\sup_z |\beta_i(z + me_k) - \beta_i(z)| \leq \sup_z |\beta_i(z + e_k) - \beta_i(z)| m = a_{ik} m. \tag{5.24}$$

It follows that

$$\begin{aligned} E(|F_i(\tau^{(1)}) - F_i(\tau^{(2)})|) &\leq E(\Sigma_k |\beta_i(Y_1(\tau_1^{(2)}) \cdots Y_k(\tau_k^{(1)}) Y_{k+1}(\tau_{k+1}^{(1)}) \cdots) \\ &\qquad - \beta_i(Y_1(\tau_1^{(2)}) \cdots Y_k(\tau_k^{(2)}) Y_{k+1}(\tau_{k+1}^{(1)}) \cdots)|) \\ &\leq E(\Sigma_k a_{ik} |Y_k(\tau_k^{(1)}) - Y_k(\tau_k^{(2)})|) \qquad (5.25) \\ &= \Sigma_k a_{ik} E(Y_k(\tau_k^{(1)} \vee \tau_k^{(2)}) - Y_k(\tau_k^{(1)} \wedge \tau_k^{(2)})) \\ &= \Sigma_k a_{ik} E(|\tau_k^{(1)} - \tau_k^{(2)}|). \end{aligned}$$

Note that the first inequality uses the continuity of β_i and the last equality follows from the fact that $Y_k(t)$ is a Poisson process with respect to the filtration $\mathcal{M}_t^{(k)} = \vee_{u_k < t} \mathcal{F}_u$, and $\tau_k^{(1)}$ and $\tau_k^{(2)}$ are $\mathcal{M}_t^{(k)}$ stopping times. Consequently,

$$\begin{aligned} E(Y_k(\tau_k^{(1)} \vee \tau_k^{(2)}) - Y_k(\tau_k^{(1)} \wedge \tau_k^{(2)})) &= E(\tau_k^{(1)} \vee \tau_k^{(2)} - \tau_k^{(1)} \wedge \tau_k^{(2)}) \\ &= E(|\tau_k^{(1)} - \tau_k^{(2)}|). \end{aligned} \tag{5.26}$$

□

PROOF OF THEOREM 5.8. For an $\mathbb{R}^\infty$-valued random variable ξ set $\| \ \|_E = \|c_i E(|\xi_i|)\|$. For $\tau^{(1)}, \tau^{(2)} \in \mathcal{T}$ inequality (5.23) and the triangle inequality imply

$$\|F(\tau^{(1)}) - F(\tau^{(2)})\|_E \leq \Sigma_k \|c_i a_{ik}/c_k\|_i c_k E(|\tau_k^{(1)} - \tau_k^{(2)}|). \tag{5.27}$$

If (5.13) holds then the Hölder inequality gives

$$\|F(\tau^{(1)}) - F(\tau^{(2)})\|_E \leq M \|\tau^{(1)} - \tau^{(2)}\|_E. \tag{5.28}$$

Similarly if we first apply the Hölder inequality to (5.23)

$$c_i E(|F_i(\tau^{(1)} - F_i(\tau^{(2)})|) \leq \|c_i a_{ik}/c_k\|_k^* \|\tau^{(1)} - \tau^{(2)}\|_E. \tag{5.29}$$

In this case (5.12) implies (5.28).

If $\tau^{(1)}(t)$ and $\tau^{(2)}(t)$ are solutions of (5.6) with values in $\mathcal{T}$ then (5.28) implies

$$\begin{aligned} \|\tau^{(1)}(t) - \tau^{(2)}(t)\|_E &\leq \int_0^t \|F(\tau^{(1)}(s)) - F(\tau^{(2)}(s))\|_E ds \\ &\leq M \int_0^t \|\tau^{(1)}(s) - \tau^{(2)}(s)\|_E ds. \end{aligned} \tag{5.30}$$

Uniqueness of the solution of (5.6) would follow by Gronwall's inequality if we knew the right side of (5.30) was finite. But

$$\begin{aligned} \|\tau^{(1)}(t) - \tau^{(2)}(t)\|_E &\leq \int_0^t \|F(\tau^{(1)}(s)) - F(\tau^{(2)}(s))\|_E ds \\ &\leq \int_0^t \|c_i b_i\|_i ds \leq Mt \end{aligned} \tag{5.31}$$

which implies the right side of (5.30) is bounded by $M^2 t^2/2$.

To prove existence let $\underline{\beta} = \{\underline{\beta}_i\}$ where $\underline{\beta}_i = \inf_{z \in E} \beta_i(z)$. Note that $|\beta_i(z) - \underline{\beta}_i| \leq b_i$. Let

$$\tau^{(n)}(t) = \int_0^t F(s\underline{\beta})\,ds \qquad 0 \leq t \leq 1/n, \tag{5.32}$$

and

$$\tau^{(n)}(t) = \tau^{(n)}(k/n) + \int_{k/n}^t F\big(\tau^{(n)}(k/n) + (s - k/n)\underline{\beta}\big)\,ds \qquad k/n \leq t \leq (k+1)/n. \tag{5.33}$$

Set $\gamma^{(n)}(t) = \tau^{(n)}([nt]/n) + (t - [nt]/n)\underline{\beta}$ and note

$$\tau^{(n)}(t) = \int_0^t F(\gamma^{(n)}(s))\,ds. \tag{5.34}$$

The fact that $\gamma^{(n)}(t), \tau^{(n)}(t) \in \mathcal{T}$ follows from the fact that $\tau_i^{(n)}(t) \geq \gamma_i^{(n)}(t)$ (recall the definition of $\underline{\beta}$) and $\{\gamma_i^{(n)}(t) \leq u\} = \{\tau([nt]/n) \leq u - (t - [nt]/n)\underline{\beta}\} \in \mathcal{F}_u$. Finally observe that

$$\begin{aligned} \|\tau^{(n)}(t) - \gamma^{(n)}(t)\|_E &\leq \int_{[nt]/n}^t \|F(\gamma^{(n)}(s)) - \underline{\beta}\|_E\,ds \\ &\leq \|c_i b_i\|_i/n \end{aligned} \tag{5.35}$$

so

$$\begin{aligned} \|\gamma^{(n)}(t) - \gamma^{(m)}(t)\|_E &\leq \|\tau^{(n)}(t) - \gamma^{(n)}(t)\|_E + \|\tau^{(m)}(t) - \gamma^{(m)}(t)\|_E \\ &\quad + \left\|\int_0^t F(\gamma^{(n)}(s))\,ds - \int_0^t F(\gamma^{(m)}(s))\,ds\right\|_E \\ &\leq 2\|c_i b_i\|_i/n + M\int_0^t \|\gamma^{(n)}(s) - \gamma^{(m)}(s)\|_E\,ds \end{aligned} \tag{5.36}$$

and hence

$$\|\tau^{(n)}(t) - \tau^{(m)}(t)\|_E \leq 2\|c_i b_i\|_i(1 + e^{Mt})/n. \tag{5.37}$$

Consequently $\tau^{(n)}(t) - \tau^{(1)}(t)$ is a Cauchy sequence in $\|\cdot\|_E$ and hence there exists a stopping time $\tau(t)$ such that $\tau_n(t) \to \tau(t)$ in probability (i.e., $\|\tau^{(n)}(t) - \tau^{(1)}(t) - (\tau(t) - \tau^{(1)}(t))\|_E \to 0$). Since $\|\tau^{(n)}(t) - \gamma^{(n)}(t)\|_E \to 0$ it follows that $\tau(t)$ satisfies (5.6). □

Proof of Theorem 5.15. In this case, the norm is

$$\|\xi\|_E = \sup_i \Sigma_j E(|\xi_{ij}|) + \sup_j \Sigma_i E(|\xi_{ij}|).$$

By Lemma 5.22 and the hypotheses, we have

$$\begin{aligned} &E\big(|F_{ij}(\tau^{(1)}) - F_{ij}(\tau^{(2)})|\big) \\ &\quad\leq \Sigma_{k,l} \sup_z |\beta_{ij}(z + e_{kl}) - \beta_{ij}(z)| E\big(|\tau_{kl}^{(1)} - \tau_{kl}^{(2)}|\big) \\ &\quad\leq \Sigma_k a_{ijk} \Sigma_l E\big(|\tau_{kl}^{(1)} - \tau_{kl}^{(2)}|\big) + \Sigma_l b_{ijl} \Sigma_k E\big(|\tau_{kl}^{(1)} - \tau_{kl}^{(2)}|\big) \\ &\quad\leq \sup_k \Sigma_l E\big(|\tau_{kl}^{(1)} - \tau_{kl}^{(2)}|\big) \Sigma_k a_{ijk} + \sup_l \Sigma_k E\big(|\tau_{kl}^{(1)} - \tau_{kl}^{(2)}|\big) \Sigma_l b_{ijl}. \end{aligned} \tag{5.38}$$

We now see that

$$\|F(\tau^{(1)}) - F(\tau^{(2)})\|_E \leq M\|\tau^{(1)} - \tau^{(2)}\|_E. \tag{5.39}$$

The remainder of the proof is similar to that of Theorem 5.8. □

COROLLARY 5.40. *Let $A_i f(l)(A_{ij} f(l)) = f(l+1) - f(l)$. Under the hypotheses of Theorem* 5.8 (*Theorem* 5.15) *all solutions of the random time change problem* $(A_i, \beta_i, i = 1, 2, \cdots)((A_{ij}, \beta_{ij}, i,j = 1, 2, \cdots))$ *have the same distribution.*

PROOF. Assume the hypotheses of Theorem 5.8. Theorem 5.8 implies existence. To show uniqueness, suppose $Z(t)$ is a solution of the random time change problem for $(A_i, \beta_i, i = 1, 2, \cdots)$ corresponding to an increasing family $\mathcal{F}_u$ and Poisson processes $Y_1, Y_2, \cdots$. If $\mathcal{F}_u$ does not satisfy $\mathcal{F}_u = \cap_{\varepsilon>0}\mathcal{F}_{u+\varepsilon}$ replace $\mathcal{F}_u$ by $\mathcal{F}_u^* = \cap \mathcal{F}_{u+\varepsilon}$. Note $\mathcal{F}_u^* = \cap_{\varepsilon>0}\mathcal{F}_{u+\varepsilon}^*$ and $\mathcal{F}_u^*$ satisfies (3.1). As in Lemma 5.22, let $F_i(\tau) = \beta_i(Y_1(\tau_1), Y_2(\tau_2), \cdots)$ and define $\tau^{(n)}(t)$ by (5.32) and (5.33). Note that the distribution of $\{Y_i(\tau_i^{(n)}(t))\}$ does not depend on what family of independent Poisson processes is used to define $\tau^{(n)}(t)$. The uniqueness in Theorem 5.8 implies

$$E\left(\left|\int_0^t \beta_i(Z(s))\,ds - \tau_i^{(n)}(t)\right|\right) \to 0$$

and hence $E(|Z_i(t) - Y_i(\tau_i^{(n)}(t))|) \to 0$. Consequently, all solutions of the random time change problem must have the same distribution. □

Theorem 5.8, with $\|\cdot\|$ the l_∞-norm, is essentially Liggett's Theorem (4.2). Liggett, however, considers speed changes for more general Markov processes than Poisson processes. We will now show how a slight modification of our proof gives existence and uniqueness under his conditions. Let $U_i(k)$ be independent discrete time Markov chains and let $V_i(t)$ be independent (of each other and of the U_i) Poisson processes. Then $Y_i(t) = U_i(V_i(t))$ are independent Markov processes with generators $A_i f(x) = \int (f(y) - f(x))P_i(x, dy)$ where the $P_i(x, \Gamma)$ are the transition functions for the U_i. (The U_i may have arbitrary measurable state spaces E_i). The following theorem gives existence and uniqueness under the conditions of Liggett's Theorem (4.2).

THEOREM 5.41. *Let U_i, V_i and Y_i be as above, and let $\mathcal{F}_u$ satisfy $\mathcal{F}_u = \cap_{\varepsilon>0}\mathcal{F}_{u+\varepsilon}$ and* (3.1). *Let $\beta_i(z) \geq 0$ be continuous functions on $E = \Pi_l E_l$ (with the discrete topology on E_i and the corresponding product topology on E). Suppose*

$$\sup_i \sup_{z^1, z^2 \in E} |\beta_i(z^1) - \beta_i(z^2)| < \infty \tag{5.42}$$

and

$$\sup_i \Sigma_k \sup_{z\in E} \sup_{y \in E_k} |\beta_i(z_1 \cdots z_{k-1}, y, z_{k+1}, \cdots) - \beta_i(z_1 \cdots z_{k-1}, z_k, z_{k+1} \cdots)| < \infty. \tag{5.43}$$

Then there exists an almost surely unique solution of

$$Z_i(t) = Y_i\left(\int_0^t \beta_i(Z(s))\,ds\right) \tag{5.44}$$

with $\tau(t) = \left(\int_0^t \beta_i(Z(s))\,ds\right) \in \mathcal{T}$ all $t \geq 0$.

PROOF. Observe that for $\tau^{(1)}, \tau^{(2)} \in \mathfrak{T}$

$$
\begin{aligned}
&E\big(|F_i(\tau^{(1)}) - F_i(\tau^{(2)})|\big)\\
&\qquad \leqslant E\big(\Sigma_k \sup_l |\beta_i(U_1(l_1)\cdots U_k(l_k+1)\cdots)\\
&\qquad\qquad - \beta_i(U_1(l_1)\cdots U_k(l_k)\cdots)||Y_k(\tau_k^{(1)}) - Y_k(\tau_k^{(2)})|\big)\\
&\qquad \leqslant \Sigma_k \sup_{z\in E}\sup_{y\in E_k}|\beta_i(z_1\cdots y\cdots)\\
&\qquad\qquad - \beta_i(z_1\cdots z_k\cdots)|E\big(|Y_k(\tau_k^{(1)}) - Y_k(\tau_k^{(2)})|\big).
\end{aligned}
\tag{5.45}
$$

The proof now proceeds as in Theorem 5.8. □

We now consider Liggett's conditions for systems with speed change and exclusion, and we will see that existence and uniqueness under the conditions of his Theorem 3.7 follow from Theorem 5.15. We consider a countable collection of particles moving among sites indexed by the integers. We assume that each site contains finitely many particles.

If $X_i(t)$ denotes the number of particles at site i at time t, then

$$X_i(t) = X_i(0) + \Sigma_j Z_{ji}(t) - \Sigma_j Z_{ij}(t) \tag{5.46}$$

where $Z_{ij}(t)$ denotes the number of particles that have jumped directly from site i to site j up to time t. We will obtain the Z_{ij} as time changed Poisson processes using Theorem 5.15.

Following Liggett we define

$$\beta_{ij}(\eta) = c(i,\eta)p(i,j,\eta). \tag{5.47}$$

Intuitively, $c(i,\eta)\,dt$ is the probability that a particle at site i jumps in $(t, t+dt]$ and $p(i,j,\eta)$ is the probability that it jumps to site j. For this particle interpretation $\eta_i = 0$ should imply $c(i,\eta) = 0$, but we will not require this for our result (therefore allowing a negative "number of particles" at a site).

Given η, define η^{i+}, η^{i-} and η^{ij} as follows:

$$
\begin{aligned}
\eta_j^{i+} &= \eta_j + 1 \qquad && j = i;\\
&= \eta_j && j \neq i;
\end{aligned}
\tag{5.48}
$$

$$
\begin{aligned}
\eta_j^{i-} &= \eta_j - 1 \qquad && j = i;\\
&= \eta_j && j \neq i;
\end{aligned}
\tag{5.49}
$$

$$
\begin{aligned}
\eta_k^{ij} &= \eta_k - 1 \qquad && k = i;\\
&= \eta_k + 1 && k = j;\\
&= \eta_k && k \neq i,j
\end{aligned}
\tag{5.50}
$$

Note that $\eta^{ij} = (\eta^{i-})^{j+} = (\eta^{j+})^{i-}$.

Liggett's conditions are somewhat different from those stated below. In particular, his functions are only defined for η in which $\eta_i = 0$ or 1. Let $\eta_i^* = (\eta_i \wedge 1) \vee 0$. We leave it to the reader to show that if c^* and p^* satisfy Liggett's conditions, then $c(i,\eta) \equiv c^*(i,\eta^*)$ and $p(i,j,\eta) \equiv p^*(i,j,\eta^*)$ satisfy ours.

THEOREM 5.51. *Let $\beta_{ij}(\cdot)$ be as above and let Y_{ij} be independent Poisson processes. Let $\mathcal{F}_u$ satisfy $\mathcal{F}_u = \bigcap_{\varepsilon>0}\mathcal{F}_{u+\varepsilon}$ and the analog of* (3.1). *Let $X(0)$ be fixed and define*

$$H_i(z) = X_i(0) + \Sigma_j z_{ji} - \Sigma_j z_{ij}. \tag{5.52}$$

Suppose there exist $c(i)$ and $\rho(i,j)$ such that

$$c(i,\eta) \leqslant c(i), p(i,j,\eta) \leqslant \rho(i,j), \tag{5.53}$$

$$\begin{aligned} &\Sigma_j \sup_\eta |c(i,\eta^{j+}) - c(i,\eta)| \leqslant c(i);\\ &\Sigma_j \sup_\eta |c(i,\eta^{j-}) - c(i,\eta)| \leqslant c(i); \end{aligned} \tag{5.54}$$

$$\begin{aligned} &\Sigma_k \sup_\eta |p(i,j,\eta^{k+}) - p(i,j,\eta)| \leqslant \rho(i,j);\\ &\Sigma_k \sup_\eta |p(i,j,\eta^{k-}) - p(i,j,\eta)| \leqslant \rho(i,j). \end{aligned} \tag{5.55}$$

If

$$\sup_i \Sigma_j c(i)\rho(i,j) < \infty \text{ and } \sup_j \Sigma_i c(i)\rho(i,j) < \infty, \tag{5.56}$$

then the random time change problem

$$Z_{ij}(t) = Y_{ij}\left(\int_0^t \beta_{ij}(H(Z(s)))\,ds\right) \tag{5.57}$$

has a unique solution.

REMARK. This result implies existence and uniqueness for the martingale problem corresponding to Z and existence for the martingale problem corresponding to X follows easily. In order to prove uniqueness for the martingale problem corresponding to X, we need an analog of Theorem 4.44. That is, given a solution of the martingale problem corresponding to X, we must construct the processes Z_{ij} and show that they are a solution to the appropriate martingale problem. This construction is given in Theorem 5.71 below.

PROOF. Considering condition (5.16), we have

$$\begin{aligned} \sup_z |\beta_{ij}(z + e_{kl}) - \beta_{ij}(z)|&\\ &= \sup_\eta |c(i,\eta^{kl})p(i,j,\eta^{kl}) - c(i,\eta)p(i,j,\eta)|\\ &\leqslant \sup_\eta |c(i,\eta^{k-})p(i,j,\eta^{k-}) - c(i,\eta)p(i,j,\eta)|\\ &\quad + \sup_\eta |c(i,\eta^{l+})p(i,j,\eta^{l+}) - c(i,\eta)p(i,j,\eta)|\\ &\equiv a_{ijk} + b_{ijl}. \end{aligned} \tag{5.58}$$

To obtain (5.19) consider

$$\begin{aligned} \Sigma_k a_{ijk} &\leqslant \Sigma_k \sup_\eta |c(i,\eta^{k-}) - c(i,\eta)|\rho(i,j)\\ &\quad + \Sigma_k c(i)|p(i,j,\eta^{k-}) - p(i,j,\eta)|\\ &\leqslant 2c(i)\rho(i,j). \end{aligned} \tag{5.59}$$

Condition (5.19) now follows by (5.56), and (5.20) is verified similarly.

Since $\beta_{ij}(z) \leqslant c(i)\rho(i,j)$, (5.56) implies (5.17) as well, and Theorem 5.51 follows from Theorem 5.15. □

Let $\beta_{ij}(\eta)$ satisfy

(5.60) $$\sup_i \Sigma_j \sup_\eta \beta_{ij}(\eta) + \sup_j \Sigma_i \sup_\eta \beta_{ij}(\eta) < \infty.$$

Let D_η be the collection of bounded functions $f(\eta)$ that depend on only finitely many η_i and define

(5.61) $$A_\eta f(\eta) = \Sigma_{ij}\beta_{ij}(\eta)\big(f(\eta^{ij}) - f(\eta)\big).$$

Note that the sum is convergent by (5.60) and the fact that f depends on only finitely many coordinates.

Let D_z^0 be the collection of functions of the form $f(z) = \Pi_{(i,j)\in I} f_{ij}(z_{ij})$ where I is finite and define

(5.62) $$A_z f(z) = \Sigma\beta_{ij}(H(z))\big(f(z + e_{ij}) - f(z)\big).$$

The domain of the closure of A_z defined on the linear span of D_z^0 will contain D_z, the collection of $f(z)$ depending on only finitely many z_{ij}, with A_z on D_z being given by (5.62).

Under condition (5.60), any solution of the martingale problem (A_z, D_z^0) (or equivalently (A_z, D_z)) with $Z_{ij}(0) = 0$, all i,j will satisfy

(5.63) $$\sup_i E\big(\Sigma_j Z_{ji}(t)\big) + \sup_i E\big(\Sigma_j Z_{ij}(t)\big) < \infty.$$

Consequently $X_i(t) = H_i(Z(t))$ is well defined and the proof of the following theorem is straightforward.

THEOREM 5.64. *Let Z be a solution of the martingale problem (A_z, D_z). Then $X = H(Z)$ is a solution of the martingale problem (A_η, D_η).*

We now must show how to construct a solution of the martingale problem (A_z, D_z) given a solution of the martingale problem (A_η, D_η). We leave the proof of the following lemma to the reader.

LEMMA 5.65. *Let X be a solution of the martingale problem (A_η, D_η). Then for $f \in D_\eta$*

(5.66)
$$f(X(t) - X(s)) - \int_s^t \Sigma\beta_{ij}(X(u))\big(f(X^{ij}(u) - X(s)) - f(X(u) - X(s))\big)\,du$$

is a martingale for $t \geqslant s$.

Let φ_1 be a bounded function on $\mathbb{Z}$ such that $\varphi_1(0) = \varphi_1(-1) = 0$ and $\varphi_1(1) = 1$ and let $\varphi_{-1}(k) = \varphi_1(-k)$. Then

(5.67) $$E\big(\lim_{n\to\infty}\Sigma_{k\leqslant nt}\varphi_1(X_j((k+1)/n) - X_j(k/n))\big) = E\big(\textstyle\int_0^t \Sigma_i \beta_{ij}(X(s))\,ds\big)$$

(5.68) $$E\big(\lim_{n\to\infty}\Sigma_{k\leqslant nt}\varphi_{-1}(X_i((k+1)/n) - X_i(k/n))\big) = E\big(\textstyle\int_0^t \Sigma_j \beta_{ij}(X(s))\,ds\big),$$

and

$$E\big(\lim_{n\to\infty}\Sigma_{k<nt}\varphi_{-1}(X_i((k+1)/n) - X_i(k/n)) \\ \varphi_1((X_j((k+1)/n) - X_j(k/n))) \\ = E\big(\textstyle\int_0^t \beta_{ij}(X(s))\,ds\big). \tag{5.69}$$

Since the limits in (5.67) and (5.68) are independent of the values of $\varphi_1(k)$ and $\varphi_{-1}(k)$ except for $k = -1, 0, 1$, we can conclude that, with probability one, the jumps in $X_i(t)$ have magnitude one. Taking all three equations together, we see that at the instant of each decrease of $X_i(t)$ (of necessity by one) there is a corresponding increase of some $X_j(t)$. Consequently, we have the natural interpretation that a particle has moved from i to j, and we define

$$Z_{ij}(t) = \lim_{n\to\infty}\Sigma_{k<nt}\varphi_{-1}(X_i((k+1)/n) \\ - X_i(k/n))\varphi_1(X_j((k+1)/n) - X_j(k/n)). \tag{5.70}$$

THEOREM 5.71. *Let X be a solution of the martingale problem (A_η, D_η), and let Z be defined by* (5.70). *Then Z is a solution of the martingale problem (A_z, D_z).*

PROOF. Let

$$G_{ij}(\eta) = \varphi_{-1}(\eta_i)\varphi_1(\eta_j), \tag{5.72}$$

and define

$$Z^{(n)}(t) = G(X(t) - X([nt]/n)) + \Sigma_{k<nt}G(X(k/n) - X((k-1)/n)). \tag{5.73}$$

Then for $f \in D_z$,

$$f(Z^{(n)}(t)) - \textstyle\int_{[nt]/n}^t \Sigma\beta_{ij}(X(s))\big[f\big(G(X^{ij}(s) - X([ns]/n)) \\ + Z^{(n)}([ns]/n)\big) - f(Z^{(n)}(s))\big]\,ds \\ - \Sigma_{k<nt}\textstyle\int_{(k-1)/n}^{k/n}\Sigma\beta_{ij}(X(s))\big[f\big(G(X^{ij}(s) - X([ns]/n)) \\ + Z^{(n)}([ns]/n)\big) - f(Z^{(n)}(s))\big]\,ds \tag{5.74}$$

is a martingale and letting n go to infinity, we have $Z^{(n)}(t) \to Z(t)$ and hence $Z(t)$ is a solution of the martingale problem (A_z, D_z). □

6. Knight's theorem. We refer the reader to Meyer (1976) for the definitions of local martingales, predictable processes, etc. used in this section.

Let $X_1, X_2, \cdots, X_N$ be continuous local martingales with respect to the same filtration $\mathcal{G}_t$ with $X_i(0) = 0$ for all i. Let $\langle X_i X_j\rangle$ be the unique predictable process such that $\langle X_i X_j\rangle_0 = 0$ and

$$X_i X_j - \langle X_i X_j\rangle \tag{6.1}$$

is a local martingale, and define

$$\gamma_i(t) = \inf\{s : \langle X_i X_i\rangle_s \geqslant t\}. \tag{6.2}$$

Note that $\langle X_i X_i\rangle$ is necessarily nondecreasing and continuous and for simplicity we will assume $\lim_{i\to\infty}\langle X_i X_i\rangle_t = \infty$.

Knight (1970) (see also Meyer (1971)) shows that $\langle X_i X_j \rangle \equiv 0$ (that is X_i and X_j are orthogonal) for $i \neq j$ implies $X_i(\gamma_i(t))$ $i = 1, 2, \cdots, N$ are independent Brownian motions. This result is closely related to Theorem 3.11, Part (b), and in this section we adapt the proof of Theorem 3.11 to prove Knight's theorem as well as its converse (corresponding to Part (a)). As a corollary, we show that $\langle X_i X_j \rangle \equiv 0$ for $i \neq j$ implies $\Pi_{i \in I} X_i$ is a local martingale for all finite $I \subset \{1, 2, \cdots, N\}$.

THEOREM 6.3. (a) *Let $\mathcal{F}_u$ be an increasing family of σ-algebras indexed by $[0, \infty)^N$ and let $Y_j, j = 1, \cdots, N$ be Brownian motions (of necessity independent) for which*

$$\Phi_\theta^I(u) = \Pi_{j \in I} \exp\{ i\theta_j Y_j(u_j) + \tfrac{1}{2}\theta_j^2 u_j \} \tag{6.4}$$

is a $\mathcal{F}_u$ martingale for all finite I. Let $\tau(t), t \in [0, \infty)$ be $\mathcal{F}_u$ stopping times such that $\tau(t)$ is a nondecreasing continuous function of t for almost every ω. Then $\Pi_{j \in I} Y_j(\tau_j(t))$ is a continuous local martingale with respect to $\mathcal{G}_t \equiv \mathcal{F}_{\tau(t)}$ for each finite subset $I \subset \{1, 2, \cdots, N\}$.

(b) *Let $X_i, i = 1, 2, \cdots, N$, be continuous local martingales with respect to $\mathcal{G}_t$, such that $X_i(0) = 0$ for all i, $\langle X_i X_j \rangle \equiv 0$ for $i \neq j$, and $\lim_{t \to \infty} \langle X_i X_i \rangle_t = \infty$ for all i. Let $\gamma_i(t)$ be given by (6.2) and define*

$$\tau_i(t) = \langle X_i X_i \rangle_t, \tag{6.5}$$

$$Y_i(t) = X_i(\gamma_i(t)), \tag{6.6}$$

$$\mathcal{F}_u = \sigma(Y_i(s_i) : s_i \leq u_i, i = 1, 2, \cdots) \vee \cap_i \mathcal{G}_{\gamma_i(u_i)}. \tag{6.7}$$

Then (6.4) is a $\mathcal{F}_u$-martingale (which implies the Y_i are independent Brownian motions) and $\tau(t)$ is a $\mathcal{F}_u$ stopping time.

REMARK. Note that if $\Phi_\theta^I(u)$ is a martingale indexed by u for each θ then $Y_1 Y_2 \cdots$ are independent processes with independent Gaussian increments and hence are independent Brownian motions.

COROLLARY 6.8. *If $X_i, i = 1, 2, \cdots, N$, are pairwise orthogonal continuous local martingales* (i.e., *$X_i X_j$ is a local martingale for $i \neq j$*), *then $\Pi_{i \in I} X_i$ is a local martingale for all finite subsets $I \subset \{1, 2, \cdots, N\}$.*

REMARK. This corollary can also be obtained directly from Itô's formula (Kunita and Watanabe (1967), Theorem (2.2)).

PROOF. (a) Since $\Phi_\theta^I(u)$ is a $\mathcal{F}_u$-martingale for all $\theta \in \mathbb{R}^I$, if $I \subset \{1, 2, \cdots, N\}$, then

$$\Pi_{j \in I} Y_j(u_j) = \left(\Pi_{j \in I} \frac{\partial}{\partial \theta_j} \right) \Phi_\theta^I(u)|_{\theta = 0} \tag{6.9}$$

is an $\mathcal{F}_u$ martingale.

Let $\psi(r) = \inf\{t : \max_{j \in I} \tau_j(t) \geq r\}$. Then $\{\psi(r) \geq t\} = \{\max_{j \in I} \tau_j(t) \geq r\} \in \mathcal{G}_t \equiv \mathcal{F}_{\tau(t)}$ and hence $\{\psi(r) \leq t\} \cap \{\tau(t) \leq u\} \in \mathcal{F}_u$ for all r, t and u. We claim that $\tau(t \wedge \psi(r))$ is a $\mathcal{F}_u$-stopping time for all t and r. Note that $\{\tau(t \wedge \psi(r)) \leq u\} = \{\tau(t)$

$\leqslant u\} \cup \{\tau(\psi(r)) \leqslant u\}$ so it is enough to show $\{\tau(\psi(r)) \leqslant u\} \in \mathcal{F}_u$. Since $\psi(r-\varepsilon) < \psi(r)$ for $\varepsilon > 0$ we have $\{\tau(\psi(r)) \leqslant u\} = \cap_\varepsilon \cup_s \{\psi(r-\varepsilon) \leqslant s\} \cap \{\tau(s) \leqslant u\} \in \mathcal{F}_u$ where the intersection is over rational ε with $0 \leqslant \varepsilon \leqslant r$ and the union is over rational $s \geqslant 0$.

The optional sampling theorem Kurtz (1980) implies

$$\Pi_{j\in I} Y_j(\tau_j(t \wedge \psi(r))) \tag{6.10}$$

is a $\mathcal{F}_{\tau(t\wedge\psi(r))}$-martingale. Let $A \in \mathcal{F}_{\tau(t)}$. Since $\{\psi(r) > t\} \in \mathcal{F}_{\tau(t\wedge\psi(r))} \subset \mathcal{F}_{\tau(t)}$, we have

$$A \cap \{\psi(r) > t\} \cap \{\tau(t\wedge\psi(r)) \leqslant u\} = A \cap \{\psi(r) > t\} \cap \{\tau(t) \leqslant u\} \in \mathcal{F}_u.$$

Therefore $A \cap \{\psi(r) > t\} \in \mathcal{F}_{\tau(t\wedge\psi(r))}$. For $s > t$,

$$\begin{aligned}\int_A \Pi_{j\in I} Y_j(\tau_j(s\wedge\psi(r)))\,dP &= \int_{A\cap\{\psi(r)>t\}} \Pi_{j\in I} Y_j(\tau_j(s\wedge\psi(r)))\,dP \\ &\quad + \int_{A\cap\{\psi(r)<t\}} \Pi_{j\in I} Y_j(\tau_j(\psi(r)))\,dP \\ &= \int_{A\cap\{\psi(r)>t\}} \Pi_{j\in I} Y_j(\tau_j(t\wedge\psi(r)))\,dP \\ &\quad + \int_{A\cap\{\psi(r)<t\}} \Pi_{j\in I} Y_j(\tau_j(\psi(r)))\,dP \\ &= \int_A \Pi_{j\in I} Y_j(\tau_j(t\wedge\psi(r)))\,dP.\end{aligned} \tag{6.11}$$

Consequently (6.10) is a $\mathcal{G}_t$ martingale and hence $\Pi_{j\in I} Y_j(\tau_j(t))$ is a $\mathcal{G}_t$ local martingale.

(b) Let $u \in [0,\infty)^N$ and $0 = t_0^j < t_1^j < \cdots < t_m^j = u_j$. Define

$$\begin{aligned}\varphi_j(s) &= \varphi_{jk}, \quad \gamma_j(t_k^j) \leqslant s < \gamma_j(t_{k+1}^j), \qquad k < m; \\ &= \theta_j, \quad \gamma_j(t_m^j) \leqslant s.\end{aligned} \tag{6.12}$$

Then Itô's formula, Kunita and Watanabe (1967), Theorem (2.2), implies

$$\Pi_{j\in I} \exp\left\{i\int_0^t \varphi_j(s)\,dX_j(s) + \tfrac{1}{2}\int_0^t \varphi_j^2(s)\,d\langle X_j X_j\rangle\right\} \equiv \Pi_{j\in I} M_j(t) \tag{6.13}$$

is a local martingale for all finite $I \subset \{1,2,\cdots,N\}$. Therefore for $v \geqslant u$, Lemma 3.18 implies

$$E\left(\Pi_{j\in I} M_j(\gamma_j(v_j)) | \mathcal{G}_{\wedge_{j\in I}\gamma_j(u_j)}\right) = \Pi_{j\in I} M_j\left(\wedge_{j\in I}\gamma_j(u_j)\right). \tag{6.14}$$

From the definition of $\varphi_j(s)$ and γ_j, we see that (6.14) implies

$$\begin{aligned}&E\Big(\chi_A \Pi_{j\in I} \exp\left\{i\theta_j(Y_j(v_j) - Y_j(u_j)) + \tfrac{1}{2}\theta_j^2(v_j - u_j)\right\} \\ &\qquad \otimes \exp\left\{\textstyle\sum_{k=0}^{m-1} i\varphi_{jk}\left(Y_j(t_{k+1}^j) - Y_j(t_k^j)\right) + \tfrac{1}{2}\varphi_{jk}^2(t_{k+1}^j - t_k^j)\right\}\Big) \\ &= E\left(\chi_A \Pi_{j\in I} \exp\left\{\textstyle\sum_{k=0}^{m-1} i\varphi_{jk}\left(Y_j(t_{k+1}^j) - Y_j(t_k^j)\right) + \tfrac{1}{2}\varphi_{jk}^2(t_{k+1}^j - t_k^j)\right\}\right)\end{aligned} \tag{6.15}$$

for all $A \in \cap_j \mathcal{G}_{\gamma_j(u_j)}$. From (6.15) it follows that

$$\begin{aligned}&E\left(\left[\Phi_\theta(u)/\Phi_\theta(v)\right]\chi_A \Pi_{j\in I} f_{jk}\left(Y_j(t_{k+1}^j) - Y_j(t_k^j)\right)\right) \\ &\qquad = E\left(\chi_A \Pi_{j\in I} f_{jk}\left(Y_j(t_{k+1}^j) - Y_j(t_k^j)\right)\right)\end{aligned} \tag{6.16}$$

for all $A \in \cap_j \mathcal{G}_{\gamma_j(u_j)}$ and all bounded measurable functions f_{jk}. (Note, for example, that functions of the form $f(y) = \int e^{i\varphi y} h(\varphi)\, d\varphi$, h continuous with compact support, are dense in L_1.) From (6.16) and the definition of $\mathcal{F}_u$ we conclude

$$E(\Phi_\theta(u)/\Phi_\theta(v) | \mathcal{F}_v) = 1 \tag{6.17}$$

and hence Φ_θ is a $\mathcal{F}_u$ martingale.

To see that $\tau(t)$ is a $\mathcal{F}_u$ stopping time, note that

$$\{\tau(t) \leqslant u\} = \cap_j \{\gamma_j(u_j) \geqslant t\} = \{\wedge_j \gamma_j(u_j) \geqslant t\} \in \mathcal{F}_u. \tag{6.18}$$ □

Meyer (1971) proves an analog of Knight's theorem for Poisson processes (see also Aalen and Hoem (1978)). The corresponding analog of Theorem 6.3 is as follows:

THEOREM 6.19. (a) *Let* $\mathcal{F}_u$ *be an increasing family of* σ*-algebras indexed by* $[0, \infty)^N$, *and let* $Y_j, j = 1, 2, \cdots, N$, *be Poisson processes (of necessity independent) for which*

$$\Phi_\theta^I(u) = \Pi_{j \in I} \exp\{i\theta_j Y_j(u_j) - u_j(e^{i\theta_j} - 1)\} \tag{6.20}$$

is a $\mathcal{F}_u$*-martingale for all finite* I. *Let* $\tau(t), t \in [0, \infty)$, *be* $\mathcal{F}_u$ *stopping times such that* $\tau(t)$ *is a nondecreasing continuous function of* t *for almost every* ω. *Then* $\Pi_{j \in I}(Y_i(\tau_j(t)) - \tau_j(t))$ *is a right continuous local martingale, with respect to* $\mathcal{G}_t \equiv \mathcal{F}_{\tau(t)}$, *for each finite subset* $I \subset \{1, 2, \cdots, N\}$.

(b) *Let* $X_i, i = 1, 2, \cdots$ *be counting processes (right continuous processes that are constant except for jumps of* $+1$) *such that no two* X_i *have jumps in common. Let* Λ_i *be continuous increasing processes, and suppose that the* $X_i - \Lambda_i$ *are all local martingales with respect to the same filtration* $\mathcal{G}_t$. *Suppose* $X_i(0) = \Lambda_i(0) = 0$ *and* $\lim_{t \to \infty} \Lambda_i(t) = \infty$ *for all* i. *Define*

$$\tau_i(t) = \Lambda_i(t), \tag{6.21}$$

$$\gamma_i(t) = \inf\{s : \Lambda_i(s) \geqslant t\}, \tag{6.22}$$

$$Y_i(t) = X_i(\gamma_i(t)), \tag{6.23}$$

$$\mathcal{F}_u = \sigma(Y_i(s_i) : S_i \leqslant u_i, i = 1, 2, \cdots) \vee \cap_i \mathcal{G}_{\gamma_i(u_i)}. \tag{6.24}$$

Then (6.20) *is a* $\mathcal{F}_u$*-martingale (which implies the* Y_i *are independent Poisson processes), and* $\tau(t)$ *is a* $\mathcal{F}_u$ *stopping time.*

PROOF. The proof of part (a) is the same as in Theorem 6.3. For part (b), the change of variable formula (Meyer (1976), page 285) implies

$$\Pi_{j \in I} \exp\{\textstyle\int_0^t i\varphi_j(s)\, dX_i(s) - \int_0^t (e^{i\varphi_j(s)} - 1)\, d\Lambda_i(s)\} \tag{6.25}$$

is a $\mathcal{G}_t$ martingale for φ_j defined as in the proof of Theorem 6.3. Part (b) then follows as in the proof of Theorem 6.3. □

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REFERENCES

[1] AALEN, ODD O. and HOEM, JAN M. (1978). Random time changes for multivariate counting processes. *Scand. Acturar. J.* 81–101.

[2] BILLINGSLEY, PATRICK (1968). *Convergence of Probability Measures*. Wiley, New York.

[3] HELMS, LESTER L. (1974). Ergodic properties of several interacting Poisson particles. *Advances in Math.* **12** 32–57.

[4] HOLLEY, RICHARD and STROOCK, DANIEL W. (1976). A martingale approach to infinite systems of interacting processes. *Ann. Probability* **4** 195–228.

[5] KNIGHT, FRANK B. (1970). An infinitesimal decomposition for a class of Markov processes. *Ann. Math. Statist.* **41** 1510–1529.

[6] KUNITA, H. and WATANABE, S. (1967). On square integrable martingales. *Nagoya Math. J.* **30** 209–245.

[7] KURTZ, THOMAS G. (1975). Semigroups of conditioned shifts and approximation of Markov processes. *Ann. Probability* **3** 618–642.

[8] KURTZ, THOMAS G. (1978a). Strong approximation theorems for density dependent Markov chains. *Stochastic Processes Appl.* **6** 223–240.

[9] KURTZ, THOMAS G. (1978b). Diffusion approximations for branching processes. In *Branching Processes* (Anatole Joffe and Peter Ney, eds.) 269–292 Advances in Probability **5**.

[10] KURTZ, THOMAS G. (1980). The optional sampling theorem for martingales indexed by directed sets. *Ann. Probability* **8** 675–681.

[11] LAMPERTI, JOHN (1967). On random time substitutions and the Feller property. In *Markov Processes and Potential Theory*. (Joshua Chover, ed.) 87–101. Wiley, New York, London.

[12] LIGGETT, THOMAS M. (1972). Existence theorems for infinite particle systems. *Trans. Amer. Math. Soc.* **165** 471–481.

[13] MEYER, P. A. (1971). Demonstration simplifiée d'un théorème de Knight. In *Séminaire de Probabilités* V, Univ. Strasbourg, Lecture Notes in Math. **191** 191–195.

[14] MEYER, P. A. (1976). Un cours sur les intégrales stochastiques. In *Séminaire de Probabilités* **X**, Univ. Strasbourg, Lecture Notes in Math., **511** 245–400.

[15] MORANDO, PHILIPPE (1969). Representation des mesures aléatoires positives à accroissements indépendents. In *Séminaire de Probabilitiés* **III**, Univ. Strasbourg, Lecture Notes in Math., **88** 217–229.

[16] MOTZKIN, T. S. and WASOW, W. (1953). On the approximation of linear elliptic differential equations by difference equations with positive coefficients. *J. Math. Phys.* **31** 253–259.

[17] PENROSE, R. (1955). A generalized inverse for matrices. *Proc. Cambridge Philos. Soc.* **51** 406–413.

[18] STROOCK, DANIEL W. and VARADHAN, S. R. S. (1969a). Diffusion processes with continuous coefficients, I. *Comm. Pure Appl. Math.* **22** 345–400.

[19] STROOCK, DANIEL W. and VARADHAN, S. R. S. (1969b). Diffusion processes with continuous coefficients, II. *Comm. Pure Appl. Math.* **22** 479–530.

[20] VOLKONSKI, V. A. (1958). Random substitution of time in strong Markov processes. *Theor. Probability Appl.* **3** 310–326.

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