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BROWNIAN MOTIONS ON THE HOMEOMORPHISMS OF THE PLANE[1]

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Let z denote a point of R_2. We study random flows $Z_{st}(z) \in R_2$, $0 \le s \le t < \infty$, $z \in R_2$, $Z_{tu}(Z_{st}(z)) = Z_{su}(z)$ for $s \le t \le u$. Such flows are called Brownian if Z is continuous in (s, t, z) and has appropriate spatial and temporal homogeneity properties and if $Z_{st}, Z_{uv}, \ldots$ are independent homeomorphisms of R_2 onto R_2 when $s \le t \le u \le v \le \ldots$. For a Brownian flow the coordinates of any k points are a $2k$-dimensional continuous Markov process, $k = 1, 2, \ldots$. If these processes are diffusions whose diffusion matrices have bounded continuous derivatives of order ≤ 2 (i.e., are C^2-bounded), then the diffusion matrices are necessarily obtained in a certain way from the covariance tensor of the field of infinitesimal displacements. A converse is given in the incompressible isotropic case: given a C^2-bounded covariance tensor of an isotropic solenoidal R_2-valued field in R_2, there exists a corresponding incompressible isotropic Brownian flow.

1. Introduction. Let H be the space of homeomorphisms of R_2 with itself, with the topology of uniform convergence on compact sets. We shall find and study H-valued stochastic processes $Z_{s,t} = Z_{st}$, $0 \le s \le t < \infty$ having the following properties. (a) Z is continuous in (s, t); (b) $Z_{tu} \circ Z_{st} = Z_{su}$, $0 \le s \le t \le u < \infty$; (c) if $s_1 \le t_1 \le s_2 \le t_2 \le \ldots$, then $Z_{s_1t_1}, Z_{s_2t_2}, \ldots$ are independent. We call Z a *Brownian motion on H*. The reason for the name is evident if we consider a Wiener process X_t, put $X_{st} = X_t - X_s$, and then compare Z_{st} with X_{st} and $\circ$ with $+$. Thus we have a homogeneous "additive" or "infinitely divisible" process, which may be called "Brownian" because the sample functions are continuous. For a discussion of homogeneous processes on groups or semigroups see, e.g., Grenander [7a]. The recent work of Bucan [1a] on random semigroups of linear operators may also be cited, although its methods and results are quite different from those of the present paper.

If $z \in R_2$, we write $z = (x^1, x^2)$ or (x, y). Letting $Z_{st}(z)$ be the value of Z_{st} at z, we write $Z_{st}(z) = (X^1_{st}(z), X^2_{st}(z)) = (X_{st}(z), Y_{st}(z))$ in coordinate form. We also put $Z_{0t} = Z_t$, $X^1_{0t} = X^1_t$, etc. From (b) we see that Z is a *flow* in R_2; that is, $Z_{tu}(Z_{st}(z)) = Z_{su}(z)$, where we think of $Z_{st}(z)$ as the position at time t of the point that was at z at time s. The flow Z derived from a Brownian motion on H is called a *Brownian flow*; this term is defined in its own right in Section 2. Note that (a) implies $Z_{st}(z)$ is continuous jointly in (s, t, z).

NOTE. The word "flow" applied to a process $(Z_{st}(z))$ does not by itself imply that Z_{st} is surjective.

We shall consider only *homogeneous flows*, without always saying so; that is, the processes $(Z_{s+h,t+h}(z + z_0) - z_0, 0 \le s \le t < \infty, z \in R_2)$ have the same law for every $h \ge 0$ and $z_0 \in R_2$. This implies in particular that $Z_{st}(z) - z$ is stationary, considered as a function of z.

Let Z be a Brownian flow, let $z_1, \ldots, z_k$ be points of R_2, and fix $s \ge 0$. The process $(Z_{st}(z_1), \ldots, Z_{st}(z_k), t \ge s)$, which has continuous paths in R_{2k}, is called a *finite-set* process (more specifically a k-point process) of Z. We shall see that each finite-set process is Markovian.

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Leaving unanswered the question whether there are Brownian flows whose finite-set processes are not diffusions, we treat only the case where they *are* diffusions, with diffusion coefficients that are C^2-bounded (i.e., have bounded continuous partial derivatives of order ≤ 2.) In this case, because of spatial homogeneity, we may as well assume zero drift.

The law of a k-point process is then determined by a $2k \times 2k$ diffusion matrix B^k. The matrix B^k determines a law for k points whose marginal law for any subset of $k-1$ points must be that determined by B^{k-1}; in other words, the sequence (B^k) has the property that we shall call *consistency*. It is easy to determine all possible consistent sequences with C^2-bounded elements. Each such sequence is determined, though (2.3) and (2.5) in terms of a matrix function $(b_{pq}(z))$, p, $q = 1, 2$ where

$$b_{pq}(z) = \lim_{t \downarrow 0} \mathscr{E}(X^p_{0t}(0))(X^q_{0t}(z) - x^q)/t, \qquad z = (x^1, x^2). \tag{1.1}$$

The class of functions (b_{pq}) is exactly the class of C^2-bounded covariance tensors of homogeneous R_2-valued fields in R_2.

Conversely, suppose (b_{pq}) *is a* C^2-bounded convariance tensor of a homogeneous R_2-valued field in R_2. Then (2.3) and (2.5) define a consistent sequence of diffusion matrices, and the question is whether the sequence determines a Brownian flow. One can at least begin the construction (Section 2). However, complete results have been obtained only in case the flow is isotropic and incompressible (i.e., preserves Lebesgue measure.) The main result, contained in Theorem 16.6 is as follows.

Let (b_{pq}) be a C^2-bounded covariance tensor as above. Suppose also that (b_{pq}) is *solenoidal,* meaning it satisfies the divergence condition (3.5), and is *isotropic* (see Section 4). Then there is a unique law for an incompressible isotropic Brownian flow whose finite-set processes are determined by (2.3) and (2.5). Every incompressible isotropic Brownian flow corresponds to such a covariance (b_{pq}).

In the incompressible isotropic case, the one-point paths are just 2-dimensional Brownian motion, and the distance between two points is a diffusion, which helps simplify the analysis.

The proof of the existence of a continuous flow Z_{st} with independent "increments" is simple. The main technical problem is then to establish the homeomorphic property. For this we approximate our flow Z by a sequence of homeomorphic flows Z^n, where, for fixed s and t, the sequence of inverse functions $(Z^n_{st})^{-1}$ is tight. From Lemma 13.1, we then deduce that Z is homeomorphic. (It is easy to show that two given points have probability zero of meeting.)

The approximation is carried out in stages. We first prove the homeomorphic property for a special class of Brownian flows (the "stream-function case") using a sequence (Z^n) of "stirring processes" as the approximating sequence. We then extend the results to Theorem 16.6.

The stirring processes used here are similar to the stirring processes in R_1 introduced by W. Lee [13] as analogues of the symmetric case of F. Spitzer's "simple exclusion" in Z_1; see [15]. The use of stirring processes gives a picture of a Brownian flow as a limiting case of a vortex model for homogeneous turbulence, where the vortices come and go very rapidly instead of remaining stable for some time. For a vortex model of turbulence see Chorin [2], especially page 113. The field of short-time displacements of a Brownian flow plays a role like the velocity field in models of homogeneous turbulence. An example is condition (3.5) for incompressibility.

In the latter part of the paper it is shown that the length of a rectifiable curve is a submartingale under an incompressible isotropic Brownian flow. The paper concludes with a remark about duality.

Notation. $\mathscr{B}(S)$ is the family of Borel sets in the topological space S. The measurable sets in a Euclidean space or an interval of R_1 are always the Borel sets. C^n_b is the set of real functions on an indicated Euclidean space that have bounded continuous partial derivatives of order $\leq n$; functions in C^2_b are called C^2-bounded. C^n_0 are the functions in C^n_b with compact support. C_R^∞ (often called $\mathcal{S}$) are the functions in C_b^∞, which, with all derivatives, go to zero at ∞ more rapidly than any rational function. $C_0^n(R_m)$ indicates that the domain is R_m. C^∞ is the class of infinitely differentiable functions.

If $(U_1(z), U_2(z), z \in R_2)$ is a homogeneous random field, U_1 and U_2 real, then the covariance tensor (b_{pq}) is given by $b_{pq}(z) = \mathscr{E}\, U_p(z')U_q(z' + z), p, q = 1, 2$.

The symbol $|S|$ will denote the cardinality of the set S, the Lebesgue measure of the Borel set S in a Euclidean space, or the length of the curve S, depending on the context.

$C(A, B)$ is the set of continuous mappings from A into B. C_{22} is the space of continuous mappings $R_2 \to R_2$ with the topology of uniform convergence on compact sets.

θ denotes a quantity ≤ 1 in modulus. c denotes a constant not depending on the parameters of the formula in which it appears.

2. Brownian flows; consistent sets of diffusions. By a *Brownian flow* in R_2 is meant a measurable mapping

(2.1) $$(s, t, z, \omega) \to Z_{st}(z, \omega), \qquad 0 \leq s \leq t < \infty, z \in R_2, \omega \in \Omega$$

with values in R_2; ω is from a probability space $(\Omega, \mathscr{F}, P)$. It is assumed that Z is homogeneous (see Section 1) and that the following properties hold.

(2.2)(a) For a.e. ω: $Z_{tu}(Z_{st}(z, \omega), \omega) = Z_{su}(z, \omega)$ for all z and all $0 \leq s \leq t \leq u < \infty$.

(2.2)(b) For a.e. ω: $Z_{st}(z, \omega)$ is continuous in (s, t, z), and $Z_{st}(\cdot, \omega)$ is a homeomorphism of R_2 onto itself for all $0 \leq s \leq t < \infty$.

(2.2)(c) For $s \leq t$ let $\mathscr{T}_{st}$ be the σ-field generated by $Z_{uv}(z)$, $s \leq u \leq v \leq t$, $z \in R_2$. Then $\mathscr{T}_{st}, \mathscr{T}_{uv}, \ldots$ are independent if $s \leq t \leq u \leq v \leq \ldots$.

A Brownian flow Z determines a Brownian motion on H in the sense of Section 1 and conversely.

Let $Z_t = Z_{0t} = (X_t^1, X_t^2) = (X_t, Y_t)$. If (2.2)(a–c) are true and if $p_k(t; z_1, \ldots, z_k; B_1 \times \cdots \times B_k) = P\{Z_t(z_1) \in B_1, \ldots, Z_t(z_k) \in B_k\}$ then

$$\begin{aligned} &P\{Z_{t+h}(z_1) \in B_1, \ldots, Z_{t+h}(z_k) \in B_k \mid \mathscr{T}_{0t}\} \\ &= P\{Z_{t,t+h}(Z_t(z_i)) \in B_i, 1 \leq i \leq k \mid \mathscr{T}_{0t}\} \\ &= p_k(h; Z_t(z_1), \ldots, Z_t(z_k); B_1 \times \cdots \times B_k), \end{aligned}$$

whence *the k-point process* $(Z_t(z_1), \ldots, Z_t(z_k), t \geq 0)$ *is Markov.*

Let us find the most general possible Brownian flow whose finite set processes are diffusions with C^2-bounded diffusion coefficients. The drift must be the same at all points of R_2 and we may take it to be 0.

Let B^k be the diffusion matrix for the motion of k points. Then B^k is $2k \times 2k$ and has the form (with the ordering $x_1^1 x_1^2 \ldots x_k^1 x_k^2$)

(2.3) $$B^k = \begin{pmatrix} B_{11}^k & \cdots & B_{1k}^k \\ B_{k1}^k & \cdots & B_{kk}^k \end{pmatrix}$$

where B_{ij}^k is a 2×2 matrix $(B_{ij}^k(p, q))$, $p, q = 1, 2$, with (putting $z_i = (x_i^1, x_i^2)$)

(2.4) $$B_{ij}^k(p, q) = \lim_{t \downarrow 0} \mathscr{E} \frac{(X_t^p(z_i) - x_i^p)(X_t^q(z_j) - x_j^q)}{t}.$$

Note that the right side of (2.4) is the same for all $k \geq 2$ (since one flow governs all points), and depends on $z_j - z_i$. Hence we have

(2.5) $$B_{ij}^k = \begin{pmatrix} b_{11}(z_j - z_i) & b_{12}(z_j - z_i) \\ b_{21}(z_j - z_i) & b_{22}(z_j - z_i) \end{pmatrix}.$$

The condition of nonnegative definiteness for the B^k is

(2.6) $$\sum_{p,q=1}^{2} \sum_{m,n=1}^{k} b_{pq}(z_m - z_n) c_m^p c_n^{q*} \geq 0$$

for complex c_m^p. This is also the condition that (b_{pq}) is the correlation tensor of an R_2-valued homogeneous random field in R_2. (See Section 1, Notation.) Note $b_{pq}(z) = b_{qp}(-z)$.

Hence *every Brownian flow with* C^2*-bounded diffusion matrices is based on a* C^2*-bounded covariance tensor through* (2.5).

We need the notion of a *consistent sequence of diffusion matrices*. For each $k = 1, 2, \ldots$ let B^k (not yet the above matrix) be a $2k \times 2k$ C^2-bounded diffusion matrix for the coordinates of k points diffusing in R_2 with zero drift. We call the B^k *consistent* if, having used B^{k+r} to determine the law of paths of points initially at $z_1, \ldots, z_{k+r}$, the law of any k-point subset, say the points initially at $z_1, \ldots, z_k$, is the same as that determined by B^k. Assuming that the motion is spatially and temporally homogeneous and that the motion of k points does not depend on the order of labeling, we see that B^k must have the form given by (2.3) and (2.5).

For each k-tuple $z_1, \ldots, z_k$, the matrix B^k determines the law of k continuous trajectories $(Z.(z_1), \ldots, Z.(z_k))$. Considering a trajectory as a point in the Polish space E of continuous mappings from $[0, \infty)$ into R_2, with the topology of uniform convergence on compact sets, we see that consistency, as defined above, is exactly the consistency condition for the Kolmogorov extension theorem. Hence an E-valued random field $(Z.(z), z \in R_2)$ is defined and hence a random field $(Z_t(z), t > 0, z \in R_2)$.

Let (b_{pq}) be a C^2-bounded covariance tensor and let matrices B^k be defined by (2.3) and (2.5). From Stroock and Varadhan [16], B^k is the matrix of a diffusion with zero drift. Let $\mathscr{A}_k$ be the generator corresponding to B^k. If $f(z_1, \ldots, z_k, z_{k+1}, \ldots, z_{k+r}) = f_0(z_1, \ldots, z_k)$, where f_0 is in C_b^2, then $\mathscr{A}_{k+r} f = \mathscr{A}_k f_0$. Since $\mathscr{A}_{k+r}$ and $\mathscr{A}_k$ both have unique solutions to the martingale problem from [16], the distribution determined by B^{k+r} for the paths starting at $z_1, \ldots, z_k$ solves the martingale problem for $\mathscr{A}_k$, and therefore the B^k are consistent. Hence a random field $(Z_t(z), t \geq 0, z \in R_2)$ is determined.

Let us suppose this field has a version that is continuous in z. Then for each t, $Z_t = Z_t(\cdot)$ is a random point in C_{22}, the space of continuous mappings $R_2 \to R_2$ with the topology of uniform convergence on compacta. Let Q_t be the distribution of Z_t.

(2.7). DEFINITION. Let Q' and Q'' be Borel probability measures in C_{22}. Then $Q' * Q''$ denotes the distribution of $\xi \circ \eta$ where ξ and η are independent points of C_{22} with the respective distributions Q' and Q''.

NOTE. $Q' * Q''$ is continuous in the pair (Q', Q'') (weak convergence).

(2.8). LEMMA. *Assume the field* $(Z_t(z))$, *constructed as above from a consistent set of diffusions, is continuous in* z. *Let* Q_t *be the distribution of* Z_t *in* C_{22}. *Then* $Q_{t+s} = Q_t * Q_s$, $s, t \geq 0$.

PROOF. Let $\varphi(z_1, \ldots, z_k)$ be bounded and Borel measurable. Let $p_k(t; z_1, \ldots, z_k; .)$ be the distribution of $Z_t(z_1), \ldots, Z_t(z_k))$. Then

(2.9) $$\int_{\zeta \in C_{22}} Q_t\,(d\zeta)\varphi(\zeta(z_1), \ldots, \zeta(z_k)) = \int p_k(t; z_1, \ldots, z_k; dz_1' \ldots dz_k')\varphi(z_1', \ldots, z_k').$$

Fixing $z_1, \ldots, z_k$ and putting $f(\zeta) = \varphi(\zeta(z_1), \ldots, \zeta(z_k))$, we have, using (2.9)

$$\int Q_{t+s}\,(d\zeta) f(\zeta) = \int p_k(t+s; z_1, \ldots, z_k; dz_1' \ldots dz_k')\cdot\varphi(z_1', \ldots, z_k')$$

$$= \int p_k(t; z_1, \ldots, z_k; dz_1'' \ldots dz_k'') \int Q_s\,(d\zeta)\varphi(\zeta(z_1''), \ldots, \zeta(z_k''))$$

$$= \int Q_s\,(d\zeta) \int Q_t\,(d\zeta')\varphi(\zeta(\zeta'(z_1)), \ldots, \zeta(\zeta'(z_k)))$$

$$= \int Q_s\,(d\zeta) \int Q_t\,(d\zeta')f(\zeta \circ \zeta') = \int_{C_{22}} Q_s * Q_t(d\zeta)f(\zeta).$$

The lemma follows from this. □

(2.10). LEMMA. *Under the conditions of Lemma* (2.8) *there exists a* C_{22}*-valued random field* Z_{st}, $0 \le s \le t < \infty$ *such that* $s \le t \le u \le v \le \ldots$ *implies* $Z_{st}, Z_{uv}, \ldots$ *are independent.* Z_{st} *has the distribution* Q_{t-s}. *For fixed* $s \le t \le u$, $Z_{tu} \circ Z_{st} = Z_{su}$ a.s.

REMARK. The continuity stipulated in Lemma 2.8 will be proved only for incompressible isotropic flows, but probably holds much more generally.

PROOF. The proof is essentially the same as in the construction of real additive processes.

First we construct a set of finite-dimensional distributions for a C_{22}-valued random function Z_{st} such that $Z_{tu} \circ Z_{st} = Z_{su}$ a.s. if $0 \le s \le t \le u < \infty$. For example if $s \le t \le u \le v$, the joint distribution of Z_{su} and Z_{tv} is that of $Z_{tu} \circ Z_{st}$ and $Z_{uv} \circ Z_{tu}$, where Z_{st}, Z_{tu}, Z_{uv} are independent. From Lemma 2.8 we see that the finite-dimensional distributions are uniquely determined and consistent. Since the Kolmogorov extension theorem is valid for processes with values in C_{22} (a Polish space), the lemma follows. □

We define the field $Z_{st}(z)$ in the obvious way from Z_{st}. We shall later show continuity in (s, t, z) for the incompressible isotropic case.

REMARK. Homogeneity in space and time of the flow Z of Lemma 2.10 follows from the spatial and temporal homogeneity properties of the finite-set processes.

I am indebted to J.C. Octoby (private communication) for an example of a continuous incompressible map of R_2 onto itself that is not a homeomorphism.

3. Incompressibility and reversibility. We consider the relation between incompressibility (preservation of Lebesgue measure) of the flow of Lemma 2.10 and reversibility of the k-point processes with respect to $2k$-dimensional Lebesgue measure. We always assume zero drift. As before let $Z_t = Z_{0t}$. Let $\varphi\colon R_2 \to R_1$ be in C_0^∞. From the moment inequalities in Friedman [7], page 107, the integral $\int_{R_2} \varphi(Z_t(z))\, dz$ exists and has a finite second moment. From translation invariance we have

$$\mathscr{E}\int_{R_2} \varphi \circ Z_t\, dz = \int_{R_2} \varphi\, dz. \tag{3.1}$$

Then, letting $\varphi \otimes \varphi(z, z') = \varphi(z)\varphi(z')$,

$$\mathscr{E}\left(\int_{R_2} (\varphi \circ Z_t - \varphi)\, dz\right)^2 = \int\int_{R_2 \times R_2} (T_t^{(2)}\, \varphi \otimes \varphi - \varphi \otimes \varphi)\, dz\, dz', \tag{3.2}$$

where $(T_t^{(k)})$ is the semigroup operator for the motion of k points. We let $\mathscr{A}_k$ be the generator of $(T_t^{(k)})$.

We say that (Z_{st}) is *incompressible* if (3.2) is 0 for each $t \ge 0$ and each $\varphi \in C_0^\infty$. This implies, for fixed s and t, that the mapping Z_{st} a.s. preserves Lebesgue measure. If (3.2) is 0 for each t then, again using [7], page 107, we find, letting $t \downarrow 0$, that

$$\int\int_{R_2 \times R_2} \mathscr{A}_2\, \varphi \otimes \varphi\, dz\, dz' = 0, \tag{3.3}$$

where

$$\mathscr{A}_k\psi(z_1, \ldots, z_k) = \frac{1}{2}\sum_{i,j=1}^{k}\sum_{p,q=1}^{2} b_{pq}(z_j - z_i)\frac{\partial^2\psi}{\partial x_i^p \partial x_j^q}(z_1, \ldots, z_k). \tag{3.4}$$

It can be shown from (3.3) that (b_{pq}) satisfies

$$\sum_p \frac{\partial b_{pq}(x^1, x^2)}{\partial x^p} = \sum_q \frac{\partial b_{pq}(x^1, x^2)}{\partial x^q} = 0. \tag{3.5}$$

This is not surprising because the same condition is satisfied by the correlation tensor of velocities in the usual models for incompressible turbulent flow; see Monin and Yaglom [14], vol. 2, page 26. If (3.5) holds, necessarily $b_{pq}(z) = b_{qp}(z)$.

From (3.5) we deduce further that $(\mathscr{A}_k\psi_1, \psi_2) = (\psi_1, \mathscr{A}_k\psi_2)$ for $\psi_1, \psi_2 \in C_0^\infty(R_{2k})$, where $(f, g) = \int_{R_{2k}} fg\, dz_1 \cdots dz_k$. From this we can show that if R_λ^k and T_t^k are the resolvent and semigroup operators for $\mathscr{A}_k$ then $(R_\lambda^k f, g) = (f, R_\lambda^k g)$ and $(T_t^k f, g) = (f, T_t^k g)$ for $f, g \in C_0^\infty$. That is, *each k-point process is reversible with respect to Lebesgue measure.* From reversibility we get, taking limits, $\int_{R_{2k}} T_t^k f\, dz_1 \cdots dz_k = \int_{R_{2k}} f\, dz_1 \cdots dz_k$ for bounded measurable $f \geq 0$.

Conversely, assuming reversibility for the two point process, we find from (3.2) that we have incompressibility and hence (3.5). To summarize:

(3.6). THEOREM. *Let Z_{st} be as in Lemma* (2.10). *Suppose the* 2*-point diffusion matrix is* C^2*-bounded and assume zero drift. Then the following conditions are equivalent.*
(3.6.1). *The flow is incompressible.*
(3.6.2). *Condition* (3.5) *holds.*
(3.6.3) *The k-point Markov processes are reversible with respect to Lebesgue measure for each* $k = 1, 2, \ldots$.

(3.7). COROLLARY. *If* (3.6.1)–(3.6.3) *hold and if Z_{st}* is a.s. *a homeomorphism, then for each s and t, $s \leq t$, the random homeomorphisms Z_{st} and $(Z_{st})^{-1}$ have the same distribution.*

PROOF. Let f and $g \in C_0^\infty(R_{2k})$. Then

$$\int_\Omega dP \int \cdots \int_{R_{2k}} f(Z_t^{-1}(z_1), \ldots, Z_t^{-1}(z_k))g(z_1, \ldots, z_k)\, dz_1 \cdots dz_k$$

$$= \int_\Omega dP \int \cdots \int f(z_1', \ldots, z_k')g(Z_t(z_1'), \ldots, Z_t(z_k'))\, dz_1' \cdots dz_k',$$

because the random transformation $z_i' = Z_t^{-1}(z_i)$, $1 \leq i \leq k$ preserves Lebesgue measure in R_{2k}. Since the symbols f and g can be interchanged in the last integral, the corollary follows. □

4. Spectral formulas: incompressible and isotropic case. For an R_2-valued field $(U_1(z), U_2(z))$, putting $b_{pq}(z) = \mathscr{E}U_p(z_1)U_q(z_1 + z)$, we have

$$b_{pq}(z) = \int_{R_2} e^{iz\cdot\lambda}\, dF_{pq}(\lambda),\ \lambda = (\lambda_1, \lambda_2), \tag{4.1}$$

where the spectral matrix (dF_{pq}) is finite and Hermitian nonnegative definite. We assume b_{pq} is C^2-bounded, which is true iff the F_{pq} have finite second moments.

We assume that (b_{pq}) is *solenoidal* (i.e., satisfies (3.5)) and isotropic. (Note: because of the solenoidal property, rotational isotropy also implies invariance under reflection.) Then (dF_{pq}) has the following form in polar coordinates (r, φ).

$$dF_{11} = \sin^2 \varphi \frac{d\varphi}{2\pi} dM(r),$$

(4.2) $$dF_{12} = -\sin \varphi \cos \varphi \frac{d\varphi}{2\pi} dM(r) = dF_{21},$$

$$dF_{22} = \cos^2 \varphi \frac{d\varphi}{2\pi} dM(r),$$

where M is a finite measure in $[0, \infty)$, and where $\int_0^\infty r^2\, dM(r) < \infty$ because of C^2-boundedness. The covariance properties can be expressed in terms of $f_L(\rho) = b_{11}(\rho, 0)$ and $f_N(\rho) = b_{22}(\rho, 0)$, using (4.7) below. We have

(4.3) $$f_L(\rho) = -\int_0^\infty \frac{J_0'(r\rho)}{r\rho} dM(r) = \frac{1}{2}\int dM - \frac{\rho^2}{16}\int r^2\, dM + o(\rho^2),$$

(4.4) $$f_N(\rho) = -\int_0^\infty J_0''(r\rho)\, dM(r) = \frac{1}{2}\int dM - \frac{3\rho^2}{16}\int r^2\, dM + o(\rho^2).$$

From (4.3) and (4.4) we find $f_N(\rho) = d(\rho f_L(\rho))/d\rho$. Here J_0 is the Bessel function of order 0; see [17], Chapter 17, 17.1, 17.11, and 17.23.

The relations (4.2) can be derived from (3.5), the assumption that the b_{pq} are twice continuously differentiable, and the assumption of rotational invariance. They can also be derived from Theorem 12 of Yaglom [18], using the statement following (4.42) of [18] about the solenoidal case. In our case invariance under rotations implies invariance under reflections, which is not true for nonsolenoidal R_2-valued fields. See Remark 3 of Section 4 of [18].

In the special case where M is concentrated at 0, we have $f_L = f_N = b_{11} = b_{22} =$ constant, and $b_{12} = 0$.

Since for all complex z we have $|\frac{1}{2} + J_0'(z)/z| \le c_1 |z|^2$, $|\frac{1}{2} + J_0''(z)| \le c_1 |z|^2$, we find

(4.5) $$|f_L(0) - f_L(\rho)| \le c_1\rho^2 \int_0^\infty r^2\, dM,$$

(4.6) $$|f_N(0) - f_N(\rho)| \le c_1\rho^2 \int_0^\infty r^2\, dM,$$

where c_1 does not depend on ρ or M. Unless M is concentrated at 0, $|f_L(\rho)|$ and $|f_N(\rho)|$ are $< \frac{1}{2}\int dM$ if $\rho > 0$. Note the formulas ($\rho^2 = x^2 + y^2$)

$$b_{11}(x, y) = \frac{x^2 f_L(\rho) + y^2 f_N(\rho)}{\rho^2},$$

(4.7) $$b_{12}(x, y) = xy(f_L(\rho) - f_N(\rho))/\rho^2,$$

$$b_{22}(x, y) = b_{11}(y, x).$$

Compare (12.29), page 39 of [14] for 3 dimensions.

In an important special case there is a *stream function* ψ: $R_2 \Rightarrow R_1$, circularly symmetric about (0, 0), such that

$$
\begin{aligned}
dM(r) &= r^3 h(r)\, dr/2\pi, \qquad r \geq 0,\\
h(|\lambda|) &= (\hat{\psi}(\lambda))^2 \qquad (4.8)\\
\hat{\psi}(\lambda) &= \int_{R_2} e^{iz\cdot\lambda}\psi(z)\, dz \text{ (real).}
\end{aligned}
$$

ψ will be in either C_0^4 or C_R^∞ in our applications. Then

$$
\begin{aligned}
f_L(\rho) &= -\frac{1}{2\pi\rho}\int_0^\infty r^2 h(r) J_0'(r\rho)\, dr, \qquad (4.9)\\
f_N(\rho) &= -\frac{1}{2\pi}\int_0^\infty r^3 h(r) J_0''(r\rho)\, dr, \qquad \rho > 0.
\end{aligned}
$$

Letting * denote convolution, put

$$
\begin{aligned}
B(z) &= \frac{1}{4\pi^2}\int e^{iz\cdot\lambda}(\hat{\psi}(\lambda))^2\, d\lambda \qquad (4.10)\\
&= \psi * \psi(z) = \frac{1}{2\pi}\int_0^\infty J_0(r|z|) r h(r)\, dr.
\end{aligned}
$$

Then

$$
\begin{aligned}
b_{11}(z) &= -\frac{\partial^2 B}{\partial y^2},\ b_{12}(z) = \frac{\partial^2 B}{\partial x \partial y}, \qquad (4.11)\\
b_{22}(z) &= -\frac{\partial^2 B}{\partial x^2}.
\end{aligned}
$$

Normalization. Since $f_L(0) = f_N(0) = \frac{1}{2}\int_0^\infty dM$, we shall often take

$$\tfrac{1}{2}\int_0^\infty dM = 1 \qquad (4.12)$$

to simplify the formulas. In case there is a stream function ψ, (4.12) becomes

$$\frac{1}{8\pi^2}\int_{R_2}(\lambda_1^2 + \lambda_2^2)(\hat{\psi})^2\, d\lambda = 1 = \frac{1}{4\pi}\int_0^\infty r^3 h(r)\, dr. \qquad (4.13)$$

5. Diffusion of two points. Consider the 4 dimensional diffusion of the coordinates of 2 points in R_2 determined by the matrix B^2 given by (2.3)–(2.5) with $k = 2$ for the incompressible isotropic case, where b_{pq} is as in Section 4.

Let $\mathscr{A}_2$ be the corresponding generator. Let ρ_t be the distance betwen the two points at time t. The formal generator for ρ_t, determined by the action of $\mathscr{A}_2$ on functions of $(x^1 - x^2)^2 + (y^1 - y^2)^2 = \rho^2$, is, assuming (4.12),

$$\mathscr{A}\varphi(\rho) = (1 - f_L(\rho))\varphi''(\rho) + \frac{1 - f_N(\rho)}{\rho}\varphi'(\rho), \qquad \rho \geq 0. \qquad (5.1)$$

We extend to $-\infty < \rho < \infty$, putting $f_L(\rho) = f_L(|\rho|)$, $f_N(\rho) = f_N(|\rho|)$. $1 - f_L(\rho)$ has bounded continuous first and second derivatives in R_1 and hence (Friedman [7], page 129) $\sqrt{1 - f_L}$ satisfies a uniform Lipschitz condition. Likewise $(1 - f_N(\rho))/\rho$ satisfies a uniform Lipschitz condition. Hence, using the martingale result of Stroock and Varadhan [16], Theorem 2.3,

ρ_t is a diffusion in $(-\infty, \infty)$ with generator (5.1) if $\varphi \in C_0^\infty$. Since $1 - f_L(\rho)$ and $1 - f_N(\rho)$ are $O(\rho^2)$ near $\rho = 0$, 0 is inaccessible and ρ_t on $(0, \infty)$ satisfies the stochastic differential equation

$$d\rho = 2^{1/2}(1 - f_L(\rho))^{1/2}\, dW + \frac{1 - f_N(\rho)}{\rho}\, dt, \qquad \rho > 0 \tag{5.2}$$

where W is Wiener. The corresponding equation for ρ^γ, $\gamma > 0$ is

$$d\rho^\gamma = 2^{1/2}\gamma\rho^{\gamma-1}(1 - f_L(\rho))^{1/2}\, dW + \gamma\rho^{\gamma-2}[1 - f_N(\rho) + (\gamma - 1)(1 - f_L(\rho))]\, dt. \tag{5.3}$$

Taking expectations in the integrated form of (5.3) and using (4.5) and (4.6) we get

$$\begin{aligned}\mathscr{E}\rho_t^\gamma &= \rho_0^\gamma + \gamma \int_0^t \mathscr{E}\rho_s^\gamma \frac{1 - f_N(\rho_s) + (\gamma - 1)(1 - f_L(\rho_s))}{\rho_s^2}\, ds \\ &\le \rho_0^\gamma + \gamma(1 + |\gamma - 1|)c_1 \int_0^t \mathscr{E}\rho_s^\gamma\, ds \int_0^\infty r^2\, dM\end{aligned}$$

whence

$$\mathscr{E}\rho_t^\gamma \le \rho_0^\gamma \exp\left(tC_\gamma c_1 \int_0^\infty r^2\, dM\right), \tag{5.4}$$

$\gamma > 0$ where C_γ depends only on γ and c_1 appears in (4.5) and (4.6).

We also record that if the vector difference between two points in our flow is denoted by (u_t, v_t), then (u_t, v_t) is a diffusion with the operator $\mathscr{A}'$:

$$\mathscr{A}'\varphi(u, v) = (1 - b_{11}(u, v))\frac{\partial^2\varphi}{\partial u^2}(u, v) + (1 - b_{22}(u, v))\frac{\partial^2\varphi}{\partial v^2}(u, v) - 2b_{12}(u, v)\frac{\partial^2\varphi}{\partial u \partial v}. \tag{5.5}$$

6. Continuity properties of $Z_{st}(z)$, incompressible isotropic case. In this section we assume that (b_{pq}) is C^2-bounded, solenoidal, and isotropic and, hence, given by (4.1)–(4.2). Let B^k be given by (2.3) and (2.5) and let $Z_t(z)$ be a version of the random field constructed from the B^k in Section 2.

(6.1). LEMMA. *For fixed t the random field $(Z_t(z))$ has a continuous version.*

PROOF. Take any $\gamma > 2$ in (5.4) and use Corollary 1 to Theorem 1 of [11].

(6.2). LEMMA. *Let (Z_{st}) be the C_{22}-valued random field of Lemma* (2.10). *Then the corresponding field $(Z_{st}(z), 0 \le s \le t < \infty, z \in R_2)$ has a continuous version that is a flow.*

(Recall that a flow need not be subjective.)

PROOF. From Lemma 2.10, it is sufficient to prove continuity. Suppose $0 \le s, t \le 1$. From (5.4)

$$\mathscr{E}|Z_{st}(z') - Z_{st}(z)|^{16} \le c|z - z'|^{16}. \tag{6.3}$$

Next, if $s < s' < t < t'$,

$$\mathscr{E}|Z_{st}(z) - Z_{s't'}(z)|^{16} \le c(\mathscr{E}|Z_{st}(z) - Z_{s't}(z)|^{16} + \mathscr{E}|Z_{s't}(z) - Z_{s't'}(z)|^{16}).$$

Now

$$\begin{aligned}\mathscr{E}|Z_{st}(z) - Z_{s't}(z)|^{16} &= \mathscr{E}\mathscr{E}\{|Z_{s't}(Z_{ss'}(z)) - Z_{s't}(z)|^{16} \mid Z_{ss'}(z)\} \\ &\le (\text{from (5.4)})\ c\mathscr{E}|Z_{ss'}(z) - z|^{16} \le c(s' - s)^8,\end{aligned}$$

since $Z_{ss'}(z)$ is Wiener for fixed z. Also

$$\mathscr{E}\,|Z_{s't}(z) - Z_{s't'}(z)|^{16} = \mathscr{E}\,\mathscr{E}\{|Z_{s't}(z) - Z_{tt'}(Z_{s't}(z))|^{16} \,\big|\, Z_{s't}(z)\} \leq c(t'-t)^8.$$

Hence we have

$$(6.4) \qquad \mathscr{E}\,|Z_{st}(z') - Z_{s't'}(z)|^{16} \leq c((s'-s)^8 + (t'-t)^8 + |z'-z|^{16}), \qquad s < s' < t < t'.$$

Similar arguments show that (6.4) holds in all cases. Applying Theorem 1 of [11], with $\alpha = 16$, $\varphi(s, t, x, y) = s^8 + t^8 + (x^2 + y^2)^8$, $\epsilon_k^{(1)} = \epsilon_k^{(2)} = (\delta_1)^k$, $\epsilon_k^{(3)} = \epsilon_k^{(4)} = (\delta_2)^k$, where $2^{-1/4} < \delta_1 < 1$, $2^{-3/4} < \delta_2 < 1$, and s, t, x, y correspond respectively to t_1, t_2, t_3, and t_4 in [11], we find that $Z_{st}(z)$ has a continuous version. It follows that C_{22}-valued process $(Z_{st}, 0 \leq s \leq t < \infty)$ has a version that is a.s. continuous in (s, t); □

From now on $(Z_{st}(z))$ *will denote a continuous version.*

Remark on isotropy. Choose a Cartesian coordinate system in R_2. Let $b(z) = (b_{pq}(z))$ be the covariance matrix defined in Section 4, and let g be a matrix representing a reflection or rotation in R_2. Let g' be the transpose of g. A calculation shows that $gbg'(z) = b(gz)$. It follows that the Itô system of stochastic differential equations for k points with diffusion matrix B^k and zero drift is invariant under reflections and rotations, $k = 1, 2, \ldots$. Hence the flow Z defined by Lemma 2.10 is isotropic if (b_{pq}) has the form of Section 4.

7. Stream functions. This section has some estimates connected with the deterministic flow corresponding to a stream function $\psi(x, y)$, which will determine the "stirring processes" of Section 8. Let $\psi_1'(x, y) = (\partial\psi/\partial x)$, $\psi_2'(x, y) = (\partial\psi/\partial y)$, $\psi_{12}'' = (\partial^2\psi/\partial x\partial y)$, etc.

A smooth steady incompressible flow in the plane can be constructed from a stream function $\psi(x, y)$ by taking $-\partial\psi/\partial y$ and $\partial\psi/\partial x$ as the velocity components in the x and y directions respectively; see [12], Chapter 4. For such a flow let $(\xi_\tau(x, y), \eta_\tau(x, y))$ be the position at time τ of the point initially at (x, y). Then (ξ_τ, η_τ) is the solution of

$$(7.1) \qquad d\xi_\tau/d\tau = -\psi_2'(\xi_\tau, \eta_\tau),\ d\eta_\tau/d\tau = \psi_1'(\xi_\tau, \eta_\tau),\ \xi_0(x, y) = x,\ \eta_0(x, y) = y.$$

Let $W_{(\tau)}(x, y) = (\xi_\tau(x, y), \eta_\tau(x, y))$, $\tau \in R_1$. It is a familiar fact that $(W_{(\tau)}, \tau \in R_1)$ is a group of homeomorphisms of R_2 preserving Lebesgue measure, provided ψ is smooth, in particular if ψ is as in (7.2).

(7.2). *Assumptions and notation.* Let $\psi: R_2 \Rightarrow R_1$ be in C_0^4, and assume that the support of ψ is interior to a circle with radius K centered at (0, 0). Let MPH mean "Lebesgue-measure preserving homeomorphism(s) of R_2 with itself."

From (7.1) we have the estimate

$$(7.3) \qquad \xi_\tau(x, y) = x - \psi_2'\tau + (1/2)[\psi_2'\psi_{12}'' - \psi_1'\psi_{22}'']\tau^2 + c\theta\cdot\tau^3, \qquad \tau \in R_1,$$

where the derivatives of ψ are evaluated at (x, y).

NOTE. Here and throughout, c is a constant, not necessarily the same in each formula, and not depending on the variables in the formula; θ is a function whose absolute value is ≤ 1. Similarly

$$(7.4) \qquad \eta_\tau(x, y) = y + \psi_1'\tau + (1/2)[\psi_1'\psi_{12}'' - \psi_2'\psi_{11}'']\tau^2 + c\theta\cdot\tau^3.$$

By similar calculations, using the fact that ξ_τ and η_τ have continuous first partial derivatives with respect to x and y (see [4], pages 22ff, especially Theorem 7.2, page 25) we find, on any bounded τ-interval,

$$(\xi_\tau(x, y) - \xi_\tau(0, 0))^2 + (\eta_\tau(x, y) - \eta_\tau(0, 0))^2$$

$$= x^2 + y^2 + \tau\{-2x(\psi_2'(x, y) - \psi_2'(0, 0))$$

$$
\begin{aligned}
&+ 2y(\psi_1'(x, y) - \psi_1'(0, 0))\} \\
&+ \tau^2\{(\psi_2'(x, y) - \psi_2'(0, 0))^2 + (\psi_1'(x, y) - \psi_1'(0, 0))^2 \\
&+ x[\psi_2'(x, y)\psi_{12}''(x, y) - \psi_1'(x, y)\psi_{22}''(x, y) \\
&- (\psi_2'(0, 0)\psi_{12}''(0, 0) - \psi_1'(0, 0)\psi_{22}''(0, 0))] \\
&+ y[\psi_1'(x, y)\psi_{12}''(x, y) - \psi_2'(x, y)\psi_{11}''(x, y) \\
&- (\psi_1'(0, 0)\psi_{12}''(0, 0) - \psi_2'(0, 0)\psi_{11}''(0, 0))]\} \\
&+ c\theta\tau^3\rho^*\rho \\
&= x^2 + y^2 + \tau F + \tau^2 G + c\theta\cdot\tau^3\rho^*\rho,
\end{aligned} \tag{7.5}
$$

where $\rho^2 = x^2 + y^2$, $\rho^* = \min(1, \rho)$. Note that

$$|F| \leq c\rho^*\rho, \qquad |G| \leq c\rho^*\rho. \tag{7.6}$$

(7.7) *Circular symmetry.* We henceforth assume

$$\psi(x, y) = g_1(\rho), \qquad \rho = \sqrt{x^2 + y^2}, \qquad \rho \geq 0. \tag{7.8}$$

Since ψ has continuous derivatives of order 4 at (0, 0), necessarily

$$g_1(\rho) = c_0 + c_2\rho^2 + c_4\rho^4 + o(\rho^4), \qquad \rho \downarrow 0. \tag{7.9}$$

Then the Fourier transform $\hat{\psi}(\lambda) = \int e^{iz\cdot\lambda}\psi(z)\, dz$ is a real function of $|\lambda|$ for $\lambda \in R_2$. Put

$$h(|\lambda|) = (\hat{\psi}(\lambda))^2. \tag{7.10}$$

Our assumptions imply $h(s) = O(s^{-8})$ as $s \to \infty$.

8. The stirring processes. Let $\psi^{\alpha\beta}(x, y) = \psi(x - \alpha, y - \beta)$, $(\alpha, \beta) \in R_2$ and let $(\xi_\tau^{\alpha\beta}, \eta_\tau^{\alpha\beta})$ be the solution of (7.1) that is initially (x, y), if ψ is replaced by $\psi^{\alpha\beta}$. Let $W_n^{\alpha\beta}$ be the MPH $(x, y) \to (\xi_{1/\sqrt{n}}^{\alpha\beta}(x, y), \eta_{1/\sqrt{n}}^{\alpha\beta}(x, y))$. Note that

$$
\begin{aligned}
\xi_\tau^{\alpha\beta}(x, y) - x &= \xi_\tau(x - \alpha, y - \beta) - (x - \alpha), \\
\eta_\tau^{\alpha\beta}(x, y) - y &= \eta_\tau(x - \alpha, y - \beta) - (y - \beta).
\end{aligned} \tag{8.1}
$$

We may think of $W_n^{\alpha\beta}$ as an instantaneous vortex centered at (α, β). The nth stirring process Z^n is constructed by applying such vortices repeatedly at random times and at random points (α, β) of R_2. The rigorous construction will be by means of a space-time Poisson point process. Roughly speaking, for the nth stirring process there is a probability $n d\alpha\, d\beta\, dt$ that a vortex is applied in any time period of length dt, with (α, β) in any rectangle of dimensions $d\alpha \times d\beta$. As we let $n \Rightarrow \infty$, the vortices become more and more frequent in each region, while the vortex $W_n^{\alpha\beta}$ moves any point distant $\leq K$ (see (7.2)) from (α, β) a distance of the order at most $1/\sqrt{n}$.

Let Ω' be the set of denumerable subsets ω of $(0, \infty) \times R_2$ having finitely many points (t, x, y) in each bounded region, no two points with the same first coordinate, and having infinitely many points in each cylinder with an open base in R_2. Let $\mathscr{F}'$ be the smallest σ-field in Ω' containing the sets $\{\omega: |\omega \cap B| = k\}$, $k = 0, 1, \cdots$, B a bounded Borel set in $(0, \infty) \times R_2$ and let P^n be the probability measure on $\mathscr{F}'$ corresponding to a Poisson point process in $(0, \infty) \times R_2$ with intensity n. Let $\mathscr{E}^n$ be the expectation symbol corresponding to P^n. However we may write $\mathscr{E}$ instead of $\mathscr{E}^n$ if the meaning is clear.

For convenience we shall later replace Ω' by a certain set $\Omega \in \mathscr{F}'$, where $P^n(\Omega' \backslash \Omega) = 0$, $n = 1, 2, \cdots$.

Define $Z_{st}^n(z, \omega) = (X_{st}^n(x, y, \omega), Y_{st}^n(x, y, \omega))$, $0 \leq s \leq t < \infty$, $(x, y) \in R_2$, $n = 1, 2, \cdots$,

$\omega \in \Omega'$ as follows. (The definition is actually the same for each n; it is the probability measure P^n that changes, rather than the random function.) The construction is similar to that in [13].

Let K be as in (7.2). Given $z_0 \in R_2$, $\omega \in \Omega'$, $s \geq 0$, let $t_1 = \inf\{t' : t' > s, \exists\ (\alpha, \beta) \in R_2, \text{dist}((\alpha, \beta), z_0) < K, (t', \alpha, \beta) \in \omega\}$. Let (t_1, α_1, β_1) be the indicated point of ω. Put $Z^n_{st}(z_0, \omega) = z_0$, $s \leq t < t_1$ and $Z^n_{st_1}(z_0, \omega) = W^{\alpha_1\beta_1}_n z_0 = z_1$, say. ($z_1 = z_0$ is possible.) Letting $t_2 = \inf\{t' : t' > t_1, \exists\ (\alpha, \beta) \in R_2, \text{dist}((\alpha, \beta), z_1) < K, (t', \alpha, \beta) \in \omega\}$, and letting (t_2, α_2, β_2) be the indicated point of ω, we take $Z^n_{st}(z_0, \omega) = z_1$, $t_1 \leq t < t_2$, $Z^n_{st_2}(z_0, \omega) = W^{\alpha_2\beta_2} z_1$ and so on. In this way we construct $Z^n_{st}(z)$ for $0 \leq s \leq t$, $z \in R_2$. Roughly speaking a point $(u, \alpha, \beta) \in \omega$ corresponds to an application of $W^{\alpha\beta}_n$ to R_2 at time u, and $Z^n_{st}(z)$ is the position at time t of the point which was at z at time s.

If there are infinitely many jumps before t, we put $Z^n_{st}(z, \omega) = 0$. It is routine to show that $Z^n_{st}(z, \omega)$ is measurable in the quadruple (s, t, z, ω).

The statement that Z^n is a flow (Z^n is MPH) means there exists $\Omega^n \in \mathscr{F}'$, $P^n(\Omega^n) = 1$ such that $Z^n_{tu}(Z^n_{st}(z, \omega), \omega) = Z^n_{su}(z, \omega)$, $0 \leq s \leq t \leq u < \infty$, $z \in R_2$, $\omega \in \Omega^n$ ($Z^n_{st}(\cdot, \omega)$ is MPH for $0 \leq s \leq t < \infty$, $\omega \in \Omega^n$).

(8.2). LEMMA. *Z^n is a flow and is a* MPH. *Moreover for each $s_1 > 0$, the random variables $Z^n_{st}(z)$, $s_1 \leq s \leq t$, $z \in R_2$ are independent of $Z^n_{st}(z)$, $0 \leq s \leq t \leq s_1$, $z \in R_2$.*

PROOF.[1] Let R_2 be divided into closed squares oriented along the axes with vertices at the points $(3iK, 3jK)$, $i, j = 0, \pm 1, \cdots$, where K was defined in (7.2). Enumerate these squares as $S_1, S_2, \cdots$, where S_1 has its lower left-hand corner at 0. The *neighbors* of S_i are the 8 other squares sharing a side or a vertex with S_i.

Let ω_n be the projection on R_2 of the set of points (u, α, β) of ω having $0 < u \leq c_n$, where we pick $c_n \leq 1/(144K^2 n)$. Let $T_0, T_1, \cdots$ be random unions of squares defined as follows. If $\omega_n \cap S_1 = \varnothing$, then $T_0 = \varnothing$. Otherwise $T_0 = S_1$.

If $T_m = \varnothing$ for some m, then $T_{m+1} = \varnothing$. If $T_m \neq \varnothing$, T_{m+1} is the set of squares $S_i \notin T_0 \cup \cdots \cup T_m$ such that S_i is a neighbor of some $S_j \in T_m$ and $\omega_n \cap S_i \neq \varnothing$. We see that $\mathscr{E}\{|T_{m+1}| \mid T_0, \cdots, T_m\} \leq |T_m| \cdot 8(3K)^2 n/(144K^2 n) = (1/2)|T_m|$, where $|T_m|$ is the number of squares in T_m, so that, a.s. (P^n), $\cup T_m$ is a finite set of squares. Starting with each S_i we construct a finite union in the same manner. Two such unions, if nonempty, either coincide or are at a distance $\geq 3K$ from each other. We enumerate the distinct nonempty unions as $T_{(1)}, T_{(2)}, \ldots$. Let $T^+_{(i)} = \{(x, y) : \text{dist}(T_{(i)}, (x, y)) \leq K\}$. It can be seen that there is a set $\Omega_{n1} \in \mathscr{F}'$, $P^n(\Omega_{n1}) = 0$, such that if $\omega \notin \Omega_{n1}$, $(\alpha, \beta) \in \omega_n$ implies $W^{\alpha\beta}_n$ leaves invariant each set $T^+_{(i)}$ and each point $(x, y) \notin \cup T^+_{(i)}$. Hence if $\omega \notin \Omega_{n1}$, then the points (α, β) of ω_n such that $W^{\alpha\beta}_n$ moves any point of $T^+_{(i)}$ are a finite set and hence Z^n_{st}, $0 \leq s \leq t \leq c_n$, restricted to $T^+_{(i)}$, is a finite composition of MPH, each leaving $T^+_{(i)}$ invariant. Hence for $\omega \notin \Omega_{n1}$, Z^n_{st} is a MPH for $0 \leq s \leq t \leq c_n$ and $Z^n_{tu}(Z^n_{st}(z, \omega), \omega) = Z^n_{su}(z, \omega)$ for $0 \leq s \leq t \leq u \leq c_n$. We make a similar construction of Z^n_{st} for $kc_n \leq s \leq t \leq (k + 1)c_n$, $k = 1, 2, \cdots$. By piecing together, we then construct Z^n_{st} for $0 \leq s \leq t < \infty$ in such a manner that the flow properties are clear. The independence property is also clear. □

(8.3). DEFINITIONS. We now take Ω to be the subset of Ω' on which we can construct Z^n_{st} as above for $kc_n \leq s \leq t \leq (k + 1)c_n$, $k \geq 0$, $n \geq 1$. Put $\mathscr{F} = \mathscr{F}' \cap \Omega$. Then $(\Omega, \mathscr{F}, P^n)$ will be our probability space for Z^n. Put $Z^n_{0t} = Z^n_t$, $X^n_{0t} = X^n_t$, $Y^n_{0t} = Y^n_t$.

(8.4). LEMMA. *$(Z^n_t(z), t \geq 0)$ is for fixed z a random walk in R_2 with jump intensity $\pi K^2 n$. If the position just before a jump*[2] *is z, the position just after a jump is $W^{\alpha\beta}_n z$, where (α, β) is uniformly distributed in the disk of radius K, center z. Moreover $\mathscr{E} Z^n_t(z) = z$.*

---

[1] For a detailed proof in a similar case, see Sections 3–5 of [8].

[2] We say there is a "jump" at z at time u if $(u, \alpha, \beta) \in \omega$ and $\text{dist}(z, (\alpha, \beta)) < K$ although $W^{\alpha\beta}_n z$ might be equal to z.

PROOF. The random walk property is clear from the construction of the process and the nature of Poisson point processes. To show that $\mathscr{E}X_t^n(0, 0) = 0$, recall the definition of $\xi_\tau^{\alpha\beta}$ from Section 7 and the beginning of Section 8. Let S_K be the open disk, center (0, 0), radius K. The expectation of the first jump of $(X_t^n(0, 0), t \geq 0)$ is

$$\frac{1}{\pi K^2}\int_{S_K}\int \xi_{1/\sqrt{n}}^{\alpha\beta}(0, 0)\, d\alpha\, d\beta = \frac{1}{\pi K^2}\int\int_{S_K} \{\xi_{1/\sqrt{n}}(-\alpha, -\beta) + \alpha\}\, d\alpha\, d\beta.$$

Since $(\alpha, \beta) \Rightarrow (\xi_{1/\sqrt{n}}(-\alpha, -\beta), \eta_{1/\sqrt{n}}(-\alpha, -\beta))$ is a measure preserving transformation of S_K into itself, we have $\int\int \xi_{1/\sqrt{n}}(-\alpha, -\beta)\, d\alpha\, d\beta = \int\int \alpha\, d\alpha\, d\beta = 0$. Since $X_t^n(0, 0)$ is the sum of a Poisson number of such jumps, $\mathscr{E}X_t^n(0, 0) = 0$ and similarly $\mathscr{E}Y_t^n(0, 0) = 0$, whence $\mathscr{E}Z_t^n(z) = z$.

It can be seen that $(Z_t^n(z_1), \ldots, Z_t^n(z_k))$ is a jump-type Feller-Markov process. We denote its transition function by

$$p_k^n(t; z_1, \cdots, z_k; \Gamma). \tag{8.5}$$

9. Correlation and spectral functions; small displacements. Recalling that $\psi_1'(x, y) = \partial\psi(x, y)/\partial x$, $\psi_2'(x, y) = \partial\psi(x, y)/\partial y$, put

$$\begin{aligned} b_{11}(x, y) &= \int_{R_2}\int \psi_2'(\alpha, \beta)\psi_2'(x + \alpha, y + \beta)\, d\alpha\, d\beta, \\ b_{12}(x, y) &= b_{21}(x, y) = -\int_{R_2}\int \psi_1'(\alpha, \beta)\psi_2'(x + \alpha, y + \beta)\, d\alpha\, d\beta, \\ b_{22}(x, y) &= \int_{R_2}\int \psi_1'(\alpha, \beta)\psi_1'(x + \alpha, y + \beta)\, d\alpha\, d\beta. \end{aligned} \tag{9.1}$$

(Integration by parts shows that b_{12} is unchanged if we exchange the subscripts of ψ_1' and ψ_2'.) Since ψ and $\hat\psi$ are functions of $\rho = |z|$ and $|\lambda|$ respectively, we get the following spectral formulas by taking Fourier transforms twice in (9.1), putting $\lambda = (\lambda_1, \lambda_2)$.

$$\begin{aligned} b_{11}(z) &= \frac{1}{4\pi^2}\int_{R_2} e^{i\lambda\cdot z}(\lambda_2)^2\hat\psi^2\, d\lambda \\ &= -\frac{1}{2\pi}\left\{\frac{y^2}{\rho^2}\int_0^\infty s^3h(s)J_0''(s\rho)\, ds \right. \\ &\quad \left. + \frac{x^2}{\rho^3}\int_0^\infty s^2h(s)J_0'(s\rho)\, ds\right\}, \\ b_{11}(0, 0) &= \frac{1}{4\pi}\int_0^\infty s^3h(s)\, ds. \end{aligned} \tag{9.2}$$

Similarly

$$\begin{aligned} b_{12}(z) &= -\frac{1}{4\pi^2}\int e^{iz\cdot\lambda}\lambda_1\lambda_2(\hat\psi)^2\, d\lambda \\ &= \frac{xy}{2\pi}\int_0^\infty \left\{\frac{s^3h(s)J_0''(s\rho)}{\rho^2} - \frac{s^2h(s)J_0'(s\rho)}{\rho^3}\right\} ds, \\ b_{12}(0, 0) &= 0; \end{aligned} \tag{9.3}$$

$$b_{22}(x, y) = b_{11}(y, x). \tag{9.4}$$

Formulas (9.2)–(9.4) correspond to (4.1), (4.2), and (4.8) with $h(|\lambda|) = (\hat\psi(\lambda))^2$. Hence we see that (b_{pq}), defined by (9.2)–(9.4), is a C^2-bounded isotropic solenoidal correlation tensor. (See the statement below (7.10).) We define f_L and f_N by (4.9) as before.

Normalization. Henceforth we shall assume (see (4.13))

$$\frac{1}{4\pi}\int_0^\infty s^3 h(s)\,ds = 1. \tag{9.5}$$

Then from (9.2)–(9.4)

$$f_L(\rho) = 1 - \beta\rho^2 + o(\rho^3),$$
$$f_N(\rho) = 1 - 3\beta\rho^2 + o(\rho^3),$$
$$\beta = \frac{1}{32\pi}\int_0^\infty s^5 h(s)\,ds.$$

(9.6). LEMMA. *We have*

$$\mathscr{E}X_t^n(0,0)(X_t^n(x,y)-x) = tb_{11}(x,y) + \theta c\left(\frac{t}{\sqrt{n}} + n^{4/3}t^{5/3}\right), \tag{9.7}$$

$$\mathscr{E}X_t^n(0,0)(Y_t^n(x,y)-y) = tb_{12}(x,y) + \theta c\left(\frac{t}{\sqrt{n}} + n^{4/3}t^{5/3}\right), \tag{9.8}$$

$$\mathscr{E}Y_t^n(0,0)(Y_t^n(x,y)-y) = tb_{22}(x,y) + \theta c\left(\frac{t}{\sqrt{n}} + n^{4/3}t^{5/3}\right), \tag{9.9}$$

provided t lies in a compact interval. $\mathscr{E}Y_t^n(0,0)(X_t^n(x,y)-x)$ *also has the estimate given on the right side of* (9.8).

PROOF. From (7.3) and (8.1)

$$\xi_\tau^{\alpha\beta}(0,0)(\xi_\tau^{\alpha\beta}(x,y)-x) = \psi_2'(-\alpha,-\beta)\psi_2'(x-\alpha,y-\beta)\tau^2 + c\theta\tau^3, \tag{9.10}$$

where $|\theta| \le 1$, if τ is confined to a compact interval.

Let K be as in (7.2). Let S' be the set of points of R_2 at a distance $< 3K$ from either (0, 0) or (x, y). Let $N_t = |\omega \cap ((0,t] \times S')|$, I_t the indicator of the event $\{N_t \ge 2\}$. Using Hölder's and then Schwarz's inequality,

$$\begin{aligned}|\mathscr{E}I_tX_t^n(0,0)(X_t^n(x,y)-x)| &\le (\mathscr{E}(I_t)^{3/2})^{2/3}(\mathscr{E}|X_t^n(0,0)(X_t^n(x,y)-x)|^3)^{1/3} \\ &\le (P(N_t \ge 2))^{2/3}(\mathscr{E}(X_t^n(0,0))^6)^{1/3}.\end{aligned} \tag{9.11}$$

Now $X_t^n(0,0)$ is distributed as a sum of i.i.d. variables, say $\Delta_1 + \cdots + \Delta_\nu$ where $\mathscr{E}\Delta_i = 0$ and ν is Poisson, mean $\lambda = \pi K^2 nt$, independent of the Δ_i. Hence

$$\begin{aligned}\mathscr{E}(X_t^n(0,0))^2 &= \lambda\mathscr{E}(\Delta_1)^2, \\ \mathscr{E}(X_t^n(0,0))^4 &= \lambda\mathscr{E}(\Delta_1)^4 + 3\lambda^2(\mathscr{E}(\Delta_1)^2)^2 \\ \mathscr{E}(X_t^n(0,0))^6 &= \lambda\mathscr{E}(\Delta_1)^6 + 15\lambda^2\mathscr{E}(\Delta_1)^2\mathscr{E}(\Delta_1)^4 \\ &\quad + 10\lambda^2(\mathscr{E}(\Delta_1)^3)^2 + 15\lambda^3(\mathscr{E}(\Delta_1)^2)^3.\end{aligned} \tag{9.12}$$

Since $|\Delta_i| \le c/\sqrt{n}$, we have $\mathscr{E}(X_t^n(0,0))^6 \le c(t/n^2 + t^2)$ if t is confined to a compact interval. Moreover

$$P^n(N_t \ge 2) = 1 - e^{-|S'|nt}(1 + |S'|nt) \le |S'|^2n^2t^2,$$

where $|S'|$ is the Lebesgue measure of S'. Hence the last expression in (9.11) is bounded by $c(n^2t^2)^{2/3}(ct)^{1/3} \le cn^{4/3}t^{5/3}$.

Next note that $N_t(\omega) = 0$ implies $X_t^n(0,0) = 0$ while $N_t(\omega) = 1$ implies $X_t^n(0,0)(X_t^n(x,y)$

$- x) = \xi_{1/\sqrt{n}}^{\alpha\beta}(0, 0)(\xi_{1/\sqrt{n}}^{\alpha\beta}(x, y) - x)$, where (u, α, β) is the point of ω in $(0, t] \times S'$.

From (9.10) and the bound just obtained for (9.11),

$$\mathscr{E}X_t^n(0, 0)(X_t^n(x, y) - x)$$
$$= P^n(N_t = 1)\left\{\frac{1}{|S'|}\cdot\frac{1}{n}\int_{S'}\int \psi_2'(-\alpha, -\beta)\psi_2'(x - \alpha, y - \beta)\, d\alpha\, d\beta + c\theta/n^{3/2}\right\} + c\theta n^{4/3}t^{5/3}$$
$$= tb_{11}(x, y) + \theta ct/n^{1/2} + \theta cn^{4/3}t^{5/3}.$$

Note that $P^n(N_t = 1) = |S'|\,nt - c\theta(|S'|\,nt)^2$. Similar arguments apply to (9.8) and (9.9). □

10. The limiting diffusion for k points. To find the generator $\mathscr{A}_k^n$ of the jump-type Markov process $(X_t^n(x_1, y_1), Y_t^n(x_1, y_1), \cdots, X_t^n(x_k, y_k), Y_t^n(x_k, y_k))$, suppose $\varphi(x_1, y_1, \cdots, x_k, y_k)$ and its partial derivatives of order ≤ 3 are bounded and continuous. From the usual calculations, Lemma 8.4, (9.12), and (9.7)–(9.9) we find, using $|\mathscr{E}UVW| \le (\mathscr{E}U^2)^{1/2}(\mathscr{E}V^4\mathscr{E}W^4)^{1/4}$ to estimate the third order terms,

$$\mathscr{A}_k^n\varphi(x_1, y_1, \cdots, x_k, y_k) = \mathscr{A}_k\varphi(x_1, y_1, \cdots, x_k, y_k) + O(1/\sqrt{n}) \tag{10.1}$$

where $|O(1/\sqrt{n})| \le c_k/\sqrt{n}$, c_k depends on k and φ but not the x_i and y_i and

$$\mathscr{A}_k\varphi = \frac{1}{2}\sum_{i=1}^k\left(\frac{\partial^2\varphi}{\partial x_i^2} + \frac{\partial^2\varphi}{\partial y_i^2}\right) + \sum_{i,j=1;\, i\neq j}^k\left\{\frac{1}{2}b_{11}\frac{\partial^2\varphi}{\partial x_i\partial x_j} + \frac{1}{2}b_{22}\frac{\partial^2\varphi}{\partial y_i\partial y_j} + b_{12}\frac{\partial^2\varphi}{\partial x_i\partial y_j}\right\}; \tag{10.2}$$

in (10.2), $\mathscr{A}_k\varphi$ and φ and its derivatives are evaluated at $(x_1, y_1, \cdots, x_k, y_k)$, and b_{pq} in the (i, j) term is evaluated at $(x_i - x_j, y_i - y_j)$. Note $b_{12}(0, 0) = 0$ from (9.3). We see that $\mathscr{A}_k$ is formally the generator of a diffusion with 0 drift and diffusion matrix B^k of order $2k$ of the type given by (2.3) and (2.5), with B_{ii}^k now

$$\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

The functions b_{pq} are those given by (9.2)–(9.4). B^k is nonnegative definite and is strictly positive definite iff the z_i appearing in it are all distinct; this can be shown from (9.2)–(9.4).

Since the elements of B^k and their partial derivatives of order ≤ 2 are bounded and continuous, it follows from Theorem 1.2, page 129 of Friedman [7] that the elements of $\sqrt{B^k}$, the unique nonnegative definite square root, satisfy a uniform Lipschitz condition in R_{2k}.

Let D (with the Skorohod topology) be the right-continuous functions with left-hand limits from $[0, 1]$ into R_1; let C be the continuous functions on that interval. Let $\mathscr{T}_t$, $0 \le t \le 1$ be the usual σ-fields generated by the functions in D. Let D^{2k}, C^{2k}, and $\mathscr{T}_{2kt}$ be the $2k$-fold Cartesian products. For given initial points $z_1, \cdots, z_k$ the process $(Z_t^n(z_1), \cdots, Z_t^n(z_k))$ induces a probability measure $P_{k;z_1,\cdots,z_k}^n$ in D^{2k}.

According to a result of Stroock and Varadhan [16], Theorem 2.3, which applies to singular diffusions, there is a unique solution $P_{k;z_1,\cdots,z_k}$ for the martingale problem for $\mathscr{A}_k$ operating on the functions $C_0^\infty(R_{2k})$, with initial points $z_1, \cdots, z_k$; this is a diffusion process with diffusion matrix B^k and zero drift, and we consider it here as a probability measure in D^{2k} which puts all its weight on C^{2k}. (Evidently it is sufficient to consider $0 \le t \le 1$.)

(10.3). THEOREM. *$P_{k;z_1,\cdots,z_k}^n$ converges weakly to $P_{k;z_1,\cdots,z_k}$ as $n \to \infty$, $k = 1, 2, \cdots$.*

PROOF. Fix $z_1, \cdots, z_k$ and drop them from the notation. We first show the compactness of P_k^n, $n = 1, 2, \cdots$, with k fixed. For this it is sufficient to show that a sequence of coordinate processes, say $X_t^n(0, 0)$, $0 \le t \le 1$, $n \ge 1$ is tight.

From Billingsley [1], Theorem 15.5, pages 127–128, including the remark referring to page 56, it is sufficient to show that for each $\epsilon > 0$ and $\eta > 0$ there exist $\delta \in (0, 1)$ and n_0 such that

$$\frac{1}{\delta} P^n \{\sup_{0\le s\le\delta} |X_s^n(0, 0)| \ge \epsilon\} \le \eta, \qquad n \ge n_0. \tag{10.4}$$

Furthermore the same theorem of [1] says that each weak limit of $(X_s^n(0, 0), 0 \le s \le 1)$ is concentrated in C. The left side of (10.4), by arguments familiar for Markov processes, is

$$\le \frac{1}{\delta} P^n\{|X_\delta^n(0, 0)| \ge \epsilon/2\}/[1 - \sup_{0\le s\le\delta} P^n\{|X_s^n(0, 0)| \ge \epsilon/2]. \tag{10.5}$$

Using the fourth moment in (9.12), putting $\lambda = \pi K^2 nt$ and using $|\Delta_i| \le c/\sqrt{n}$, we have

$$P^n\{|X_s^n(0, 0)| \ge \epsilon/2\} \le \frac{1}{\epsilon^4}\left(\frac{c_1\delta}{n} + c_2\delta^2\right), \qquad 0 \le s \le \delta$$

which, with (10.5), proves (10.4). Hence $(P_k^n, n \ge 1)$ is tight and each weak limit is concentrated in C^{2k}.

If $\varphi \in C_0^\infty(R_{2k})$ let φ_t^n and $\varphi_t'^n$ be φ and $\mathscr{A}_k^n \varphi$ respectively, evaluated at $X_t^n(x_1, y_1)$, $Y_t^n(x_1, y_1), \cdots, X_t^n(x_k, y_k), Y_t^n(x_k, y_k)$. Now $\varphi_t^n - \int_0^t \varphi_s'^n ds$ is a P_k^n-martingale adapted to $\mathscr{T}_{2kt}$ and is, for fixed t, bounded on D^{2k} and continuous in the topology of D^{2k} at each point of C^{2k}. If Q_k is a weak limit of a subsequence of (P_k^n), we have shown above that $Q_k(C^{2k}) = 1$. It is readily shown from this and (10.1) and (10.2) that Q_k solves the martingale problem for $\mathscr{A}_k$ in D^{2k}, and hence the restriction of Q_k to C^{2k} solves it for $\mathscr{A}_k$ in C^{2k}, with σ-fields $\mathscr{T}_{2kt} \cap C^{2k}$. Then Q_k must be the unique diffusion corresponding to $\mathscr{A}_k$. □

(10.6). REMARK. It can be verified that k points ($k \ge 2$) whose diffusion is governed by (10.2) cannot have a joint Gaussian distribution. Hence our processes are not Gaussian, even though single-point motions are Wiener.

(10.7). DEFINITION. Let $p_k(t; z_1, \cdots, z_k; \Gamma)$, $\Gamma \in \mathscr{B}_{2k}$, be the transition law for the diffusion just defined.

11. Weak convergence of the fields $(Z_{st}^n(z), z \in R_2)$, s and t fixed.

In this section we show that the sequence just named converges weakly to the field $(Z_{st}(z), z \in R_2)$ corresponding to (b_{pq}) given by (9.2)–(9.4). Any such Z_{st} will be designated as a *stream function case*, corresponding to the stream function ψ. Recall that Q_t is the distribution of $Z_t = Z_{0t}$. Let Q_t^n be the distribution of Z_t^n.

(11.1). LEMMA. *Let ρ_t^n be the distance at time t, under the stirring Z^n, between two points whose initial distance is ρ_0. Then $\mathscr{E}(\rho_t^n)^4 \le \rho_0^4 e^{ct}$.*

PROOF. For convenience suppose the points are initially at $(0, 0)$ and (x, y). Let S' be the subset of R_2 defined in the proof of Lemma 9.6. Let σ_1 be the smallest s such that some $(s, \alpha, \beta) \in \omega$ with $(\alpha, \beta) \in S'$, and let $s_1 = (\rho_{\sigma_1}^n)^2$. Then s_1 is given by the expression after the first equality sign in (7.5) with $\tau = 1/\sqrt{n}$, $\rho = \rho_0$, $\rho^* = \min(1, \rho_0)$, (α, β) uniformly distributed in S', and derivatives $\partial\psi(x, y)/\partial x$, $\partial^2\psi(x, y)/\partial x\partial y$, etc., replaced by $\partial\psi(x - \alpha, y - \beta)/\partial x$, $\partial^2\psi(x - \alpha, y - \beta)/\partial x\partial y$, etc. From (7.5) and (7.6) we have

$$s_1 = \rho_0^2 + F_{\alpha\beta}/\sqrt{n} + G_{\alpha\beta}/n + c\theta\rho_0^*\rho_0/n^{3/2}, \qquad |F_{\alpha\beta}|, |G_{\alpha\beta}| \le c\rho_0^*\rho_0.$$

We easily verify $\mathscr{E}^n F_{\alpha\beta} = 0$, whence

$$\mathscr{E}^n s_1^2 = \rho_0^4 + \frac{c\theta}{n}(\rho_0^{*2}\rho_0^2 + \rho_0^*\rho_0^3) \le \rho_0^4(1 + c/n).$$

Continuing, if s_m is the squared distance after m jumps,

$$\mathscr{E}^n(s_{m+1}^2 | s_1, \cdots, s_m) \le s_m^2(1 + c/n). \tag{11.2}$$

Hence $\mathscr{E}^n(s_m)^2 \le \rho_0^4(1 + c/n)^m$. Since the number of jumps in $[0, t]$ is dominated by a Poisson random variable J with mean cnt, $\mathscr{E}(\rho_t^n)^4 \le \rho_0^4 \mathscr{E}(1 + c/n)^J \le \rho_0^4 e^{ct}$. □

(11.3). THEOREM. *For $s \le t$ fixed, the distribution of Z_{st}^n, considered as a random point in C_{22}, converges weakly to Q_{t-s}, where Q_t is defined above* (2.7) *and the functions b_{pq} are given in* (9.2)–(9.4).

PROOF. It is sufficient to consider $s = 0$. If $z_1, \cdots, z_k \in R_2$, the joint distribution of $Z_t^n(z_1), \cdots, Z_t^n(z_k)$ converges to that of $Z_t(z_1), \cdots, Z_t(z_k)$ by Theorem 10.3. Weak convergence follows if we can show relative compactness, which holds (see Fahrmeir [6]) if

$$\lim_{h\downarrow 0} \sup_n Q_t^n\{\xi \in C_{22}: w_S(h, \xi) > \epsilon\} = 0 \tag{11.4}$$

for each compact $S \subset R_2$ and $\epsilon > 0$, where

$$w_S(h, \xi) = \sup_{z,z' \in S, |z-z'| \le h} |\xi(z) - \xi(z')|.$$

For convenience take $t = 1$ and S the unit square. From Lemma 11.1, $Q_1^n\{|\xi(z_1) - \xi(z_2)| > b\} \le c|z_1 - z_2|^4/b^4$. Using this and amplifying somewhat the proof of Theorem 1 of [11], we find that if $\epsilon > 0$ and $0 < \delta < 1$ are given, we can find m_0 depending on ϵ and δ but not n such that

$$Q_1^n\{|\xi(x', y') - \xi(x, y)| \le \epsilon \ \forall \text{ dyadic } x, y, x', y' \in [0, 1] \text{ such that } |x - x'| \le 2^{-m_0} \text{ and } |y - y'| \le 2^{-m_0}\} \ge 1 - \delta.$$

Since ξ is continuous, this implies (11.4). □

12. The inverse process (stream function case). If the stream function ψ of Section 7 is replaced by $-\psi$, each mapping $W_n^{\alpha\beta}$ (see Section 8) is replaced by its inverse, which we call $W_n'^{\alpha\beta}$. Since the operators $\mathscr{A}_k$ of (10.2) depend on ψ only through $|\hat{\psi}|^2$, the limiting flow has the same law for $-\psi$ as for ψ.

Fix $T > 0$. Let Ω be as in (8.3). For each $\omega \in \Omega$ let ω' be the set of all points $(T - u, x, y)$ such that $(u, x, y) \in \omega$ and $0 < u < T$. If $0 \le s \le t \le T$, define

$$Z_{st}'^n(z, \omega) = {}_1Z_{st}^n(z, \omega'),$$

where ${}_1Z_{st}^n$ is the same function as Z_{st}^n except that $-\psi$ replaces ψ. Then the process $Z_{st}'^n$, $0 \le s \le t \le T$ has the same law as Z_{st}^n, $0 \le s \le t \le T$, except that $-\psi$ replaces ψ. Let $Z_t'^n = Z_{0t}'^n$.

(12.1). LEMMA. *For $\omega \in \Omega$, $Z_T'^n(\cdot, \omega)$ is the inverse mapping of $Z_T^n(\cdot, \omega)$.*

The proof follows readily from the construction in Section 8.

13. Homeomorphic property; incompressibility (stream function case). Let H be the subspace of functions in C_{22} that are homeomorphisms of R_2 with itself. Routine arguments show that H is a Borel set in C_{22}.

Let φ be the mapping of H onto H defined by $\varphi\xi = \xi^{-1}$. It can be seen that φ is a homeomorphism. If P is any probability measure on $\mathscr{B}(C_{22})$ with $P(H) = 1$, let $P\varphi^{-1}$ denote the distribution of $\varphi\xi$ in C_{22} if ξ has the distribution P. Then $P\varphi^{-1}$ is also concentrated on H.

The following lemma can evidently be extended to a wider setting than the present one. It is basic for our results.

(13.1). LEMMA. *Let $q_1, q_2, \cdots$ be a tight sequence of Borel probability measures in C_{22} such that $q_n(H) = 1$, $n = 1, 2, \cdots$. Suppose also that the sequence $(q_n\varphi^{-1})$ is tight. Then* *each weak limit of a subsequence of either sequence is concentrated on H.*

PROOF. Let π_n be the distribution in $C_{22} \times C_{22}$ under q_n of the pair $(\xi, \varphi\xi)$, $\xi \in H$. Then $\pi_n(H \times H) = 1$. Let $f(z, \xi, \xi') = \xi(\xi'(z))$, $g(z, \xi, \xi') = \xi'(\xi(z))$, $\xi, \xi' \in C_{22}$, $z \in R_2$. Then f and g are continuous. Since (q_n) and $(q_n\varphi^{-1})$ are tight, so is (π_n). Let π be the weak limit of a subsequence of (π_n). Let $d_0(z, z') = \min(1, d(z, z'))$, where d is the Euclidean metric. Then for each $z \in R_2$

$$\int_{C22 \times C22} d_0(z, f(z, \xi, \xi'))\, d\pi = \lim \int d_0\, d\pi_n = 0,$$

and similarly if f is replaced by g. Hence, if $A = \{(\xi, \xi'): \xi, \xi' \in C_{22}, \xi(\xi'(z)) = \xi'(\xi(z)) = z\ \forall \text{ rational } z \in R_2\}$, then $\pi(A) = 1$. Since $A \subset H \times H$, the lemma is proved. □

(13.2). THEOREM. *Suppose (Z_{st}) is constructed from a stream function $\psi \in C_0^4$ with circular symmetry about the origin. Then almost surely Z_{st} is a homeomorphism of R_2 onto R_2 for every $0 \le s \le t < \infty$. That is, considering (Z_{st}) as a C_{22}-valued process, the values are in H.*

PROOF. We first show from Lemma 13.1 that Z_t, t fixed, is a.s. a homeomorphism of R_2 onto R_2. The rest of the argument is general for continuous nonrandom flows in R_2 and may well be known.

Fix $T > 0$. The random mappings Z_T^n and $Z_T'^n$ (see Section 12) are a.s. inverses. From Theorem 11.3 and the fact that $Z_T'^n$ is distributed like Z_T^n except $-\psi$ replaces ψ, the sequences (Z_T^n) and $(Z_T'^n)$ are both tight. Hence, from Lemma 13.1, Z_T is a.s. a homeomorphism.

For each $0 \le s \le t < \infty$ let $(f_{st}(z), z \in R_2)$ be a continuous nonrandom mapping of R_2 into R_2 such that $f_{tu}(f_{st}(z)) = f_{su}(z)$, $0 \le s \le t \le u < \infty$, $z \in R_2$, and suppose f_{0T} is a homeomorphism of R_2 onto R_2. We show that f_{st}, $0 \le s \le t \le T$, are all homeomorphisms of R_2 onto R_2. Taking T = 1, 2, 3, $\cdots$, is clear that the theorem will follow. The argument for $T = 1$ is now given.

Suppose f_{01} is a homeomorphism of R_2 onto R_2.

(a) f_{0t} is injective for $0 \le t \le 1$ since $f_{01}(z) = f_{t1}(f_{0t}(z))$ and f_{01} is a homeomorphism.

(b) Let $H_t = f_{0t}(R_2)$. Then f_{t1} is injective from H_t onto R_2, since for $z \in H_t$ we have $f_{t1}(z) = f_{t1}(f_{0t}(f_{0t}^{-1}(z))) = f_{01}(f_{0t}^{-1}(z))$.

(c) H_t, with the relative topology of R_2, is homeomorphic with R_2. For let $S \subset R_2$ be closed. Then $f_{01}(S)$ is closed. Since f_{t1} is continuous, the set $\{z: z \in H_t, f_{t1}(z) \in f_{01}(S)\} = f_{0t}(S)$ is relatively closed in H_t, proving (c).

(d) $H_t = R_2$. According to Brouwer's domain invariance theorem (see [5], page 358), H_t is an open subset of R_2. If $H_t \ne R_2$, let $Q \in R_2 \backslash H_t$ be a boundary point of H_t. Let $z_n \Rightarrow Q$, $z_n \in H_t$. Let $S = \cup \{z_n\}$; this set is closed in H_t. Then $f_{t1}(S)$ is a bounded subset of R_2. But $f_{t1}(S) = f_{t1}(f_{0t}(f_{0t}^{-1}(S))) = f_{01}(f_{0t}^{-1}(S))$. The closed set $f_{0t}^{-1}(S)$ cannot be compact, since then $S = f_{0t}(f_{0t}^{-1}(S))$ would be compact. Since f_{01} is a homeomorphism of R_2 with R_2, $f_{01}(f_{0t}^{-1}(S))$ is not bounded, a contradiction.

Hence f_{0t} is a homeomorphism of R_2 with itself for $0 \le t \le 1$. Since $f_{st}(z) = f_{0t}(f_{0s}^{-1}(z))$ for $0 \le s \le t \le 1$, we see that f_{st} is a homeomorphism of R_2 with itself for $0 \le s \le t \le 1$. □

From Lemma 6.2 and Theorem 13.2 we may consider (Z_{st}) as an H-valued process, where H is the set of homeomorphisms in C_{22}. Noting that the map $\xi \Rightarrow \xi^{-1}$ is continuous on H we have:

(13.3). LEMMA. *(Stream-function case.) (Z_{st}) and $(Z_{st})^{-1}$* a.s. *have continuous sample functions.*

(13.4). THEOREM. *(Stream-function case.) Almost surely, a continuous version of (Z_{st}) is incompressible simultaneously for all $0 \le s \le t < \infty$.*

PROOF. For fixed s and t, the a.s. incompressibility of Z_{st} follows from Theorem 3.6, since (b_{pq}) satisfies (3.5). Now let $\varphi \in C_0$: $R_2 \to R_1$. If S is the support of φ, Lemma (13.3) implies that $\bigcup_{0 \le s \le t \le t_0} Z_{st}^{-1}(S)$ is bounded for each $t_0 > 0$ and hence $\int_{R_2} \varphi \circ Z_{st}(z)\, dz$ is a.s. continuous in s and t, proving the result. □

(I am indebted to T. Liggett for the shortened form of the last step.)

14. Extension lemma. Let $\mathscr{M}$ be the set of finite Borel measures M in $[0, \infty)$ with $\int_0^\infty r^2\, dM < \infty$. To any such M there corresponds a covariance (b_{pq}) via (4.1) and (4.2). Let Z be the flow constructed from (b_{pq}) as in Section 2 and let $Q_t[M]$ be the corresponding distribution of Z_t.

Let $\mathscr{M}_0$ be the set of $M \in \mathscr{M}$ corresponding to a stream function ψ as in Section 7 (in particular satisfying (7.8)), and let $\mathscr{M}_B$ be the set of $M \in \mathscr{M}$ such that $Q_t = Q_t[M]$ has $Q_t(H) = 1$ for each $t \ge 0$. We have seen that $\mathscr{M}_0 \subset \mathscr{M}_B$. We shall see later that $\mathscr{M}_B = \mathscr{M}$.

(14.1). LEMMA. *Suppose $M_1, M_2, \cdots \in \mathscr{M}_B$. Suppose $M_n \to M$ weakly and* sup $\int_0^\infty r^2\, dM_n < \infty$. *Then $M \in \mathscr{M}_B$, and if $t \ge 0$ is fixed, $Q_t[M_n]$ converges weakly to $Q_t[M]$.*

PROOF. Fix t. There is no harm in assuming $\frac{1}{2} \int_0^\infty dM = 1$, and also $\frac{1}{2} \int_0^\infty dM_n = 1$, since only a sequence of multipliers $C_n \to 1$ is involved. From (5.4) ($\mathscr{E}_n$ corresponds to M_n)

$$\mathscr{E}_n \rho_t^4 \le \rho_0^4 e^{c_2 t} \tag{14.2}$$

where c_2 does not depend on n. The same arguments as those used in Theorem (11.3) show that $(Q_t[M_n])$ is tight. Let Z_t^n correspond to $Q_t[M_n]$ and let $Z_t'^n$ be the inverse function of Z_t^n. (Note that the meaning of Z_t^n is not the same as in Section 8.) From Corollary 3.7, $Z_t'^n$ has the same distribution at Z_t^n. Hence, from Lemma 13.1, every weak limit of a subsequence of $Q_t[M_n]$ is concentrated on H.

In what follows, a superscript or subscript n will indicate that M is replaced by M_n in formulas such as (4.1), (4.2), or (10.2); it will not refer to the stirring processes.

Let (b_{pq}^n) correspond to M_n, with the generator $\mathscr{A}_k^n$ for k-point diffusions; similarly (b_{pq}) and $\mathscr{A}_k$ for M. From (4.1)–(4.2), $b_{pq}^n(z)$ converges to $b_{pq}(z)$ uniformly on compact sets. Because $\mathscr{A}_k^n$ and $\mathscr{A}_k$ have unique solutions to the martingale problem (see [16]), the k-point diffusions for $\mathscr{A}_k^n$, starting at $z_1, \cdots, z_k$, converge weakly to the diffusion for $\mathscr{A}_k$. (We have tightness because the marginal one-point diffusions of each $\mathscr{A}_k^n$ are Brownian.) Hence, for fixed t, the finite-dimensional distributions of the fields $\{Z_t^n(z),\ z \in R_2\}$ converge to those of $\{Z_t(z)\}$. Thus $Q_t[M_n]$ converges weakly to $Q_t[M]$. □

15. Extension to $\psi \in C_R^\infty$. Given $\psi \in C_R^\infty$, define h and M by (4.8), assuming (4.12), and define B by (4.10), so that $b_{11} = -\partial^2 B/\partial y^2$, $b_{22} = -\partial^2 B/\partial x^2$, $b_{12} = \partial^2 B/\partial x \partial y$. Let $\sigma(u)$, $-\infty < u < \infty$, be a monotone C^∞-function such that $\sigma(u) = 1$ for $u \le 0$, $\sigma(u) = 0$ for $u \ge 1$ and put $\theta_n(x, y) = \sigma((x^2 + y^2)^{1/2} - n)$, $n = 1, 2, \cdots$. Then each $\theta_n \in C_0^\infty$ and the kth order partial derivatives of θ_n are bounded by a constant c_k' not depending on n.

Put $\psi_n = \psi \theta_n$; let $\hat{\psi}_n$ be the Fourier transform of ψ_n; h_n, M_n, and B_n are defined correspondingly by (4.8) and (4.10). Then $M_n \in \mathscr{M}_B$ from Theorem 13.2, since $\psi_n \in C_0^4$. Also $B_n = \psi_n * \psi_n$ and the corresponding covariance tensor (b_{pq}^n) is given by

$$b_{11}^n = -\frac{\partial^2 B_n}{\partial y^2}, \qquad b_{12}^n = \frac{\partial^2 B_n}{\partial x \partial y}, \qquad b_{22}^n = -\frac{\partial^2 B_n}{\partial x^2}. \tag{15.1}$$

(15.2). LEMMA. *The partial derivatives of B_n of each fixed order converge to the corresponding derivatives of B uniformly on compact subsets of R_2.*

PROOF. The proof is sufficiently well illustrated by considering $\partial^2 B_n/\partial x^2$. We have

$$B_n(x, y) = \int_{(\alpha,\beta)\in R_2} \psi(\alpha, \beta)\theta_n(\alpha, \beta)\psi(x-\alpha, y-\beta)\theta_n(x-\alpha, y-\beta)\, d\alpha\, d\beta,$$

$$\text{(15.3)}\quad \begin{aligned} \frac{\partial^2 B_n}{\partial x^2} - \frac{\partial^2 B}{\partial x^2} &= \int \psi(\alpha, \beta)\Bigg\{\theta_n(\alpha, \beta)\psi(x-\alpha, y-\beta)\frac{\partial^2\theta_n(x-\alpha, y-\beta)}{\partial x^2} \\ &\quad + 2\theta_n(\alpha, \beta)\frac{\partial\psi(x-\alpha, y-\beta)}{\partial x}\frac{\partial\theta_n(x-\alpha, y-\beta)}{\partial x} \\ &\quad + \theta_n(\alpha, \beta)\frac{\partial^2\psi(x-\alpha, y-\beta)}{\partial x^2}\theta_n(x-\alpha, y-\beta) \\ &\quad - \frac{\partial^2\psi}{\partial x^2}(x-\alpha, y-\beta)\Bigg\}\, d\alpha\, d\beta \\ &= I_1^n(T) + I_2^n(T), \end{aligned}$$

where $I_1^n(T) = \int_{\alpha^2+\beta^2\le T^2}$, $I_2^n(T) = \int_{\alpha^2+\beta^2>T^2}$. Given $\epsilon > 0$, we can pick T so that $|I_2^n(T)| < \epsilon$ uniformly in x and y. Next, pick an arbitrary $D > 0$ and keep $|z| \le D$. We can then find n_0 depending only on T and D such that $I_1^n(T) = 0$ for $n \ge n_0$. This finishes the proof. □

(15.4). LEMMA. *If $\psi \in C_R^\infty$ and M is defined by* (4.8), *then $M \in \mathcal{M}_B$ as defined in Section* 14.

PROOF. Assume (4.12). Define $\psi_n = \theta_n\psi$ as above. We have $b_{11}^n(x, y) = -(\partial^2/\partial y^2)B_n(x, y) \Rightarrow b_{11}(x, y)$ uniformly on compact sets. Hence $f_L^n(0) \Rightarrow 1$, $f_N^n(0) \Rightarrow 1$ from (4.12). Since b_{11}^n is the characteristic function of F_{11}^n, Lemma (15.2) implies that F_{11}^n converges weakly to F_{11}, where $dF_{11} = \sin^2\varphi(d\varphi/2\pi)\, dM(r)$ from (4.2). Hence M_n converges weakly to M.

Moreover,

$$\begin{aligned} \int r^2\, dM_n(r) &= \frac{1}{2\pi}\int_0^\infty r^5 h_n(r)\, dr \\ &= \frac{1}{4\pi^2}\int_{R_2} (\lambda_1^2 + \lambda_2^2)^2(\hat\psi_n(\lambda))^2\, d\lambda \\ &\le \frac{2}{4\pi^2}\int_{R_2} (\lambda_1^4 + \lambda_2^4)(\hat\psi_n)^2\, d\lambda \\ &= 2\left(\frac{\partial^4 B_n(0,0)}{\partial x^4} + \frac{\partial^4 B_n(0,0)}{\partial y^4}\right) \text{ (see (4.10))}, \end{aligned}$$

which is bounded by Lemma 15.2. The result follows from Lemma 14.1. □

16. Main result.

(16.1). LEMMA. *Each $M \in \mathcal{M}$ is in $\mathcal{M}_B$.*

PROOF. Let $M \in \mathcal{M}$. Let G be the measure in R_2 given by $dG = (d\varphi/2\pi)\, dM(r)$. Let ν_n be the circular normal distribution in R_2 where each component has mean 0 and variance $1/n$. Let $G_n^* = G * \nu_n$, with corresponding density g_n^*. Then $g_n^* \in C^\infty$ and is a function of $x^2 + y^2$. Also $\sqrt{g_n^*} \in C^\infty$ because $g_n^* > 0$. Let θ_n and σ be as in Section 15. Let $a_n \downarrow 0$ be such that $\int_{|\lambda|<a_n} g_n^*(\lambda)\, d\lambda < 1/n$. Let $s_n(\lambda) = 1 - \sigma((2/a_n)(|\lambda| - a_n/2))$. Then $s_n \in C^\infty$ and vanishes for $|\lambda| \le a_n/2$, $0 \le s_n \le 1$, and $s_n(\lambda) = 1$ if $|\lambda| \ge a_n$. Let

(16.2)
$$\varphi_n(\lambda) = \frac{4\pi^2 g_n^*(\lambda)(\theta_n(\lambda))^2(s_n(\lambda))^2}{|\lambda|^2}.$$

Since $\sqrt{\varphi_n} \in C_R^\infty$, so does

$$\psi_n = \frac{1}{4\pi^2}\sqrt{\varphi_n}^{\wedge} = \frac{1}{2\pi}\left(\frac{\theta_n s_n \sqrt{g_n^*}}{|\lambda|}\right)^{\wedge}.$$

According to Lemma 15.4, if we put

(16.3)
$$dM_n(r) = r^3 h_n(r)\, dr/2\pi,$$

where

$$h_n(|\lambda|) = \varphi_n(\lambda) = (\hat{\psi}_n(\lambda))^2,$$

then each $M_n \in \mathscr{M}_B$. By Lemma 14.1 we need only show that $M_n \to M$ and $\sup \int r^2\, dM_n < \infty$. Let $dG_n(\lambda)$ be the measure $(d\varphi/2\pi)\, dM_n$. From (16.2) and (16.3) if $f(\lambda) = f_0(|\lambda|)$ is bounded and continuous,

(16.4)
$$\begin{aligned}\int_{R_2} f(\lambda)\, dG_n(\lambda) &= \int_0^\infty f_0(r) r^3 h_n(r)\, dr/2\pi \\ &= \left(\frac{1}{2\pi}\right)^2 \int_{R_2} f(\lambda)|\lambda|^3 \frac{\varphi_n(\lambda)}{|\lambda|}\, d\lambda \\ &= \int_{R_2} f(\lambda) g_n^*(\lambda)(\theta_n(\lambda))^2(s_n(\lambda))^2\, d\lambda.\end{aligned}$$

We must show that the last integral in (16.4) converges to $\int_0^\infty f_0(r)\, dM(r) = \int_{R_2} f(\lambda)\, dG(\lambda)$. Multiplying by a constant if necessary, consider G as a probability measure corresponding to an R_2-valued random variable Z; then g_n^* corresponds to $Z + Z_n$ where Z_n is independent of Z and circular normal, mean 0, variance $1/n$ for each component. Then $\text{Prob}\{|Z + Z_n| > n\} \to 0$, whence $\int_{|\lambda| \ge n} g_n^*(\lambda)\, d\lambda \to 0$. From the definition of a_n we have $\int_{|\lambda| \le a_n} g_n^*(\lambda)\, d\lambda < 1/n$. Since $dG_n(\lambda) = g_n^*(\lambda)\, d\lambda$ for $a_n \le |\lambda| \le n$ (see (16.4)), we have

(16.5)
$$\lim \int f\, dG_n = \lim \int f g_n^*\, d\lambda = \int f\, dG = \int_0^\infty f_0(r)\, dM(r).$$

Moreover,

$$\int_0^\infty r^2\, dM_n(r) = \int_{R_2} |\lambda|^2 g_n^*(\lambda)(\theta_n(\lambda)s_n(\lambda))^2\, d\lambda \le \int_{R_2} |\lambda|^2 g_n^*(\lambda)\, d\lambda = \mathscr{E}|Z + Z_n|^2$$

where Z and Z_n are as above. Since $\sup_n \mathscr{E}|Z + Z_n|^2 < \infty$, the theorem follows from Lemma 14.1. □

(16.6). THEOREM. *The incompressible isotropic Brownian flows in R_2 are in one-to-one correspondence with the isotropic solenoidal covariance tensors (b_{pq}) of* (4.1)–(4.2). *The relation is through the diffusion matrices given by* (2.3) *and* (2.5) *for the finite-set processes.*

PROOF. Let (b_{pq}) be given by (4.1)–(4.2), with $\int_0^\infty r^2\, dM(r) < \infty$. Construct (Z_{st}) as in Section 2, using B^k given by (2.3) and (2.5). From Lemmas 2.8, 2.10, and 16.1, Z has a version that is a continuous flow with independent increments and is homeomorphic. From the remarks at the end of Sections 2 and 6, Z is homogeneous in space and time and isotropic. From Theorem 3.6 Z_{st} is a.s. incompressible for fixed s and t, and the same argument as in Theorem 13.4 shows that Z is a.s. incompressible simultaneously for all $s \le t$. Conversely, given an incompressible isotropic Brownian flow, we have seen in Section 2 that the construction is based on a C^2-bounded covariance tensor which, because of Theorem 3.6 and isotropy, must be given by (4.1) and (4.2). □

17. Lengths of curves and an application. Let (Z_{st}) be a Brownian flow corresponding to $M \in \mathscr{M}$. Let S be a continuous curve of length $|S| < \infty$, and let $S_t = Z_t(S) = Z_{0t}(S)$.

(17.1). THEOREM. *S_t has a finite length $|S_t|$; the process $(|S_t|)$ is a lower semicontinuous* (LSC) *submartingale and* $\mathscr{E}|S_t| \le |S| e^{ct}$.

(17.2) COROLLARY. $P\{\sup_{0\le t\le T} |S_t| \ge b\} \le |S| e^{cT}/b,\ b > 0.$

The corollary follows from (17.1) and a familiar submartingale inequality, since the supremum of an LSC function on an interval is the same as its supremum on the rational points in the interval.

PROOF OF (17.1). Parameterize S as z_u, $0 \le u \le 1$. For $n = 1, 2, \ldots$ let

$$\rho_{nit} = |Z_t(z_{i/2^n}) - Z_t(z_{(i+1)/2^n})|,$$

$$\rho_{ni} = \rho_{ni0}, \qquad i = 0, 1, \ldots, 2^n - 1,$$

$$L_{nt} = \sum_{i=0}^{2^n-1} \rho_{nit}.$$

Then almost surely: $L_{nt} \uparrow |S_t|$ for each $t \ge 0$.

From (5.4) $\mathscr{E}\rho_{nit} \le \rho_{ni} e^{ct}$ and $\mathscr{E}L_{nt} \le \sum_{i=0}^{2^n-1} \rho_{ni} e^{ct}$. Letting $n \uparrow \infty$ we have, from monotone convergence, $\mathscr{E}|S_t| \le e^{ct}|S|$. Let $\mathscr{F}_t$ be the σ-field generated by $Z_s(z)$, $0 \le s \le t$, $z \in R_2$. From the integrated form of (5.3) we see that $\mathscr{E}\rho_t^\gamma \ge \rho_0^\gamma$ if $\gamma \ge 1$. It follows that $(\rho_{nit}, \mathscr{F}_t)$ is a submartingale and hence $(L_{nt}, \mathscr{F}_t)$ is a submartingale. Since $L_{nt} \uparrow |S_t|$ and (L_{nt}) is continuous, the theorem is proved. □

We can apply (17.2) to get a bound on the spread of the interior of a simple closed curve during a finite interval of time. Let S be a rectifiable simple closed curve, let S^+ be the union of S with its interior, $S_t^+ = Z_t(S^+)$. Assume $(0, 0) \in S$. Let D_R be the disk, radius R, center $(0, 0)$. Then

$$\begin{aligned} P\{S_t^+ \subset D_R, 0 \le t \le T\} &= P\{S_t \subset D_R, 0 \le t \le T\} \\ &\ge 1 - P(\sup_{0\le t\le T}|S_t| > R) - P(\max_{0\le t\le T}|Z_t(0,0)| > R/2) \\ &\ge 1 - |S|e^{cT}/R - P(\max_{0\le t\le T}|Z_t(0,0)| > R/2), \end{aligned}$$

which can be estimated since $Z_t(0, 0)$ is Brownian.

18. Duality (association). Let (Z_{st}) be an incompressible Brownian flow, so that Z_{st} and Z_{st}^{-1} have the same distribution. Fix t and consider Z_t as a random point of $H \subset C_{22}$.

Let $S_{AB} = \{\omega : \omega \in C_{22}, \omega(A) \cap B \ne \varnothing\}$, $A, B \subset R_2$. Then S_{AB} is closed if A is compact and B is closed; S_{AB} is open if B is open and A is arbitrary. Since $S_{AB} = \cup S_{A_iB}$ or $\cup S_{AB_i}$ if $A = \cup A_i$ or $B = \cup B_i$ respectively, it follows that $S_{AB} \in \mathscr{B}(C_{22})$ if A and B are countable unions of compact (in fact of closed) sets. Then $P\{Z_t(A) \cap B \ne \varnothing\} = P\{A \cap Z_t^{-1}(B) \ne \varnothing\}$, whence

$$P\{Z_t(A) \cap B \ne \varnothing\} = P\{A \cap Z_t(B) \ne \varnothing\} \tag{18.1}$$

if A and B are countable unions of closed sets. If B is a small finite set (in particular a single point), the right side of (18.1) is a computationally feasible expression for the left side. Such duality (or association) relations have been studied for interactive systems on lattices (see, e.g., [9] and [10]). For nonlattice systems see Cocozza, Galves, and Roussignol [3]. These relations go back to the self-dual simple exclusion process of F. Spitzer [15].

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REFERENCES

[1] Billingsley, P. (1968). *Convergence of Probability Measures.* Wiley, New York.

[1a] Bucan, G. P. (1979). On the mixed product of non-homogeneous stochastic semigroups. *Theor. Probability Appl.* **24** 168–175. (Russian)

[2] Chorin, A. (1975). *Lectures on Turbulence Theory.* Mathematics Lecture Series No. 5., Publish or Perish, Inc., Boston.

[3] Cocozza, C., Galves, A. and Roussignol. M. (1978). Méthodes d'association pour des systèmes infinis de particules sur R. *C. R. Acad. Sci. Paris Sér. A* **286** 1011–1014.

[4] Coddington, E. and Levinson, N. (1955). *Theory of Ordinary Differential Equations.* McGraw-Hill, New York.

[5] Dugundji, J. (1965). *Topology.* Allyn and Bacon, Boston.

[6] Fahrmeir, L. (1973). Konvergenz stetiger stochastischer Prozesse mit Polnischem Zustands-und Parameterraum. *Z. Wahrscheinlichkeitstheorie und verw. Gebiete* **28** 37–42.

[7] Friedman, A. (1975). *Stochastic Differential Equations and Applications, Vol.* 1. Academic, New York.

[7a] Grenander, U. (1963). *Probabilities on Algebraic Structures.* Almquist and Wiksell, Stockholm; Wiley, New York.

[8] Harris, T. E. (1972). Nearest-neighbor Markov interaction processes on multidimensional lattices. *Advances in Math.* **9** 66–89.

[9] Harris, T. E. (1976). On a class of set-valued Markov processes. *Ann. Probability* **4** 175–194.

[10] Holley, R. and Liggett, T. (1975). Ergodic theorems for weakly interacting systems and the voter model. *Ann. Probability* **3** 643–663.

[11] Jadrenko, M. (1967). Local properties of sample functions of random fields. *Visnik Kiiv. Univ. Ser. Mat. Meh.* **9** 103–112. (English translation in *Selected Trans. Math. Statist. Prob.* **10** 233–245. American Math. Soc., Providence.

[12] Lamb, H. (1945). *Hydrodynamics.* Dover Publications, New York. (Reprint of Sixth Edition (1932), Cambridge Univ.)

[13] Lee, W. (1974). Random stirring of the real line. *Ann. Probability* **2** 580–592.

[14] Monin, A. and Yaglom, A. (1975). Statistical fluid mechanics. In *Mechanics of Turbulence, Vol. 2.* MIT Press, Cambridge, Mass.

[15] Spitzer, F. (1970). Interaction of Markov processes. *Advances in Math.* **5** 246–290.

[16] Stroock, D. and Varadhan, S. (1972). On the support of diffusion processes with applications to the strong maximum principle. In *Proceedings Sixth Berkeley Symp. Math. Statist. Probability* **3** 333–359. Univ. California, Berkeley.

[17] Whittaker, E. T. and Watson, G. N. (1945). *A Course of Modern Analysis.* MacMillan, New York.

[18] Yaglom, A. (1957). Some classes of random fields in n-dimensional space related to stationary random processes. *Theor. Probability Appl.* **2** 292–337. (Russian.)

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