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ASYMPTOTIC DISTRIBUTION AND MOMENTS OF NORMAL EXTREMES

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Let $X(n)$ be the largest observation in a random sample of size n from a standard normal population. In this paper we investigate the limiting behavior of the distribution and the moments of $X(n)$ for large n. The results of this paper provide rates of convergence of the distribution and the moments of $X(n)$.

1. Introduction. Let $X(n)$ be the largest observation in a sample of size n from a standard normal population. Denote by F the standard normal distribution. Then the distribution function of $X(n)$ is given by $P[X(n) < x] = F^n(x)$. It can be shown that, as $n \to \infty$, $F^n(t + t^{-1}x) \to \exp[-\exp(-x)]$ where $t \equiv t_n$ is given by

$$1 - F(t) = 1/n. \tag{1.1}$$

(For example, see Cramér (1946) page 374.)

Some results on the rate of convergence of $F^n(t + t^{-1}x)$ can be obtained from Theorem 2.10.1 in Galambos (1978) page 113. In this paper a more specific result concerning the rate of convergence of $F^n(t + t^{-1}x)$ is given. The results of this paper and Galambos' are different in the sense that Galambos' result pertains to arbitrary F and finite and infinite values of n while our result is an asymptotic one for the special normal F. Galambos' result is in terms of $n[1 - F(a_n + b_nx)]$ for which no rate of convergence is given. It is applicable for values of x satisfying certain restrictions which are satisfied for all but small values of n. The results of this paper are asymptotic but in terms of explicit expressions. These results are more useful for analytical studies than for calculations.

Recently Hall (1979) has shown that if the constants a_n and b_n are chosen in an optimal way then the rate of convergence of $F^n(a_n + b_nx)$ to $\exp[-\exp(-x)]$ is proprotional to $1/\log n$.

Let, for $r > 0$, $m_r(n)$ and m_r be the rth moments of $F^n(t + t^{-1}x)$ and $\exp[-\exp(-x)]$ respectively; i.e.,

$$m_r(n) = \int_{-\infty}^{\infty} x^r \, dF^n(t + t^{-1}x) \quad \text{and} \quad m_r = \int_{-\infty}^{\infty} x^r \, d\exp[-\exp(-x)].$$

Pickands (1968) has shown that $m_r(n) \to m_r$ as $n \to \infty$. In Section 3 we give a result on the rate of convergence of $m_r(n)$. For $r = 1, 2$ results on the rate of convergence of the moments can be found in Cramér (1946). McCord (1964) has given similar results for some classes of distributions which do not include normal or gamma distributions.

2. Distribution of $X(n)$. Let F and f stand for the distribution function and the density function of the standard normal variable. We need the following facts about Mills' ratio. For details see Johnson and Kotz (1970) page 278. Mills' ratio $R(x)$ is defined by

$$R(x) = [1 - F(x)]/f(x), \qquad x > 0. \tag{2.1}$$

$R(x)$ satisfies the following inequalities.

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(2.2) $$x/(x^2 + 1) < R(x) < 1/x$$

and

(2.3) $$2[(x^2 + 4)^{1/2} + x]^{-1} < R(x) < 4[(x^2 + 8)^{1/2} + 3x]^{-1}.$$

LEMMA 2.1. *Let* $H(t, x) = F(t + t^{-1}x)$, $h(t, x) = n \log H(t, x) + e^{-x}$, $a(x) = e^{-x}(x^2 + 2x)/2$, *and* $b(x) = -e^{-x}(x^4 + 4x^3 + 8x^2 + 16x)/8$. *Then as* $n \to \infty$, $t^2[t^2h(t, x) - a(x)] \to b(x)$.

PROOF. Note that $t \to \infty$ iff $n \to \infty$. Also as $n \to \infty$, we have $H(t, x) \to 1$ and $nf(t)\,t^{-1} \to 1$. Write

$$L_1 = \lim h(t, x)t^2 = \lim \frac{\log H(t, x) + n^{-1}e^{-x}}{n^{-1}t^{-2}}.$$

The denominator can be replaced by $t^{-3}f(t)$ since $n^{-1}t^{-2}[t^{-3}f(t)]^{-1} = [nt^{-1}f(t)]^{-1} \to 1$. Also

$$\frac{dn^{-1}}{dt} = \frac{d[1 - F(t)]}{dt} = -f(t) \quad \text{and} \quad \frac{df(t)}{dt} = -tf(t).$$

Therefore

$$L_1 = \lim \frac{\log H(t, x) + n^{-1}e^{-x}}{t^{-3}f(t)}$$
$$= \lim \frac{[H(t, x)]^{-1}f(t + t^{-1}x)(1 - t^{-2}x) - f(t)e^{-x}}{-3t^{-4}f(t) - t^{-2}f(t)}$$

by L'Hospital's rule. It follows that

$$L_1 = \lim \frac{\exp(-x-2^{-1}t^{-2}x^2)(1 - t^{-2}x) - H(t, x)e^{-x}}{-t^{-2}}.$$

One more application of L'Hospital's rule gives $L_1 = a(x)$. Write

$$L_2 = \lim t^2[t^2h(t, x) - a(x)] = \lim \frac{\log H(t, x) + n^{-1}e^{-x} - t^{-2}n^{-1}a(x)}{t^{-4}n^{-1}}.$$

Replace the denominator $t^{-4}n^{-1}$ by $t^{-5}f(t)$, apply L'Hospital's rule, and multiply by $H(t, x)$. Then we have

$$L_2 = \lim\left[\frac{\exp(-x - 2^{-1}t^{-2}x^2)(1 - t^{-2}x) - H(t, x)e^{-x} + t^{-2}H(t, x)a(x)}{-t^{-4}} - \frac{2a(x)H(t, x)}{t^{-1}f(t)n}\right].$$

The limit of the second term is $2a(x)$. To find the limit of the first term apply L'Hospital's rule and use the fact that $t^rf(t) \to 0$ for all r. Then

$$L_2 = \lim \frac{\exp(-x-2^{-1}t^{-2}x^2)x^2(1 - t^{-2}x) + \exp(-x-2^{-1}t^{-2}x^2)2x - 2H(t, x)a(x)}{4t^{-2}} - 2a(x).$$

One more application of L'Hospital's rule gives the limit in the lemma.

THEOREM 2.1. *Let* $G(x) = \exp[-\exp(-x)]$; *then, for* $n \to \infty$,

$$t^2[t^2\{H^n(t, x) - C(x)\} - a(x)G(x)] \to [b(x) + 2^{-1}a^2(x)]G(x).$$

PROOF.

$$t^2[t^2\{H^n(t, x) - G(x)\} - a(x)G(x)] = t^2[t^2(e^{h(t,x)} - 1) - a(x)]G(x)$$

(2.4) $$= [t^2\{t^2h(t, x) - a(x)\} + t^4h^2(t, x)$$

$$\times \left\{ \frac{1}{2} + h(t, x) \sum_{i=3}^{\infty} \frac{h^{i-3}(t, x)}{i!} \right\} \Bigg] G(x).$$

By Lemma 2.1, $h(t, x) \to 0$,

$$\left| \sum_{i=3}^{\infty} \frac{h^{i-3}(t, x)}{i!} \right| < \exp[h(t, x)] \to 1, \quad \text{and} \quad t^2 h(t, x) \to a(x).$$

The theorem follows from (2.4).

3. Moments of $X(n)$.

LEMMA 3.1. *For fixed r and d, $r > 0$, $0 < d < 1$, $x^r t^2[t^2\{H^n(t, x) - G(x)\} - a(x)G(x)]$ is bounded for $x > -dt^2$ by integrable functions independent of n.*

PROOF. Using the Taylor expansion of $\log H(t, x)$ and (2.1), we get

$$t^2[t^2 h(t, x) - a(x)]$$
$$= t^4[-n\, R(t + t^{-1}x)\, f(t + t^{-1}x) + e^{-x}] - t^2 a(x) + C(t, x)$$

where by the inequalities (2.2),

$$| C(t, x) | < t^4 n\, R^2(t + t^{-1}x) f^2(t + t^{-1}x) F^{-1}(t + t^{-1}x)$$
$$< t^3(t^2 + 1)(t + t^{-1}x)^{-2} f(t) F^{-1}[(1 - d)t] e^{-x}$$
$$< t(t^2 + 1)(1 - d)^2 f(t) F^{-1}[(1 - d)t] e^{-x} \quad \text{for} \quad x > -dt^2.$$

Hence $| C(t, x) | < e^{-x}$ for large t. Writing $n^{-1} = R(t) f(t)$ and using (2.3) for $R(t)$ and $R(t + t^{-1}x)$ we get for $t + t^{-1}x > 0$,

$$t^4[-n\, R(t + t^{-1}x) f(t + t^{-1}x) + e^{-x}] - t^2 a(x)$$

$$\geq \frac{[\{(t + t^{-1}x)^2 + 8\}^{1/2} + 3(t + t^{-1}x)](2t^2 - x^2 - 2x) - 4t^2[(t^2 + 4)^{1/2} + t] \exp(-2^{-1}t^2x^2)}{2[\{(t + t^{-1}x)^2 + 8\}^{1/2} + 3(t + t^{-1}x)]}$$

$$\times e^{-x}$$

To find a bound for the numerator, write

$$[(t + t^{-1}x)^2 + 8]^{1/2} = t(1 + t^{-2}x)[1 + 4t^{-2}(1 + t^{-2}x)^{-2} + \cdots]$$

which is valid for $x > -dt^2$, $t > \sqrt{8}(1 - d)^{-1}$, and $0 < d < 1$; and $(t^2 + 4)^{1/2} = t(1 - 2t^{-2} + \cdots)$ for $t > 2$, and $\exp(-2^{-1}t^{-2}x^2) = 1 - 2^{-1}t^{-2}x^2 + \cdots$.

Using the above expansions in the numerator of (3.1) we see that it is greater than $tk_1(x)$ for some function k_1 such that $k_1(x)e^{-x}$ is integrable. Also for $x > -dt^2$, the denominator is less than $10t(1 - d)$ if $t > \sqrt{8}(1 - d)^{-1}$. Thus, for $x > -dt^2$, and t sufficiently large, $t^2[t^2 h(t, x) - a(x)] > 10^{-1}(1 - d)^{-1} k_1(x) e^{-x}$.

Similarly treating the other terms in (2.4) we can find integrable functions such that, for $x > -dt^2$, $t^2[t^2\{H^n(t, x) - G(x)\} - a(x)G(x)]$ is bounded below and above by these functions.

LEMMA 3.2. *For $0 < d < 1$, and $i, j \geq 0$, $\int_{-\infty}^{-dt^2} t^i |x|^j H^n(t, x)\, dx \to 0$ as $n \to \infty$.*

PROOF.

$$\int_{-\infty}^{-dt^2} t^i |x|^j F^n(t + t^{-1}x)\, dx = \int_{-\infty}^{(1-d)t} t^{i+j+1} |y - t|^j F^n(y)\, dy$$

(3.2)

$$\leq t^{i+j+1}F^{n-1}(0)\int_{-\infty}^{0}|y-t|^{j}F(y)\,dy+\int_{0}^{(1-d)}t^{i+2j+2}|z-1|^{j}F^{n}(tz)\,dz.$$

The first term on the right-hand side of (3.2) converges to zero since $t^kF^{n-1}(0)$ converges to zero and $\int_{-\infty}^{0} y^rF(y)\,dy$ is finite for all nonnegative r. In the second term the integrand converges to zero uniformly. To see this note that, by (2.2),

$$t^kF^n(tz)\leq t^kF^n[(1-d)t]\leq t^k\exp[-n\{1-F((1-d)t)\}]$$
$$=t^k\exp\left[-\frac{R\{(1-d)t\}f\{(1-d)t\}}{R(t)f(t)}\right]\to 0.$$

Therefore, the last term in (3.2) also converges to zero. This proves the lemma.

LEMMA 3.3. $\int_{-\infty}^{-dt^2} t^i|x|^jG(x)\,dx\to 0$ *as* $n\to\infty$.

PROOF. $\int_{-\infty}^{-dt^2} t^i|x|^jG(x)\,dx\leq t^i\exp[-2^{-1}\exp(dt^2)]\int_{-\infty}^{-1}|x|^j\exp[-2^{-1}\exp(-x)]\,dx\to 0$ as $n\to\infty$.

THEOREM 3.1. $t^4[m_r(n)-m_r]+2^{-1}t^2r[m_{r+1}+2m_r]\to r[(r+3)m_{r+2}+(12+4r)m_{r+1}+(16+4r)m_r]/8$ *as* $n\to\infty$ *where* $m_r(n)$ *and* m_r *are the* rth *moments of* $H^n(t,x)$ *and* $G(x)$ *respectively.*

PROOF. $m_r(n)-m_r=\int_{-\infty}^{\infty}x^rd[H^n(t,x)-G(x)]=-r\int x^{r-1}[H^n(t,x)-G(x)]\,dx.$ Therefore by Theorem 2.1 and Lemmas 3.1, 3.2 and 3.3 we get

$$t^4[m_r(n)-m_r]+2^{-1}t^2r[m_{r+1}+2m_r]$$
$$=-r\int[t^4x^{r-1}[H^n(t,x)-G(x)]-t^2x^{r-1}a(x)G(x)]\,dx$$
$$\to -r\int[b(x)+2^{-1}a^2(x)]x^{r-1}G(x)\,dx.$$

This, together with the fact that $\int x^ke^{-2x}G(x)\,dx=\int x^ke^{-x}dG(x)=-km_{k-1}+m_k$, gives the result of the theorem.

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