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APPROXIMATION OF PRODUCT MEASURES WITH AN APPLICATION TO ORDER STATISTICS

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Firstly, a well-known upper estimate concerning the distance of independent products of probability measures is extended to the case of signed measures. The upper bound depends on the total variation of the signed measures and on the distances of the single components where the distances are measured in the sup-metric. Under certain regularity conditions, the upper estimate can be sharpened by using asymptotic expansions. These expansions hold true over the set of all integrable function. An application of these results together with an asymptotic expansion of the distribution of a single order statistic yields an asymptotic expansion of the joint distribution of order statistics under the exponential distribution.

1. Introduction. Let d denote the sup-metric on the set of all finite measures on a measurable space $(\mathscr{X}, \mathscr{A})$; that is, $d(\mu_1, \mu_2) = \sup\{|\mu_1(A) - \mu_2(A)| : A \in \mathscr{A}\}$ for finite measures $\mu_i \,|\, \mathscr{A}$, $i = 1, 2$. Independent product measures are denoted by $\times_{i=1}^k \mu_i$ (or μ^k if all components μ_i are equal to μ). Estimates of the distance between product measures—in terms of distances between the single components—were frequently proved in the stochastical literature. In Hoeffding and Wolfowitz (1958), (4.4) and (4.5), the following inequalities are proved for probability measures P and Q:

$$d(P, Q) \le d(P^k, Q^k) \le k\, d(P, Q). \tag{1.1}$$

The upper estimate is extended to the case of nonidentical probability measures $P_i \,|\, \mathscr{A}_i$, $Q_i \,|\, \mathscr{A}_i$, $i = 1, \dots, k$, in the article of Blum and Pathak (1972), Lemma (1.3) (see also Sendler (1975), Lemma 2.1). A different lower bound was found by Behnen and Neuhaus (1975) (see proof of Proposition, pages 1351–1352). Combining these two results one obtains

$$1 - \exp(-\tfrac{1}{2} \textstyle\sum_{i=1}^k d(P_i, Q_i)^2) \le d(\times_{i=1}^k P_i, \times_{i=1}^k Q_i) \le \sum_{i=1}^k d(P_i, Q_i). \tag{1.2}$$

In Theorem 2.1, the second inequality in (1.2) will be extended to the case of signed measures.

The main objective of Section 2 will be to derive estimates which depend on terms of the form $\int (g_i - 1)^2 \, dP_i$ where g_i is a density of Q_i with respect to P_i (in short: $g_i \in dQ_i/dP_i$). These terms are suggested by asymptotic considerations: given a family of probability measures with real parameter θ, it follows, e.g., from Korollar 2.25 in Witting and Nölle (1970) that under appropriate regularity conditions the following relation holds:

$$\lim_{k \in N} d(P_\theta^k, P_{\theta+\eta k^{-1/2}}^k) = 2\Phi(\eta I^{1/2}) - 1 \tag{1.3}$$

where Φ denotes the distribution function of the standard normal distribution N, I is the Fisher-information at θ, and $\eta > 0$. Notice that

$$\lim_{k \in N} \frac{1}{2}\left(k \int (g_k - 1)^2 \, dP_\theta\right)^{1/2} = \eta I^{1/2}$$

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where $g_k \in dP_{\theta+\eta k^{-1/2}}/dP_\theta$. Thus, there is a close connection of (1.3) to a result which is given in Reiss (1980), proof of Lemma 6.3: let $g_k \in dQ_k/dP_k$. If $k^{1/2}(g_k - 1)$ is bounded and $\liminf_{k\in N} k \int (g_k - 1)^2\, dP_k > 0$ then

$$\left| d(P_k^k, Q_k^k) - 2\Phi\left(\frac{1}{2}\left(k \int (g_k - 1)^2\, dP_k\right)^{1/2}\right) + 1 \right| = O(k^{-1/2}). \tag{1.4}$$

Concerning historical remarks on the asymptotic normality of order statistics we refer to Reiss (1975). In that article, asymptotic expansions of the joint distribution of several order statistics were proved under the assumption that, roughly speaking, the number of order statistics is smaller than $n^{1/2}$ (with n denoting the size of the sample). Counterexamples show that this restriction cannot be omitted if normal approximation is considered. Theorem 3.4 will provide us with approximations to the joint distribution of order statistics (under the exponential distribution) in cases where the number of order statistics is larger than $n^{1/2}$.

2. The basic results. The following theorem extends the right-hand side of (1.2) to signed measures. Given a signed measure $\nu/\mathscr{A}$, let $X = X^+ + X^-$ be a Hahn decomposition with respect to ν, and $\|\nu\| = \nu(X^+) + \nu(X^-)$.

THEOREM 2.1. *For all measurable spaces $(X_i, \mathscr{A}_i)$ and finite signed measures $\nu_i/\mathscr{A}_i$, $\mu_i/\mathscr{A}_i$, $i = 1, \ldots, k$ the following inequality holds*:

$$d(\times_{i=1}^k \nu_i, \times_{i=1}^k \mu_i) \le \sum_{i=1}^k \|\times_{j=1}^{i-1} \nu_i\| \, \|\times_{j=i+1}^k \mu_j\| \, d(\nu_i, \mu_i)$$

(*with the convention that* $\|\times_{j=1}^{O} \nu_j\| = \|\times_{k+1}^k \mu_i\| = 1$).

PROOF. Because of

$$(\mu_1 \times \mu_2)(A) = \int \mu_1(A_{x_2})\, d\mu_2(x_2)$$

for $A \in \mathscr{A}_1 \times \mathscr{A}_2$, and

$$|\nu_1(A_{x_2}) - \mu_1(A_{x_2})| \le d(\nu_1, \mu_1)$$

we obtain

$$d(\mu_1 \times \mu_2, \nu_1 \times \mu_2) \le \|\mu_2\| \, d(\nu_1, \mu_1).$$

Thus, we also get

$$d(\nu_1 \times \mu_2, \nu_1 \times \nu_2) \le \|\nu_1\| \, d(\nu_2, \mu_2)$$

whence the triangular inequality implies the assertion for $k = 2$. By using the induction scheme and noting that

$$d(\times_{i=1}^{k+1} \nu_i, \times_{i=1}^{k+1} \mu_i) = d((\times_{i=1}^k \nu_i) \times \nu_{k+1'} (\times_{i=1}^k \mu_i) \times \mu_{k+1})$$

and

$$\|\times_{i=1}^k \nu_i\| \, \|\nu_{k+1}\| = \|\times_{i=1}^{k+1} \nu_i\|$$

the proof can be completed in a straightforward way.

For probability measures $\mu_i = P_i$ we state the following

COROLLARY 2.2. *For all probability spaces $(X_i, \mathscr{A}_i, P_i)$ and finite signed measures $\nu_i/\mathscr{A}_i$, $i = 1, \ldots, k$, the following inequality holds*:

$$d(\times_{i=1}^k \nu_i, \times_{i=1}^k P_i) \le \exp[2 \textstyle\sum_{i=1}^k d(P_i, \nu_i)] \sum_{i=1}^k d(P_i, \nu_i).$$

PROOF. Since for every $j = 1, \dots, k$

$$\|\nu_j\| \le P_j(X_j^+) + P_j(X_j^-) + 2d(P_j, \nu_j) = 1 + 2d(P_j, \nu_j)$$

we have for every $i = 1, \dots, k$

$$\|\times_{j=1}^{i} \nu_j\| \le \prod_{j=1}^{i} (1 + 2d(P_j, \nu_j)) \le \exp[2 \textstyle\sum_{j=1}^{k} d(P_j, \nu_j)] .$$

Thus, Theorem 2.1 implies the assertion.

REMARK 2.3. The lower bound in (1.2) can also be derived from the following inequalities which are due to Kraft ((1955), Lemma 1):

$$1 - \int (pq)^{1/2}\, d\mu \le d(P, Q) \le \left(1 - \left(\int (pq)^{1/2}\, d\mu\right)^2\right)^{1/2} \tag{2.4}$$

for probability measures P, Q if μ is a σ-finite measure, $p \in dP/d\mu$ and $q \in dQ/d\mu$. Applying (2.4) to $p_i \in dP_i/d\mu_i$, $q_i \in dQ_i/d\mu_i$ we get

$$d(\times_{i=1}^{k} P_i, \times_{i=1}^{k} Q_i) \ge 1 - \prod_{i=1}^{k} \int (p_i\, q_i)^{1/2}\, d\mu_i$$

$$\ge 1 - \prod_{i=1}^{k} (1 - d(P_i, Q_i)^2)^{1/2} \ge 1 - \exp[-\tfrac{1}{2} \textstyle\sum_{i=1}^{k} d(P_i, Q_i)^2] .$$

In (1.2), the upper bound is equal to $\sum_{i=1}^{k} d(P_i, Q_i)$, whereas the lower bound is smaller than $\frac{1}{2} \sum_{i=1}^{k} d(P_i, Q_i)^2$. In particular cases (e.g., in cases which are dealt with in Section 3), sharper estimates can be proved which, roughly speaking, depend on $(\sum_{i=1}^{k} d(P_i, Q_i)^2)^{1/2}$. For this purpose, we prove

THEOREM 2.5. *Let $(X_i, \mathscr{A}_i, P_i)$ be probability spaces, $\nu_i/\mathscr{A}_i$ signed measures with $\nu_i(X_i) = 1$ and $g_i \in d\nu_i/dP_i$ for $i = 1, \dots, k$. Let $f_i = g_i - 1$. For every $m \in \{0, \dots, k\}$ and $X_{i=1}^{k} \mathscr{A}_i$-measurable functions ψ the following inequality holds (if the integrals given below exist):*

$$\left| \int \psi\, d \times_{i=1}^{k} \nu_i - \int \psi\Big(\sum_{j=0}^{m} \sum_{1 \le i_1 < \dots < i_j \le k} \prod_{r=1}^{j} f_{i_r} \circ \pi_{i_r}\Big)\, d \times_{i=1}^{k} P_i \right| \tag{2.6}$$
$$\le \left[\frac{1}{(m+1)!} \int \psi^2\, d(\times_{i=1}^{k} P_i) \exp(\textstyle\sum_{i=1}^{k} P_i(f_i^2))\right]^{1/2} (\textstyle\sum_{i=1}^{k} P_i(f_i^2))^{(m+1)/2}$$

where π_i denotes the ith *projection, the term for $j = 0$ is equal to one, and $P_i(f_i^2) \equiv \int f_i^2\, dP_i$.*

PROOF. Since $P_i(f_i) = 0$ it is easy to see that the functions $\prod_{r=1}^{j} f_{i_r} \circ \pi_{i_r}$, $1 \le i_1 < \dots < i_j \le k$, $j = 1, \dots, k$, form a multiplicative system with respect to $\times_{i=1}^{k} P_i$. Put

$$h_m = \sum_{j=0}^{m} \sum_{1 \le i_1 < \dots < i_j \le k} \prod_{r=1}^{j} f_{i_r} \circ \pi_{i_r}.$$

Using the identity

$$\prod_{i=1}^{k} (1 + a_i) = \sum_{j=0}^{k} \sum_{i \le i_1 < \dots < i_j \le k} \prod_{r=1}^{j} a_{i_r}$$

which holds for all real numbers a_i, $i = 1, \dots, k$, and applying the Schwarz inequality we obtain

$$\left| \int \psi\, d \times_{i=1}^{k} \nu_i - \int \psi h_m\, d \times_{i=1}^{k} P_i \right| \le \int |\psi|\, |h_k - h_m|\, d \times_{i=1}^{k} P_i$$

$$\le \left[\int \psi^2\, d \times_{i=1}^{k} P_i\right]^{1/2} \left[\int (h_k - h_m)^2\, d \times_{i=1}^{k} P_i\right]^{1/2}$$

$$= \left[\int \psi^2 \, d \times_{i=1}^{k} P_i \right]^{1/2} [\textstyle\sum_{j=m+1}^{k} \sum_{1 \le i_1 < \cdots < i_j \le k} \prod_{r=1}^{j} P_{i_r}(f_{i_r}^2)]^{1/2}.$$

Thus, the assertion follows from the inequalities

$$\textstyle\sum_{1 \le i_1 < \cdots < i_j \le k} \prod_{r=1}^{j} a_{i_r} \le \frac{1}{j!} (\sum_{i=1}^{k} |a_i|)^j$$

and

$$\textstyle\sum_{i=m}^{\infty} \frac{z^i}{i!} \le \frac{1}{m!} \exp(z) z^m \text{ for } z \ge 0.$$

ADDENDUM 2.7. For every indicator function ψ we get

$$\left[\frac{1}{2(m+1)!} \exp(\textstyle\sum_{i=1}^{k} P_i(f_i^2)) \right]^{1/2} (\textstyle\sum_{i=1}^{k} P_i(f_i^2))^{(m+1)/2}$$

at the right-hand side of (2.6).

Hint. Use the equality

$$\sup_{\mathscr{A} \in \times_{i=1}^{k} \mathscr{A}_i} \left| (\times_{i=1}^{k} \nu_i)(A) - \int_A h_m \, d \times_{i=1}^{k} P_i \right| = \frac{1}{2} \int |h_k - h_m| \, d \times_{i=1}^{k} P_i$$

in the proof of Theorem 2.5.

3. Asymptotic expansion of the joint distribution of order statistics. Let M denote the exponential distribution (with the distribution function $G(x) = 1 - e^{-x}$ for $x \ge 0$). Let $\mathbb{B}$ denote the Borel-algebra of the set of all real numbers $\mathbb{R}$. The rth order statistic $Z_{r:n}: \mathbb{R}^n \to \mathbb{R}$ of a sample of size n is defined by $Z_{r:n}(x_1, \ldots, x_n) = z_{r:n}$ where $z_{1:n} \le \cdots \le z_{n:n}$ are the components of $(x_1, \ldots, x_n)$ arranged in the increasing order. The starting point of our calculations will be an asymptotic expansion of the distribution of a single order statistic $Z_{r:n}$ under M. Denote by P^*T the measure which is induced by the probability measure P and the measurable map T.

LEMMA 3.1. *For every $m \in \{0, 1, 2, \ldots\}$ there exists a constant $C_m > 0$ such that for every $r \in \{1, \ldots, n\}$*

$$\sup_{B \in \mathbb{B}} \left| M^{n*}\left[\left(\frac{(n-r)n}{r} \right)^{1/2} \left(Z_{r:n} - G^{-1}\left(\frac{r}{n}\right) \right) \right](B) - \int_B (1 + \textstyle\sum_{i=1}^{m} L_{r,n,i}) \, dN \right| \le C_m \left(\frac{n}{r(n-r)} \right)^{(m+1)/2}$$

where $L_{r,n,i}$ are polynomials of a degree equal to $3i$, $\int L_{r,n,i} \, dN = 0$, and the coefficients of $L_{r,n,i}$ are of order $O((n/(r(n-r)))^{i/2})$.

In particular,

$$L_{r,n,1}(x) = \frac{2n - r}{6(r(n-r)n)^{1/2}} x^3 - \left(\frac{n}{r(n-r)} \right)^{1/2} x \tag{3.2}$$

and

$$L_{r,n,2}(x) = \frac{1}{rn(n-r)} \left[\frac{x^6}{72} (2n - r)^2 - \frac{x^4}{24} (14n^2 - 10nr + r^2) + \frac{x^2}{2} n(2n - r) - \frac{1}{12} (n^2 - nr + r^2) \right].$$

PROOF. We shall only sketch the proof because the method is essentially the same as in Reiss (1976), Theorem 2.7: let Q denote the uniform distribution on (0, 1), and $P_{r,n} = Q^{n*}[(n^{3/2}/(r(n-r))^{1/2})(Z_{r:n} - r/n)]$. Firstly, an asymptotic expansion of the density of $P_{r,n}$ is proved which holds true over $A_{r,n} = [-\log(\min\{r, n-r+1\}), \log(\min\{r, n-r+1\})]$. Integrating over $A_{r,n}$ we get an asymptotic expansion of $P_{r,n}$ which holds uniformly over $\mathbb{B} \cap A_{r,n}$. An appropriate estimate of $P_{r,n}(\mathbb{R}\backslash A_{r,n})$ is obtained by using the exponential bound theorem. Combining both results we get an asymptotic expansion of $P_{r,n}$ uniformly over $\mathbb{B}$. An application of the probability integral transformation leads to Lemma 3.1.

We remark that a detailed proof of the step from the uniform to the exponential distribution in the proof of Lemma 3.1 is given in Reiss (1977), Section 16.

The results of Section 2 can be applied to the joint distribution of an appropriate set of differences of order statistics if the exponential distribution M is the underlying distribution. An asymptotic expansion of the distribution of a single difference can be obtained by combining Lemma 3.1 and the following

LEMMA 3.3. *Put* $Z_{0:n} = 0$. *For every* $r, s, n \in \mathbb{N} \cup \{0\}$ *with* $0 \le r < s \le n$

$$M^{n*}(Z_{s:n} - Z_{r:n}) = M^{n-r*}Z_{s-r:n-r}.$$

PROOF. Let π_i denote the ith projection from $\mathbb{R}^{s-r}$ to $\mathbb{R}$ for $i \in \{1, \ldots, s-r\}$. Applying the well-known formula

$$M^{n*}((Z_{r:n} - Z_{r-1:n})/(n-r+1))_{r=1}^{n} = M^n$$

we get

$$\begin{aligned}
M^{n*}(Z_{s:n} - Z_{r:n}) &= M^{n*}\left(\textstyle\sum_{i=1}^{s-r} (Z_{r+i:n} - Z_{r+i-1:n})\right)\\
= (M^{n*}(Z_{r+i:n} - Z_{r+i-1:n})_{i=1}^{s-r})^{*}\textstyle\sum_{i=1}^{s-r} \pi_i) &= M^{s-r*}\left(\textstyle\sum_{i=1}^{s-r}\dfrac{\pi_i}{n-r-i+1}\right)\\
= [\times_{i=1}^{s-r} (M^{n-r*}(Z_{i:n-r} - Z_{i-1:n-r})]^{*}(\textstyle\sum_{i=1}^{s-r} \pi_i)&\\
= M^{n-r*}\left(\textstyle\sum_{i=1}^{s-r} (Z_{i:n-r} - Z_{i-1:n-r})\right) &= M^{n-r*}Z_{s-r:n-r}.
\end{aligned}$$

Given $k \in \{1, \ldots, n\}$ and $0 = r_0 < \cdots < r_{k+1} = n$ let $\mathbf{r} := (r_1, \ldots, r_{k+1})$,

$$Q_{\mathbf{r}} := M^{n*}\left[\left(\frac{(n-r_{i-1})(n-r_i)^{1/2}}{r_i - r_{i-1}}\left(Z_{r_i:n} - Z_{r_{i-1}:n} - G^{-1}\left(\frac{r_i - r_{i-1}}{n - r_{i-1}}\right)\right)\right)\right]_{i=1}^{k},$$

and

$$L_{\mathbf{r},i,j} = L_{r_i - r_{i-1},\, n - r_{i-1},\, j}$$

where $L_{r,n,i}$ are the polynomials as given in Lemma 3.1.

THEOREM 3.4. *For every* $m \in \{0, 1, \ldots\}$

$$\sup_{B\in\mathbb{B}^k} |Q_{\mathbf{r}}(B) - (\times_{i=1}^{k} \nu_{i,m})(B)|$$
$$\le C_m \exp\left[2C_m \sum_{i=1}^{k}\left(\frac{n-r_{i-1}}{(r_i - r_{i-1})(n-r_i)}\right)^{(m+1)/2}\right] \sum_{i=1}^{k}\left(\frac{n-r_{i-1}}{(r_i-r_{i-1})(n-r_i)}\right)^{(m+1)/2}$$

where C_m *is the constant as given in Lemma* 3.1, *and* $\nu_{i,m}$ *is the signed measure with the* N-*density* $1 + \sum_{j=1}^{m} L_{\mathbf{r},i,j}$.

PROOF. Since $Z_{r_i:n} - Z_{r_{i-1}:n}$, $i = 1, \ldots, k$, are M^n-independent, the assertion follows at once from Corollary 2.2, Lemma 3.1 and Lemma 3.3.

We remark that

$$\sum_{i=1}^{k} \left(\frac{n - r_{i-1}}{(r_i - r_{i-1})(n - r_i)} \right)^{(m+1)/2} \leq 2 \sum_{i=1}^{k+1} (r_i - r_{i-1})^{-(m+1)/2}.$$

Applying Addendum 2.7 and Theorem 3.4 we obtain at once

COROLLARY 3.5. *For every* $m \in \{0, 1 \dots\}$ *and* $w \in \{0, \dots, k\}$

$$\sup_{B \in B^k} \left| Q_r(B) - \int_B \left(\sum_{u=0}^{w} \sum_{1 \leq i_1 < \dots < i_u \leq k} \prod_{s=1}^{u} \sum_{j=1}^{m} (L_{r,i_s,j} \circ \pi_{i_s})\right) dN^k \right|$$

$$\leq \sup_{B \in B^k} \left| Q_r(B) - (\times_{i=1}^{k} \nu_{i,m})(B) \right|$$

$$+ \left[\frac{1}{2(w+1)!} \exp\left(\sum_{i=1}^{k} \int (\textstyle\sum_{j=1}^{m} L_{r,i,j})^2 \, dN\right)^{1/2} \left(\sum_{i=1}^{k} \int (\textstyle\sum_{j=1}^{m} L_{r,i,j})^2 \, dN\right)^{(w+1)/2}.$$

In the particular cases of $m = 1$ and $w = 0, 1$ we obtain the following approximations.

$$\sup_{B \in B^k} | Q_r(B) - N^k(B) | \leq C_1 \exp[2C_1\rho_r]\rho_r + 3^{-1/2} \exp[\rho_r/3]\rho_r^{1/2}, \tag{3.6}$$

and

$$\sup_{B \in B^k} \left| Q_r(B) - \int_B (1 + \textstyle\sum_{i=1}^{k} L_{r,i,1} \circ \pi_i) \, dN^k \right| \leq (C_1 \exp[2C_1\rho_r] + \exp[\rho_r/3]/3)\rho_r \tag{3.7}$$

where

$$\rho_r := \sum_{i=1}^{k} \frac{n - r_{i-1}}{(r_i - r_{i-1})(n - r_i)}.$$

Notice that the error bound for the normal approximation to Q_r in (3.6) is better than that in Theorem 3.4 (applied to $m = 0$).

The corresponding results for the joint distribution of order statistics—in place of differences of order statistics—are obtained by straightforward transformations: let $g(x) = e^{-x}$ for $x > 0$, and

$$P_r := M^{n*}\left[n^{1/2} g\left(G^{-1}\left(\frac{r_i}{n}\right)\right)\left(Z_{r_i:n} - G^{-1}\left(\frac{r_i}{n}\right)\right)\right]_{i=1}^{k},$$

$$T_r := \left(n^{-1/2}(n - r_i) \sum_{j=1}^{i} \left(\frac{r_j - r_{j-1}}{(n - r_{j-1})(n - r_j)} \right)^{1/2} \pi_j \right)_{i=1}^{k},$$

and N_r the k-variate normal distribution with mean vector zero and covariances $(r_i/n)(1 - (r_j/n)$ for $1 \leq i \leq j \leq k$. Since $P_r = Q_r * T_r$ and $N_r = N^k * T_r$ it is easy to see that Theorem 3.4 and Corollary 3.5 hold true for P_r, N_r, $(\times_{i=1}^{k} \nu_{i,m}) * T_r$ and $L_{r,i,j} \circ S_{r,i}$ in place of Q_r, N^k, $\times_{i=1}^{k} \nu_{i,m}$ and $L_{r,i,j}$ where

$$S_{r,i} = \left(\frac{n(n - r_{i-1})(n - r_i)}{r_i - r_{i-1}} \right)^{1/2} \left(\frac{\pi_i}{n - r_i} - \frac{\pi_{i-1}}{n - r_{i-1}} \right)$$

with the convention that $\pi_0 \equiv 0$.

These calculations show that Corollary 3.5 is essentially the main result in Reiss (1975) in the particular case of the exponential distribution. An extension of Corollary 3.5 to other probability measures can be obtained by using the probability integral transformation as it was done in Reiss (1975) with the uniform distribution as a starting point. We remark that the theorem in Reiss (1975) additionally provides us with sharp error bounds in the case of probabilities of moderate deviation; on the other hand, those cases are excluded where $\rho_r \geq (\log n)^{-5}$. If $m \geq 2$, then Theorem 3.4 yields that $\times_{i=1}^{k} \nu_{i,m}$ is an approximation to Q_r in cases where Corollary 3.5 is not applicable for any m and w. This gain is achieved at the cost of the simplicity of the approximation. Whereas $\times_{i=1}^{k} \nu_{i,m}$ has a polynomial of degree 3km as density with respect to N^k, the degree of the corresponding polynomial in Corollary 3.5 does not depend on k.

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