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ISOTROPIC GAUSSIAN PROCESSES ON THE HILBERT SPHERE[1]

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The subject of this work is a study of four properties of an isotropic Gaussian process on an infinite dimensional sphere in Hilbert space. The process is deterministic in the sense that its values on an arbitrary nonempty open subset of the sphere determine its values throughout the sphere. An harmonic property is defined, and is characterized in terms of the covariance function of the process. If f is a function of a real variable which is square-integrable with respect to the Gaussian density, then the average of f over the values of the process on an n-dimensional subsphere converges with probability 1 for $n \to \infty$. Under further conditions on the process the average of f has, with appropriate normalization, a limiting Gaussian distribution for $n \to \infty$.

1. Introduction and summary. Let S be the unit sphere in an infinite-dimensional real separable Hilbert space, and let $X(\mathbf{t})$, $\mathbf{t} \in S$, be a Gaussian process with mean 0. X is said to be isotropic if its covariance function $EX(\mathbf{s})X(\mathbf{t})$, $\mathbf{s}, \mathbf{t} \in S$, is invariant under orthogonal transformations of S, or equivalently, is a function of the inner product $(\mathbf{s}, \mathbf{t})$. The starting point of this study is Schoenberg's theorem characterizing covariance functions on S (Schoenberg (1942)). Using this theorem we obtain a decomposition of X into a sum of independent isotropic processes with covariances proportional to $(\mathbf{s}, \mathbf{t})^m$, $m \geq 0$. Using the decomposition we prove that X is deterministic in that it is determined for all S by its values on an arbitrary neighborhood of any point in S. The first result of such a type was obtained by Lévy for the Brownian motion over Hilbert space (see Lévy (1966)). Similar results for stable Gaussian processes over l_p have been obtained by Bretagnolle, Dacunha-Castelle, and Krivine (1966), Bretagnolle and Dacunha-Castelle (1969), and Berman (1969).

We introduce the concept of harmonicity. The process is said to be harmonic of order m if the value of the process at any point $\mathbf{t}$ of the sphere may, for any set of m distinct concentric subspheres with center at $\mathbf{t}$, be expressed as a linear combination of certain averages of the process over the latter subspheres. Then we study the relation between the order of harmonicity and the form of the covariance function.

Let $f(x)$ be a real valued measurable function. The last three sections of the paper are about the limiting behavior of the average of $f(X(\mathbf{t}))$ over finite-dimensional subspheres of S of dimension n, for $n \to \infty$. We prove an ergodic theorem, namely, that the average converges with probability 1. This is much more general than results that have been known up to now, which have been limited to the function $f(x) = x$. (See McKean (1963), Berman (1969), and Jadrenko (1972).) Finally we show that the average of $f(X(\mathbf{t}))$ has, under general conditions, a limiting Gaussian distribution for $n \to \infty$. As far as we can determine, there is no precedent for the latter result.

The infinite dimensionality of the parameter space is used throughout the paper, and, as far as this writer can determine, the results have no related analogues in the case of finite dimensionality. As noted above, the entire investigation rests on the form of the covariance

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function obtained by Schoenberg. One of the striking facts implied by his results is that the isotropic covariance over infinite dimensional space is of a very simple form, generally simpler than that over a finite dimensional space. The reason for this is that isotropy over a larger space implies more restrictions than isotropy over a smaller space. The results that we obtain on determinism and its relation to the harmonic property follow from the special form of the covariance and are true only in the infinite dimensional case.

The proofs of the ergodic theorem and the central limit theorem are based on the expansion of the function f in Hermite polynomials. This approach was suggested to the author by Henry P. McKean. It has also been used recently by Taqqu in a series of papers on a related class of limit theorems for stationary Gaussian processes on a one-dimensional parameter set (1975), (1977), (1979). Some of the calculations in our proofs are similar to special cases of those considered by Taqqu (1977). His processes and ours share the property of long term dependence.

2. Orthogonal decomposition of the process. Let S be the sphere of radius 1 in a separable Hilbert space H. If $\mathbf{s}$ and $\mathbf{t}$ are two members of H, then their inner product is denoted $(\mathbf{s}, \mathbf{t})$. An orthogonal sequence in S will be represented as $\{\mathbf{e}_n, n \geq 1\}$.

Let $X(\mathbf{t})$, $\mathbf{t} \in S$, be a real Gaussian stochastic process such that $EX(\mathbf{t}) = 0$ for all $\mathbf{t}$. X is said to be isotropic if for any orthogonal transformation U on S,

$$EX(U\mathbf{s})X(U\mathbf{t}) = EX(\mathbf{s})X(\mathbf{t})$$

for all $\mathbf{s}$ and $\mathbf{t}$. In particular $EX^2(\mathbf{t})$ is constant in $\mathbf{t}$ so that we take it to be equal to 1. It follows that $EX(\mathbf{s})X(\mathbf{t})$ is a function of $(\mathbf{s}, \mathbf{t})$. According to the classical theorem of Schoenberg (1942) this function has the representation

$$P((\mathbf{s}, \mathbf{t})) = \textstyle\sum_{m=0}^{\infty} c_m(\mathbf{s}, \mathbf{t})^m \tag{2.1}$$

where

$$c_m \geq 0 \quad \text{and} \quad \textstyle\sum_{m=0}^{\infty} c_m = 1. \tag{2.2}$$

Conversely, every series (2.1) which satisfies (2.2) is the covariance of an isotropic process on S. In particular, $(\mathbf{s}, \mathbf{t})^m$ is such a covariance.

The representation (2.1) of the covariance implies an orthogonal decomposition of the process itself. For each $m \geq 0$, let $\xi_m(\mathbf{t})$ be a Gaussian process with mean 0 and covariance

$$E\xi_m(\mathbf{s})\xi_m(\mathbf{t}) = (\mathbf{s}, \mathbf{t})^m, \tag{2.3}$$

and take these processes to be mutually independent. If we define the process X as

$$X(\mathbf{t}) = \textstyle\sum_{m\geq 0} (c_m)^{1/2}\xi_m(\mathbf{t}), \tag{2.4}$$

then it follows from (2.3) that the latter process is Gaussian with the covariance function (2.1), so that it is equivalent in distribution to the original process X. We will use the version (2.4) throughout the paper.

Note that for $m = 0$ the right-hand side of (2.3) is equal to 1, so that $\xi_o = \xi_o(\mathbf{t})$ is almost surely the same for all $\mathbf{t}$

3. Averaging and filtering. An operation which we use several times in this work is that of ergodic averaging. If $\{\mathbf{e}_n, n \geq 1\}$ is an orthogonal sequence in S, then, by isotropy, the Gaussian sequence $\{X(\mathbf{e}_n), n \geq 1\}$ has the constant covariance sequence equal to $P(0)$; therefore, it is stationary, and so

$$\bar{X} = \lim_{n\to\infty} \frac{1}{n} \textstyle\sum_{j=1}^{n} X(\mathbf{e}_j)$$

exists with probability 1. By the same reasoning it follows that if $\mathbf{f}$ is a fixed point of S such that

$$(\mathbf{f}, \mathbf{e}_n) = 0, \qquad n \geq 1,$$

then, for any real t, $|t| \leq 1$, the Gaussian sequence $\{X(t\mathbf{f} + (1 - t^2)^{1/2}\mathbf{e}_n), n \geq 1\}$ has constant covariance $P(t^2)$, and the average has a limit with probability 1:

$$\bar{X}(t\mathbf{f}) = \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} X(t\mathbf{f} + (1 - t^2)^{1/2}\mathbf{e}_j). \tag{3.1}$$

Let us apply this averaging operation to the particular process ξ_m with covariance (2.3). By direct calculation we find, for real s and t,

$$E\bar{\xi}_m(s\mathbf{f})\bar{\xi}_m(t\mathbf{f}) = \lim_{n\to\infty} n^{-2} \sum_{i,j=1}^{n} (s\mathbf{f} + (1 - s^2)^{1/2}\mathbf{e}_i,\ t\mathbf{f} + (1 - t^2)^{1/2}\mathbf{e}_j)^m = (st)^m.$$

This implies

$$E[\bar{\xi}_m(t\mathbf{f}) - t^m\bar{\xi}_m(\mathbf{f})]^2 = 0,$$

so that, with probability 1,

$$\bar{\xi}_m(t\mathbf{f}) = t^m\bar{\xi}_m(\mathbf{f}).$$

Furthermore, by the definition of $\bar{\xi}_m$, we have $\bar{\xi}_m(\mathbf{f}) = \xi_m(\mathbf{f})$. It follows that

$$\bar{\xi}_m(t\mathbf{f}) = t^m\xi_m(\mathbf{f}) \tag{3.2}$$

with probability 1 for each t and $\mathbf{f}$.

Let us now apply the ergodic averaging operation to both sides of equation (2.4):

$$\bar{X}(t\mathbf{f}) = \sum_{m\geq 0} (c_m)^{1/2}\bar{\xi}_m(t\mathbf{f}).$$

Thus, from (3.2), we obtain

$$\bar{X}(t\mathbf{f}) = \sum_{m\geq 0} (c_m)^{1/2}t^m\xi_m(\mathbf{f}). \tag{3.3}$$

Comparing this with the representation (2.4), we see that the terms in the expansions of $X(\mathbf{f})$ and $\bar{X}(t\mathbf{f})$ differ only by powers of t. For $t = 0$, (3.3) and the equivalent form (3.1) reduce to $(c_0)^{1/2}\xi_0$.

LEMMA 3.1. *If $\{\mathbf{e}_n\}$ is an infinite orthogonal sequence, and $(\mathbf{f}, \mathbf{e}_n) = 0$ for $n \geq 1$, then for m such that $c_m > 0$, the random variable $\xi_m(\mathbf{f})$ is in the Hilbert space spanned by the random variables*

$$X(t\mathbf{f} + (1 - t^2)^{1/2}\mathbf{e}_n), \qquad n \geq 1, \qquad t \in I,$$

for every nonempty open subinterval I contained in $[-1, 1]$.

PROOF. It follows from the definition (3.1) that $\bar{X}(t\mathbf{f})$, $t \in I$, is certainly in the indicated Hilbert space of random variables. According to (3.3) $(c_m)^{1/2}\xi_m(\mathbf{f})$ is the general coefficient in the power series expansion of $\bar{X}(t\mathbf{f})$, $|t| \leq 1$, and so is uniquely determined by the values of the series in an arbitrary nonempty open interval I.

The component process ξ_m can be filtered from the process X by taking successive derivatives of $\bar{X}(t\mathbf{f})$ with respect to t. For example, it follows from (3.3) that

$$\xi_m(\mathbf{f}) = \frac{\left(\frac{d}{dt}\right)^m \bar{X}(t\mathbf{f})|_{t=0}}{(c_m)^{1/2}\cdot m!}, \qquad m \geq 0.$$

4. Determinism of X. Now we prove a result on the deterministic character of X.

THEOREM 4.1. *Suppose for some $\mathbf{f}$, $X(\mathbf{t})$ is given for all $\mathbf{t}$ in some open neighborhood of $\mathbf{f}$ in S; then X is determined throughout S.*

PROOF. It suffices to take $\mathbf{f} = \mathbf{e}_1$ (by isotropy), and to consider a neighborhood of $\mathbf{e}_1$ of the form $\{\mathbf{t} : (\mathbf{t}, \mathbf{e}_1) > d\}$ for some $d < 1$. This neighborhood contains every point of the form $s\mathbf{e}_1$

$+ (1 - s^2)^{1/2}\mathbf{e}_n$ for every $n \geq 2$ and every s, $d < s \leq 1$. Under the hypothesis of the theorem, the random variables $X(s\mathbf{e}_1 + (1 - s^2)^{1/2}\mathbf{e}_n)$ are given for all $d < s \leq 1$, and all $n \geq 2$. Therefore, by Lemma 3.1, $\xi_m(\mathbf{e}_1)$ is determined for all m such that $c_m > 0$.

Since every open neighborhood of $\mathbf{e}_1$ contains some neighborhood of each of its points, the argument above also implies that $\xi_m(\mathbf{t})$ is determined for all $\mathbf{t}$ in some open neighborhood of $\mathbf{e}_1$ for every m such that $c_m > 0$. Therefore, it suffices to prove the theorem for the particular case of the process $\xi_m(\mathbf{t})$.

We will show that if $\xi_m(\mathbf{t})$ is determined for all $\mathbf{t}$ in a neighborhood of $\mathbf{e}_1$, then it is determined at an arbitrary point $\mathbf{g}$ in S. In proving this it suffices to consider only the case where $\mathbf{g}$ is in the subspace spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$. Indeed, if ξ_m is determined at some point $\mathbf{g}$ by its values around $\mathbf{e}_1$, then, by isotropy, it is also determined at every point which can be obtained from $\mathbf{g}$ by a rotation of S which leaves $\mathbf{e}_1$ invariant; therefore, since every point in S can be obtained from the subspace spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$ by such a rotation, it suffices to consider only points $\mathbf{g}$ in the subspace.

Define the Gaussian process $Z(t)$ with the real parameter t as

$$Z(t) = \xi_m(\mathbf{e}_1\cos t + \mathbf{e}_2\sin t), \qquad 0 \leq t \leq 2\pi.$$

The process has mean 0 and covariance function $EZ(s)Z(t) = \cos^m(t - s)$, which is analytic, and so $Z(t)$ has analytic sample functions. Since ξ_m is determined in a neighborhood of $\mathbf{e}_1$, $Z(t)$ is determined in some interval $0 \leq t \leq h$, for $h > 0$. Therefore, by analyticity, $Z(t)$ is determined for all t, $0 \leq t \leq 2\pi$, which implies that ξ_m is determined on the intersection of S with the subspace spanned by $\mathbf{e}_1$ and $\mathbf{e}_2$.

5. The harmonic property. In this section we introduce the concept of harmonicity of a Gaussian process with an infinite dimensional time parameter, which is implicit in earlier results of the author (1969). (There is also a reference to the early work of Lévy which, unfortunately, contained a basic error.) There it was shown that a certain class of processes had the property that the value of the process at a point of the parameter space was equal to some average of the process over an infinite dimensional sphere centered at that point. We now extend this concept to the Gaussian process on the sphere and to harmonicity of order m, for $m \geq 1$, and then show how it is related to the form of the sequence $\{c_n\}$ which characterizes the covariance (2.1).

DEFINITION 5.1. X is harmonic of order m if and only if for each set of real numbers $u_1, \cdots, u_m$ such that $-1 < u_1 < \cdots < u_m < 1$, there exists a set of real numbers $d_1, \cdots, d_m$ such that with probability 1

$$X(\mathbf{e}_1) = \textstyle\sum_{j=1}^{m} d_j \bar{X}(u_j\mathbf{e}_1), \tag{5.1}$$

where $\bar{X}$ is defined by (3.1).

If X is harmonic of order m, then, by isotropy, $\mathbf{e}_1$ may be replaced in (5.1) by any point of S; $\bar{X}(u_j\mathbf{e}_1)$ is then replaced by an ergodic average over the subsphere in the corresponding orthogonal subspace.

THEOREM 5.1. *If, in the representation* (2.1), $c_n = 0$ *for all* $n \geq m$, *then* X *is harmonic of order* m.

PROOF. Consider the Gaussian process $Y(u) = \bar{X}(u\mathbf{e}_1)$, $|u| \leq 1$. If $c_n = 0$ for all $n \geq m$, then, by the representation (3.3),

$$Y(u) = \textstyle\sum_{j=0}^{m-1} (c_j)^{1/2}\xi_j(\mathbf{e}_1)u^j, \qquad |u| \leq 1.$$

This is a polynomial in u of degree $m - 1$; therefore, its value at any point u is determined by its values at any other m distinct points. In particular, for any set $u_1, \cdots, u_m$ such that $-1 < u_1 < \cdots < u_m < 1$, we can solve the system

$$Y(u_i) = \textstyle\sum_{j=0}^{m-1} (c_j)^{1/2}\xi_j(\mathbf{e}_1)u_i^j, \qquad 1 \leq i \leq m,$$

for $\{\xi_j(\mathbf{e}_1)\}$, write the latter as linear combinations of $\{Y(u_i)\}$ and substitute in the equation

$$Y(1) = \sum_{j=0}^{m-1} (c_j)^{1/2}\xi_j(\mathbf{e}_1)$$

to obtain

$$Y(1) = \sum_{j=1}^{m} d_j Y(u_j),$$

or, equivalently

$$\bar{X}(e_1) = \sum_{j=1}^{m} d_j \bar{X}(u_j \mathbf{e}_1),$$

for some $d_1, \cdots, d_m$. Noting that $\bar{X}(\mathbf{e}_1) = X(\mathbf{e}_1)$, we observe that the relation above is identical with (5.1), and the proof is complete.

THEOREM 5.2. *If X is harmonic of order m, then $c_n = 0$ for all but m indices $n \geq 0$.*

PROOF. By the representation (3.3) and Definition 5.1, for every set of u's in the latter statement, there exist d's such that $\sum_{n\geq 0} (c_n)^{1/2}[1 - \sum_{j=1}^{m} d_j u_j^n]\xi_n(\mathbf{e}_1) = 0$ with probability 1. Since the ξ's are independent and each has positive variance, the latter equation implies

$$\sum_{n\geq 0} c_n[1 - \sum_{j=1}^{m} d_j u_j^n]^2 = 0$$

which implies

$$\sum_{j=1}^{m} d_j u_j^n - 1 = 0 \qquad (5.2)$$

for all n in the set

$$N = \{n : c_n > 0\}.$$

Let $n_1 < n_2 < \cdots$ be the successive nonnegative integers belonging to the set N. We will show that no member of N is larger than n_m, and this will complete the proof.

Let us assume the contrary, namely, that N has a member $n_{m+1} > n_m$; we shall deduce a contradiction. Consider the homogeneous system of $m + 1$ equations

$$\sum_{j=0}^{m} d_j u_j^n = 0, \qquad n = n_1, \cdots, n_{m+1}, \qquad (5.3)$$

where $u_0 = 1$; here the coefficient matrix is

$$U = (u_j^{n_{i+1}})_{i,j=0,1,\cdots,m}. \qquad (5.4)$$

We assert: *for arbitrary $n_1 < \cdots < n_{m+1}$, there exists a set of m distinct real numbers u_1, $\cdots$, u_m such that $|u_j| < 1$ and such that* $\det U \neq 0$ *for $u_0 = 1$.*

Note that $\det U = 0$ if $u_i = u_j$ for some pair i and j.

The proof is by induction on m. The result is true for $m = 1$: If

$$U = \begin{pmatrix} 1 & u_1^{n_1} \\ 1 & u_1^{n_2} \end{pmatrix},$$

then $\det U = u_1^{n_2} - u_1^{n_1}$, which does not vanish for all u_1 because $n_1 < n_2$. Suppose now that the assertion is true for an integer $m - 1$, where $m \geq 2$. The determinant of the matrix U in (5.4), which corresponds to the case of the integer m, may be expanded in minors of the last column. The determinant is then of the form of a polynomial in u_m of degree n_{m+1} and where the coefficients are determinants of matrices of the form (5.4) in the $m - 1$ variables $u_1, \cdots, u_{m-1}$. If the determinant of m variables were identically equal to 0, then, as a polynomial in u_m, the coefficients would have to be identically 0, and this cannot occur under the induction hypothesis.

It follows from the assertion above that there exists a set of distinct u's with $u_o = 1$ such that the homogeneous system (5.3) has only the trivial solution $d_j \equiv 0$. But this contradicts the result above that for each set of u's there is a set $d_1, \cdots, d_m$ such that (5.2) holds; indeed, the latter implies the existence of a nontrivial solution of (5.3) with $d_0 = -1$.

COROLLARY 5.1. *If X is harmonic of order 1, then $c_n = 0$ for all $n \geq 1$, and so $X(t)$ has the same value at every $\mathbf{t} \in S$.*

PROOF. By Theorem 5.2, at most one coefficient c_m is different from 0, and so, by (2.4), $X(\mathbf{t}) = \xi_m(\mathbf{t})$. On the one hand, by Definition 5.1, $\xi_m(\mathbf{e}_1)$ may be expressed as a multiple of

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=2}^{n} \xi_m(\mathbf{e}_j).$$

On the other hand, if $m \geq 1$, then $\xi_m(\mathbf{e}_1)$ is independent of the ergodic average because $E\xi_m(\mathbf{e}_1)\xi_m(\mathbf{e}_j) = (\mathbf{e}_1, \mathbf{e}_j)^m = 0$ if $j > 1$ and so $c_m > 0$ only for $m = 0$.

It is interesting to think of the latter result as a stochastic analogue of the classical result that the only bounded functions which are everywhere harmonic are constants.

Theorem 5.1 implies the following harmonic property of any isotropic Gaussian process X on S: for any sequence of distinct subspheres S_j of S, centered at $\mathbf{e}_1$,

$$S_j = \{\mathbf{t} : \mathbf{t} \in S, (\mathbf{t}, \mathbf{e}_1) = a_j\}, \qquad j \geq 1,$$

where $\{a_j\}$ is a sequence of distinct real numbers in $(-1, 1)$, there exists an array of real numbers $\{b_{n,j};\ 1 \leq j \leq n,\ n \geq 1\}$ such that

$$X(e_1) = \lim_{n\to\infty} \sum_{j=1}^{n} b_{n,j} \bar{X}(a_j \mathbf{e}_1)$$

with probability 1. Indeed $X(\mathbf{e}_1)$ may be approximated by the first m terms of its expansion (2.4). (The convergence with probability 1 follows from the independence of the terms.) The finite sum formed from the first m terms of the expansion is, by Theorem 5.1, harmonic of order m; therefore, it is representable as a linear combination of $\bar{X}(a_j \mathbf{e}_1)$ for any m distinct a_j. If we now extend our definition of harmonicity of order m to $m = \infty$, it follows that

COROLLARY 5.2. *If $c_n > 0$ for infinitely many n, then X is harmonic of order $m = \infty$.*

A special class of processes X over l_p was shown by Berman (1969) to have a spherical averaging property with respect to a single sphere. The current result applies to any isotropic process on a sphere in l_2 but the averaging is over an infinite sequence of spheres.

We conclude this section with some comments on related work. Jadrenko introduced a definition of Markovity for isotropic Gaussian processes on the Hilbert sphere (1972). He defined the process to be Markovian if the values of the process at two points $\mathbf{s}$ and $\mathbf{t}$ of S are conditionally independent given the values on some subsphere of S such that $\mathbf{s}$ and $\mathbf{t}$ lie on opposite sides of the subsphere. He showed that such a property implies $c_m = 0$ for all but one index $m \geq 0$, and derived certain properties of the process stated in terms of the index m. We will show that the only possible case is $m = 0$, for which X is the trivial process $\xi_0(t)$ having the same value at every point. For $m \geq 1$, (2.3) implies that (i) $\xi_m(\mathbf{e})$ and $\xi_m(\mathbf{f})$ are independent if $(\mathbf{e}, \mathbf{f}) = 0$; (ii) $\xi_m(\mathbf{e}) = \xi_m(-\mathbf{e})$ if m is even; (iii) $\xi_m(\mathbf{e}) = -\xi_m(-\mathbf{e})$ if m is odd. Since the Markov property stated above implies that $\xi_m(\mathbf{e})$ and $\xi_m(-\mathbf{e})$ are conditionally independent given $\{\xi_m(\mathbf{f}): (\mathbf{e}, \mathbf{f}) = 0\}$, the Markov property contradicts (i), (ii) and (iii) unless $m = 0$.

6. Spherical averages of nonlinear functions of X. The results up to this point can be formulated not only for Gaussian processes, but also for general processes with mean 0 and covariance function of the form (2.1). This can be done by replacing independence by orthogonality, and almost sure convergence by mean square convergence. In the remainder of this work we shall derive several limit theorems which depend strictly on the Gaussian character of the process.

We use the customary symbol ϕ for the standard Gaussian density:

$$\phi(x) = (2\pi)^{-1/2} e^{-x^2/2};$$

$\phi(x, y; r)$ for standard bivariate Gaussian density with correlation coefficient r; and $\phi(x_1, \cdots, x_m; (r_{ij}))$ for the m-variate Gaussian density with 0 mean vector, unit standard deviations, and covariances r_{ij} for $i \neq j$.

We now record some results about the multivariate Gaussian distribution and the Hermite polynomials. Let $H_k(x)$ be the Hermite polynomial of degree k defined by

$$\phi^{(k)}(x) = (-1)^k H_k(x)\phi(x). \tag{6.1}$$

In particular, we have

$$H_0(x) = 1, \qquad H_1(x) = x, \qquad H_2(x) = x^2 - 1. \tag{6.2}$$

The sequence $\{H_n(x)\}$ satisfies

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_m(x)H_n(y)\phi(x, y; r)\, dx\, dy = n!r^n\delta_{m,n}, \tag{6.3}$$

where δ is the Kronecker delta; this is an immediate consequence of (6.1) and the well-known expansion (see Cramér (1946), page 290)

$$\phi(x, y; r) = \textstyle\sum_{n\geq 0} \frac{1}{n!}\, \phi^{(n)}(x)\phi^{(n)}(y)r^n, \qquad |r| < 1. \tag{6.4}$$

For our later use we need this result about $H_2(x)$ and the 4-variate Gaussian density.

LEMMA 6.1. *The multiple integral*

$$\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} \textstyle\prod_{i=1}^{4} H_2(x_i)\phi(x_1, x_2, x_3, x_4: (r_{ij}))\, dx_1\, dx_2\, dx_3\, dx_4$$

is a linear combination of the terms

$$r_{12}r_{34}r_{13}r_{24}, \quad r_{12}r_{34}r_{14}r_{23}, \quad r_{13}r_{24}r_{14}r_{23}, \quad r_{12}^2r_{34}^2, \quad r_{13}^2r_{24}^2, \quad r_{14}^2r_{23}^2. \tag{6.5}$$

PROOF. See Taqqu (1977), Lemma 3.2.

Let $\{\mathbf{e}_n : n \geq 1\}$ be an infinite orthogonal sequence in S, and let S_n be the intersection of S with the subspace spanned by $\mathbf{e}_1, \cdots, \mathbf{e}_n$. X is stochastically continuous on S_n, so that it has a measurable version on S_n. If $f(x)$ is a real valued measurable function, then there also exists a measurable version of the composite process $f(X(\mathbf{t}))$ on S_n. If

$$\int_{-\infty}^{\infty} |f(x)|\phi(x)\, dx < \infty, \tag{6.6}$$

then by Fubini's theorem, $f(X(\mathbf{t}))$ is integrable over S_n, and so its average over S_n is defined as

$$\int_{S_n} f(X(\mathbf{t}))\, d\mathbf{t}/A(S_n), \tag{6.7}$$

where the integral is taken with respect to Lebesgue measure on S_n, and where $A(S_n)$ is the surface area of S_n.

It is convenient for the purpose of calculating the moments of the spherical average above to refer to the uniform distribution over S_n; indeed, the average of any function over S_n is the expectation with respect to this distribution.

LEMMA 6.2. *Let $T_1, T_2, \cdots$ be independent and uniformly distributed random points on S_n, for some $n \geq 2$. Then the random variables $\{(T_i, T_j) : 1 \leq i < j < \infty\}$ have the common density* *function*

$$\frac{n-1}{n} \frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n+1}{2}\right)} \pi^{-1/2}(1-x^2)^{(n-3)/2}, \qquad |x|<1. \tag{6.8}$$

PROOF. If T' is a fixed point of S_n, then, by a standard calculation involving n-dimensional polar coordinates, it can be shown that (T_1, T') has the density (6.8), which is evidently independent of T'. If T_1 and T_2 are independent, then the conditional density of (T_1, T_2) given T_2 is also equal to (6.8), and is identical to its unconditional density.

In the calculations below we will refer to expectations of two kinds. The first is taken with respect to the probability measure on the space of the process X, and the expectation operator is denoted by E. The second is taken with respect to the probability measure on the space of a sequence of independent and uniformly distributed random points $\{T_j\}$ in S_n; and the expectation operator is denoted $\mathscr{E}_n$. If Q is a function defined on $S_n \times \cdots \times S_n$ (k factors), then the expected value of $Q(T_1, \cdots, T_k)$ is denoted $\mathscr{E}_n Q(T_1, \cdots, T_k)$, for any $k \geq 1$. In particular $\mathscr{E}_n h(T_1)$ represents the spherical average of a function $h(\mathbf{t})$ over S_n. If we drop the subscript and let T be a random point with the same (uniform) distribution as T_1, then the average (6.7) is denoted by

$$\mathscr{E}_n f(X(T)). \tag{6.9}$$

$\{X(t), t \in S\}$ and $\{T_n\}$ are assumed to be independently distributed.

LEMMA 6.3. *If f is continuous, and for some positive integer p, $f(x) = O(|x|^p)$ for $|x| \to \infty$, then, for every $m \geq 1$,*

$$E\{\mathscr{E}_n f(X(T))\}^m = \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(x_1) \cdots f(x_m) \times \mathscr{E}_n\{\phi(x_1, \cdots, x_m; (P(T_i, T_j))\} \, dx_1 \cdots dx_m,$$

where $T_1, \cdots, T_m$ are as in Lemma 6.2.

PROOF. By the independence of the T's we have, for fixed $X(\cdot)$,

$$\{\mathscr{E}_n f(X(T))\}^m = \mathscr{E}_n f(X(T_1)) \cdots f(X(T_m)).$$

Apply E and interchange the order of E and $\mathscr{E}_n$ on the right-hand side; this is permitted under the hypothesis on f.

Now we consider particular cases of Lemma 6.3 when f is an Hermite polynomial.

LEMMA 6.4. *The random variables $\mathscr{E}_n H_k(X(T))$, $k \geq 1$, have expected values* 0, *variances equal to*

$$k! \frac{n-1}{n} \frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n+1}{2}\right)} \pi^{-1/2} \int_{-1}^{1} P^k(x)(1-x^2)^{(n-3)/2} \, dx, \qquad k \geq 1, \tag{6.10}$$

and are uncorrelated. (*Note that $H_o = 1$.*)

PROOF. $H_k(x) = O(|x|^k)$ for $|x| \to \infty$ so that Lemma 6.3 may be applied.

It follows from (6.1) that

$$E[\mathscr{E}_n H_k(X(T))] = \int_{-\infty}^{\infty} H_k(x)\phi(x) \, dx = (-1)^k \int_{-\infty}^{\infty} \phi^{(k)}(x) \, dx = 0;$$

(6.3) implies that the random variables $\mathscr{E}_n H_k(X(T))$, $k \geq 1$, have covariances equal to 0.

To verify (6.10) we apply Lemma 6.3 for $m = 2$. The second moment is

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} H_k(x_1)H_k(x_2)\mathscr{E}_n[\phi(x_1, x_2; P(T_1, T_2))]\, dx_1\, dx_2,$$

which, by an interchange of order of integration and expectation and by (6.3), is equal to

$$k!\,\mathscr{E}_n P^k((T_1, T_2)).$$

An application of Lemma 6.2 now completes the proof.

Finally we have this special result for $k = 2$ and $m = 4$:

LEMMA 6.5. *$E\{\mathscr{E}_n H_2(X(T))\}^4$ is dominated by a constant, which is independent of n, times*

$$\left\{ 2\frac{n-1}{n} \frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n+1}{2}\right)} \pi^{-1/2} \int_{-1}^{1} P^2(x)(1-x^2)^{(n-3)/2}\, dx \right\}^2. \tag{6.11}$$

PROOF. By Lemmas 6.1 and 6.3, the fourth moment is a linear combination (with coefficients independent of n) of terms

$$\mathscr{E}_n[P((T_1, T_2))P((T_1, T_3))P((T_3, T_4))P((T_2, T_4))] \tag{6.12}$$

and

$$\mathscr{E}_n P^2((T_1, T_2))P^2((T_3, T_4)). \tag{6.13}$$

By the Cauchy-Schwarz inequality, the expression (6.12) is at most

$$\{\mathscr{E}_n[P^2((T_1, T_2))P^2((T_3, T_4))]\mathscr{E}_n[P^2((T_1, T_3))P^2((T_2, T_4))]\}^{1/2}$$

which, by the symmetry of the joint distribution of the T's, is equal to (6.13). Since the T's are mutually independent the expression (6.13) is equal to

$$[\mathscr{E}_n P^2((T_1, T_2))]^2$$

which, by Lemma 6.2, is equal to (6.11).

Several calculations in the proof of the ergodic theorem and central limit theorem depend on the asymptotic estimate of the variance (6.10) and the term (6.11). The coefficient of the integral in (6.10) is asymptotic to $k!(n/2\pi)^{1/2}$. The integral, by a change of variable, is equal to

$$n^{-1/2}\int_{-n^{1/2}}^{n^{1/2}} P^k(x/n^{1/2})\left(1-\frac{x^2}{n}\right)^{(n-3)/2} dx,$$

which is asymptotic to

$$n^{-1/2}\int_{-n^{1/2}}^{n^{1/2}} P^k(x/n^{1/2})\exp(-\tfrac{1}{2}x^2)\, dx. \tag{6.14}$$

7. The ergodic theorem.

According to the last remark in Section 2, the process $\xi_o(\mathbf{t})$ is a fixed standard Gaussian random variable, and we write $\xi_o = \xi_o(\mathbf{t})$. By (2.4) the process X may be decomposed into a sum of two independent terms, $(c_o)^{1/2}\xi_o + X'(\mathbf{t})$, where $X'(\mathbf{t}) = \sum_{m\geq 1}(c_m)^{1/2}\xi_m(\mathbf{t})$. In proving the ergodic theorem, it is convenient to consider first the case of a process of the form X' where the c_o-term is absent, and then deduce the general case from this.

We begin with a special version of the ergodic theorem where f is the Hermite function of degree 1 or 2.

LEMMA 7.1. *If $c_o = 0$, then, with probability* 1,

$$\lim_{n\to\infty} \mathscr{E}_n H_k(X(T)) = 0, \qquad k = 1, 2.$$

PROOF. *Case* $k = 1$. Here $\mathscr{E}_n H_1(X(T)) = \mathscr{E}_n X(T)$, which, as the integral of a Gaussian process, has a Gaussian distribution. By Lemma 6.4, the mean is equal to 0. The estimate of (6.10) based on (6.14) implies that the variance of $\mathscr{E}_n X(T)$ is asymptotic to

$$\int_{-n^{1/2}}^{n^{1/2}} P(y/n^{1/2})\phi(y)\, dy. \tag{7.1}$$

By virtue of (2.2) and the assumption $c_o = 0$ it follows that

$$|P(x)| \le |x| \quad \text{for} \quad |x| \le 1, \tag{7.2}$$

so that (7.1) is at most equal to

$$n^{-1/2} \int_{-\infty}^{\infty} |y|\, \phi(y)\, dy.$$

Since $\mathscr{E}_n X(T)$ has a Gaussian distribution its central moment of order $2m$ is proportional to the mth power of its variance; therefore, for $m = 3$ we have

$$E[\mathscr{E}_n X(T)]^6 = O(n^{-3/2}),$$

which implies that the series $\sum_n [\mathscr{E}_n X(T)]^6$ converge with probability 1. This completes the proof of the lemma for $k = 1$.

Case $k = 2$. According to Lemma 6.5, $E[\mathscr{E}_n H_2(X(T))]^4$ is of the order of (6.11), which, by (7.2) and the estimate (6.14) is $O(n^{-2})$. Therefore, the series $\sum_n [\mathscr{E}_n H_2(X(T))]^4$ converges with probability 1, which completes the proof.

THEOREM 7.1. (*Ergodic theorem*). *If f satisfies*

$$\int_{-\infty}^{\infty} f^2(x)\phi(x)\, dx < \infty, \tag{7.3}$$

then, with probability 1,

$$\lim_{n\to\infty} \mathscr{E}_n f(X(T)) = \int_{-\infty}^{\infty} f((c_o)^{1/2}\xi_0 + x(\textstyle\sum_{m\ge 1} c_m)^{1/2})\phi(x)\, dx. \tag{7.4}$$

PROOF. We observe that (7.3) implies (6.6), so that $\mathscr{E}_n f(X(T))$ is well defined. Let us show that if the result is true for the case $c_o = 0$ then it is true also for $c_o > 0$. According to the opening remarks of this section we have the decomposition $X(\mathbf{t}) = (c_o)^{1/2}\xi_o + X'(\mathbf{t})$. Condition X by fixing ξ_o, and apply the result for the case $c_o = 0$ to the function $f'(x) = f((c_o)^{1/2}\xi_o + x(\sum_{m\ge 1} c_m)^{1/2})$ and to the process

$$Y(t) = X'(t)/(\textstyle\sum_{m\ge 1} c_m)^{1/2}.$$

Then $\mathscr{E}_n f'(Y(T)) \to \int_{-\infty}^{\infty} f'(x)\phi(x)\, dx$ with conditional probability 1; hence, by bounded convergence, this also holds unconditionally.

Now we consider the case $c_0 = 0$. First we collect some preliminary facts about Hermite polynomial expansions. For any function f satisfying (7.3), let f_k be the Fourier coefficient with respect to the kth normalized Hermite polynomial:

$$f_k = (k!)^{-1/2} \int_{-\infty}^{\infty} f(x)H_k(x)\phi(x)\, dx. \tag{7.5}$$

Then $f(x)$ has the expansion in $L_2(\phi)$,

$$f(x) = \sum_{k\geq 0} f_k H_k(x)/(k!)^{1/2}. \tag{7.6}$$

It follows from (6.1) and (6.4) that

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x)f(y)\phi(x, y; r)\,dx\,dy = \sum_{k\geq 0} f_k^2 r^k. \tag{7.7}$$

We formally replace x by $X(T)$ in (7.6) and apply the averaging operator $\mathscr{E}_n$:

$$\mathscr{E}_n f(X(T)) = \sum_{k\geq 0} f_k \mathscr{E}_n H_k(X(T))/(k!)^{1/2}. \tag{7.8}$$

This equation holds in the sense that for each n the series on the right-hand side converges in mean square to the random variable on the left:

$$\lim_{m\to\infty} E[\mathscr{E}_n f(X(T)) - \sum_{k=0}^{m} f_k \mathscr{E}_n H_k(X(T))/(k!)^{1/2}]^2 = 0. \tag{7.9}$$

To prove this we apply Lemma 6.3 for $m = 2$ and with

$$f(x) - \sum_{k=0}^{m} f_k H_k(x)/(k!)^{1/2}$$

in place of $f(x)$; then the expectation in (7.9) is, by application of (7.7), equal to

$$\sum_{k=m+1}^{\infty} f_k^2 \mathscr{E}_n P^k((T_1, T_2)),$$

which tends to 0 for $m \to \infty$ because

$$\sum_{k=0}^{\infty} f_k^2 = \int_{-\infty}^{\infty} f^2(x)\phi(x)\,dx < \infty.$$

Using the explicit forms (6.2), we write (7.8) as

$$\mathscr{E}_n f(X(T)) = \int_{-\infty}^{\infty} f(x)\phi(x)\,dx + f_1 \mathscr{E}_n X(T) + \tfrac{1}{2} f_2 \mathscr{E}_n H_2(X(T)) + \sum_{k\geq 3} f_k \mathscr{E}_n H_k(X(T))/(k!)^{1/2}. \tag{7.10}$$

Lemma 7.1 implies that the second and third terms on the right-hand side of (7.10) converge to 0 with probability 1 for $n \to \infty$. By Lemma 6.4, the last term in (7.10) has expectation 0 and variance equal to

$$\frac{n-1}{n}\frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n+1}{2}\right)}\pi^{-1/2}\sum_{k\geq 3} f_k^2 \int_{-1}^{1} P^k(x)(1-x^2)^{(n-3)/2}\,dx, \tag{7.11}$$

which, by the inequality (7.2), is dominated by

$$2\frac{n-1}{n}\frac{\Gamma\left(\frac{n}{2}+1\right)}{\Gamma\left(\frac{n+1}{2}\right)}\pi^{-1/2}\int_{0}^{1} x^3(1-x^2)^{(n-3)/2}\,dx\cdot\sum_{k\geq 3} f_k^2.$$

By the estimates leading to (6.14), we infer that the expression displayed above is of the order $n^{-3/2}$ for $n \to \infty$. By applying the argument used in the proof of Lemma 7.1 we conclude that the last term in (7.10) converges to 0 with probability 1. This completes the proof of (7.4) for the case $c_o = 0$.

8. Central limit theorems. The Hermite expansion (7.8) can also be used to prove versions of the central limit theorem for the average $\mathscr{E}_n f$. Using the explicit forms (6.2), we write

$$(8.1)\qquad \mathscr{E}_n f(X(T)) = \int_{-\infty}^{\infty} f(x)\phi(x)\,dx + f_1 \mathscr{E}_n X(T) + \textstyle\sum_{k\geq 2} f_k \mathscr{E}_n H_k(X(T))/(k!)^{1/2}.$$

According to the proof of Lemma 7.1, $\mathscr{E}_n X(T)$ has a Gaussian distribution with mean 0 and variance asymptotic to (7.1). If the last term in (8.1) has a variance of smaller order than (7.1) for $n \to \infty$, then the difference

$$\mathscr{E}_n f(X(T)) - \int_{-\infty}^{\infty} f(x)\phi(x)\,dx,$$

divided by the square root of (7.1), has a limiting standard Gaussian distribution.

If, in (2.1), $c_0 > 0$; then, by Theorem 7.1, $\mathscr{E}_n f(X(T))$ has a limit which is a random variable whose distribution is not necessarily Gaussian. Therefore we will consider only the case $c_0 = 0$, and define

$$(8.2)\qquad b = \min(k : c_k > 0).$$

Various types of conditions on b or f are sufficient to ensure the validity of the estimates following (8.1). We present two theorems to illustrate two sets of hypotheses. It will be clear from the proofs that other variations of these hypotheses may be used.

THEOREM 8.1. *Assume in* (8.2) *that* $b \geq 2$. *If* b *is even, then*

$$(8.3)\qquad n^{b/4}\left(\mathscr{E}_n f(X(T)) - \int_{-\infty}^{\infty} f(x)\phi(x)\,dx\right)$$

has a limiting Gaussian distribution with mean 0 *and variance*

$$(8.4)\qquad c_b f_1^2 \int_{-\infty}^{\infty} x^b \phi(x)\,dx.$$

If b *is odd, then* b *is to be replaced by* $b+1$ *in the statement above.*

PROOF. *Case with even* b. By application of (8.1) the random variable (8.3) may be expressed as the sum of two random variables,

$$(8.5)\qquad n^{b/4} f_1 \mathscr{E}_n X(T)$$

and

$$(8.6)\qquad n^{b/4} \textstyle\sum_{k\geq 2} f_k \mathscr{E}_n H_k(X(T))/(k!)^{1/2}.$$

By the remark following (8.1), the random variable (8.5) has a Gaussian distribution with mean 0 and variance asymptotic to

$$(8.7)\qquad f_1^2 n^{b/2} \int_{-n^{1/2}}^{n^{1/2}} P(y/n^{1/2})\phi(y)\,dy.$$

This converges to the expression (8.4) for $n \to \infty$. Indeed, by (2.1) and (8.2), we have $P(x) \sim c_b x^b$ for $x \to 0$; and, by (2.2), $|P(x)| \leq |x|^b$ for $|x| \leq 1$.

To complete the proof for even b we will show that the second moment of (8.6) converges to 0. By the argument leading to (7.11), we find that the second moment is equal to $n^{b/2}$ times the expression (7.11) with summation over $k \geq 2$ in place of $k \geq 3$. The latter product, by virtue of the inequality $|P(x)| \leq |x|^b$, is of the order $n^{-b/2}$, which tends to 0 for $n \to \infty$.

Case with odd b. We replace b by $b+1$ in (8.5), (8.6) and (8.7). P has the expansion

$$P(x) = c_b x^b + c_{b+1} x^{b+1} + O(|x|^{b+2}).$$

Since b is odd, the leading term of $P(y/n^{1/2})$ in (8.7) vanishes upon integration, so that the asymptotic variance of (8.5),

$$f_1^2 n^{(b+1)/2} \int_{-n^{1/2}}^{n^{1/2}} P(y/n^{1/2})\phi(y)\,dy$$

converges to (8.4) with $b+1$ in place of b.

To complete the proof we verify that when b is replaced by $b+1$ in (8.6), the random variable has a second moment of order $n^{(1-b)/2}$ which tends to 0. This follows from the relation $P(x) \sim c_b x^b$ and the fact that the series

$$\sum_{k\geq 2} f_k^2 \int_{-1}^{1} P^k(x)(1-x^2)^{(n-3)/2}\,dx$$

appearing in the expression in the second moment is dominated by

$$\int_{-1}^{1} P^2(x)(1-x^2)^{(n-3)/2}\,dx \cdot \sum_{k\geq 2} f_k^2.$$

Now we state and prove a version of the central limit theorem which covers the case $b = 1$, but where a restriction is placed on f.

THEOREM 8.2. *If $f_2 = 0$, then the conclusion of Theorem 8.1 is valid for all $b \geq 1$.*

PROOF. Since the case $b \geq 2$ has already been covered, it suffices to take $b = 1$ and show that

$$(8.8)\qquad n^{1/2}\left(\mathscr{E}_n f(X(T)) - \int_{-\infty}^{\infty} f(x)\phi(x)\,dx\right)$$

has a limiting Gaussian distribution with mean 0 and variance (8.4) with $b = 2$. By the current hypothesis $f_2 = 0$, the last term in (8.1) is to be summed over $k \geq 3$, and is identical with the last term of (7.10). Our estimate of the variance of the latter, given by (7.11), is $O(n^{-3/2})$. It follows from (8.1) that the limiting distribution of (8.8) is identical with that of $n^{1/2} f_1 \mathscr{E}_n X(T)$. The latter distribution is found exactly as in the proof of Theorem 8.1 for odd b.

We remark that the hypothesis $f_2 = 0$ means that f is orthogonal to the subspace of $L_2(\phi)$ spanned by the single element $H_2(x) = x^2 - 1$. Every odd function and every constant function in $L_2(\phi)$ satisfy this condition.

We note that the variance (8.4) of the limiting distribution is proportional to $c_b f_1^2$. Thus the variance is equal to 0 if $f_1 = 0$ or if b is odd and $c_{b+1} = 0$. It would be of interest to investigate other normalizations of $\mathscr{E}_n f(X(T))$ which, in such cases, would lead to nondegenerate limiting distributions. I do not know if such distributions are necessarily Gaussian. If $f_k = 0$ for $k = 0, 1, \cdots, m$, then the dominant term in the limiting distribution of $\mathscr{E}_n f(X(T))$ is, by (8.1), that of $\mathscr{E}_n H_k(X(T))$. A nondegenerate limiting distribution for the latter has not been found by the method we have used above.

Taqqu (1979) recently found a class of nonnormal limiting distributions in the case of sums of nonlinear functions of a stationary Gaussian process with a real time parameter. His derivation is also based on the Hermite expansion.

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