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ON THE ACCOMPANYING LAWS THEOREM IN BANACH SPACES[1]

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In this paper we show that a necessary and sufficient condition on a Banach space B for the validity of the accompanying laws theorem is that c_0 is not finitely representable in B or, equivalently, that B is of cotype q for some $q > 0$. The proof is based on a result of Maurey and Pisier on the geometry of these spaces and on a theorem about approximation in L_p of Banach valued triangular arrays by finite dimensional ones.

Introduction. The classical way to prove the general central limit theorem in the line uses the *method of accompanying laws* which consists essentially in reducing the problem of weak convergence of sums to that of weak convergence of some associated infinitely divisible p.m.'s, the accompanying laws. In fact, if $\{\xi_{nj}: j = 1, \ldots, k_n, n \in N\}$ is an infinitesimal system (i.e., the variables are row-wise independent and $\max_j P\{|\xi_{nj}| > \epsilon\} \to 0$ for all $\epsilon > 0$) and $S_n = \sum_j \xi_{nj}$, then the method is based on the observation that $\mathscr{L}(S_n) \simeq e(\sum_j \mathscr{L}(\xi_{nj}))$ in distribution as $n \to \infty$. Here $e(\mu) = e^{-\mu(R)} \sum_{n=0}^{\infty} \mu^n/n!$ and $\mu^n = \mu * \mu * \ldots * \mu$ is the nth convolution power of μ, i.e., $e(\mu)$ is the compound Poisson p.m. with Lévy measure μ. As usual, $\mathscr{L}(S_n)$ denotes the law of the random variable S_n. In this article we study the relation between tightness and convergence of row sums and accompanying laws of triangular arrays of B-valued random variables (rv's), B a separable Banach space. Before describing our results we recall the main facts on accompanying laws in Euclidean spaces:

1.1. THEOREM. *Let $\{\xi_{nj}\}$ be a triangular array of row-wise independent random variables with values in R^k. Then*:

(a) *If $\{e(\sum_j \mathscr{L}(\xi_{nj}))\}$ is relatively shift compact, then for every $\tau > 0$ the set $\{\mathscr{L}(S_n - \sum_j E\xi_{nj} I_{\{|\xi_{nj}| \le \tau\}})\}$ is relatively compact* (*with respect to the topology of weak convergence of finite Borel measures*).

(b) *If $\{\mathscr{L}(S_n)\}$ is relatively compact and the ξ_{nj} symmetric, then $\{e(\sum_j \mathscr{L}(\xi_{nj}))\}$ is relatively compact.*

(c) *If moreover $\{\xi_{nj}\}$ is infinitesimal and either one of the two sequences $\{\mathscr{L}(S_n - \sum_j E\xi_{nj} I_{\{|\xi_{nj}| \le \tau\}})\}$ and $\{e(\sum_j \mathscr{L}(\xi_{nj} - E\xi_{nj} I_{\{|\xi_{nj}| \le \tau\}}))\}$ is relatively compact, so is the other and both have the same limits and are along the same subsequences.*

Part (c) was proved by Varadhan (1962) in the case of separable Hilbert space. The main contribution to this subject in Banach spaces is due to LeCam (1970) who proved (a) of Theorem 1.1 for any separable Banach space (we will refer to this as *LeCam's theorem*).

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He remarked that (b) is true in R^k and fails in the Banach space c_0 of real sequences converging to zero. De Acosta, et al, (1978, Section 3) proved a desymmetrized version of LeCam's theorem and showed that (b) is true in some special Banach spaces.

It may be convenient to observe also that Theorem 1.1 is not needed in its full strength for proving the general central limit theorem, neither in the line nor in Banach spaces (see, e.g., Araujo and Giné (1980, Sections 2.2–4 and 3,5) and de Acosta, et al, (1978, Section 2)). However, the question of determining how much of Theorem 1.1 is true in what Banach spaces is, in our opinion, interesting in its own right.

Now we describe the contents of this article. Our main result states that the converse to LeCam's theorem (namely that tightness of row sums implies tightness of accompanying laws) holds in B for symmetric infinitesimal triangular arrays if and only if c_0 is not finitely representable in B (definitions follow below). The noninfinitesimal, nonsymmetric cases are also considered. These results are proved in Section 4. Section 3 contains a theorem on L_p approximation of convergent row sums of infinitesimal B-valued arrays by tight row sums of finite dimensional ones, related to those of Pisier (1975, Théorème 3.1) and Mandrekar and Zinn (1977, proof of Theorem 2.10). This result, which is of independent interest, is used to prove the accompanying laws theorem and also an improvement of a central limit theorem in cotype p spaces due to de Acosta, et al, (1978, Theorem 6.6). The accompanying laws theorem relies also on a geometrical characterization of those spaces where some moments of the norm for the law $e(\sum_j \mathscr{L}(X_j))$ are dominated by those of $\sum_j X_j$ (Section 2). This, in turn, depends heavily on a deep theorem of Maurey and Pisier (1975, Corollary 1.3 and remarks thereafter) which characterizes probabilistically those Banach spaces in which c_0 is not finitely representable.

We start by introducing some standard notation used in the study of triangular arrays. By a symmetric infinitesimal array $\{X_{nj}\}$ we mean a family of B-valued random variables (rv's) $\{X_{nj}: j = 1, \ldots, k_n, n \in N\}$ which are symmetric, row-wise independent (i.e., $\{X_{nj}\}_{j=1}^{k_n}$ is an independent set of rv's for each n) and satisfy the infinitesimality condition

$$\lim_{n\to\infty} \max_j P\{\| X_{nj} \| > \epsilon\} = 0$$

for every $\epsilon > 0$. We associate with each triangular array the truncations

$$X_{nj\delta} = X_{nj} I_{\{\|X_{nj}\| \le \delta\}}, \qquad \tilde{X}_{nj\delta} = X_{nj} - X_{nj\delta},$$

the row sums

$$S_n = \sum_{j=1}^{k_n} X_{nj}, \qquad S_{n,\delta} = \sum_{j=1}^{k_n} X_{nj\delta}, \qquad \tilde{S}_{n,\delta} = \sum_{j=1}^{k_n} \tilde{X}_{nj\delta},$$

the row sums of the laws

$$\mu_n = \sum_{j=1}^{k_n} \mathscr{L}(X_{nj}), \qquad \tilde{\mu}_{n,\delta} = \sum_{j=1}^{k_n} \mathscr{L}(\tilde{X}_{nj}\delta)$$

and the *accompanying Poisson laws*

$$e(\mu_n) = \mathscr{L}(\textstyle\sum_{j=1}^{k_n} \sum_{i=0}^{N_j} X_{nji}) \tag{1.1}$$

where $\mathscr{L}(X_{nji}) = \mathscr{L}(X_{nj})$ for $i \ge 1$, $X_{nj0} = 0$, $\mathscr{L}(N_j) = e(\delta_1)$ (=Poisson with parameter 1) and $\{X_{nji}, N_j: j = 1, \ldots k_n, i \ge 0\}$ are independent for each n. As mentioned before, $e(\mu) = e^{-\mu(B)} \sum_{n=1}^{\infty} \mu^n/n!$, $\mu^n = \mu * \ldots * \mu$, and it is obvious that $e(\mu_n)$ is the law of the rv at the right-hand side of (1.2). If ν is a σ-finite Borel measure on B, and $A \in \mathscr{B}$, then $\nu|_A$ will denote its restriction to A. Weak convergence of probability measures will be denoted by $\to_w$ or w-lim. Finally, we will say that a set of finite measures on B is relatively compact (or uniformly bounded and tight) if it is relatively compact in the topology of weak convergence.

We conclude the section by giving some concepts from the geometry of Banach space. We also summarize some results due to Maurey and Pisier (1976, Theorem 1.2, Corollary 1.3 and Remarks following it) for easy reference and refer the reader to their work for the proofs.

A Banach space B contains l_n^∞ uniformly or, what is the same, c_0 *is finitely representable in* B if there exists $\tau \geq 1$ such that for each $n \in N$, there are n vectors $y_{n1}, \ldots, y_{nn}$ in B satisfying

$$\max_{i\leq n} |t_i|/\tau \leq \| \textstyle\sum_{i=1}^n t_i y_i \| \leq \tau \max_{i\leq n} |t_i|.$$

A Banach space B is of *cotype* p (Rademacher) if there exists $c > 0$ such that for $\{x_i\}_{i=1}^n \in B$, $n \in N$,

$$E \| \textstyle\sum_{i=1}^n x_i \epsilon_i \|^p \geq c \sum_{i=1}^n \| x_i \|^p$$

where $\{\epsilon_i\}$ is a *Rademacher sequence*: the ϵ_i are i.i.d. and $P\{\epsilon_i = 1\} = P\{\epsilon_i = -1\} = ½$. We denote by $\{\bar{\xi}_j\}$ the independent symmetrizations of $\{\xi_j\}$, the independent Poisson random variables with parameter ½.

Throughout the paper B will denote a real separable Banach space with topological dual B', duality (,) and Borel sets $\mathscr{B}$.

1.2. THEOREM. (a) *The following are equivalent for* $q \geq 2$, *and a sequence* $\{\xi_i\}_{i=1}^\infty$ *of independent identically distributed mean zero random variables* $\cdot\ni\cdot$ $P(|\xi_1| > t) > 0$ *for all* t *and* $E|\xi|^p < \infty$ *for all* $p (1 \leq p < \infty)$

(i) c_0 *is not finitely representable in* B;

(ii) *There exists a constant* $C = C(B, q, \{\xi_i\}) < \infty$ *such that for all sequences of points* $\{x_n\} \subseteq B$

$$E \| \textstyle\sum_1^n x_i \xi_i \|^q \leq CE \| \sum_1^n x_i \epsilon_i \|^q. \tag{1.2}$$

(b) c_0 *is not finitely representable in* B iff *it is of cotype* q *for some* $q > 0$.

REMARK. The part (a) is crucial for our work. The part (b) is interesting as a source of examples. It is easy to show that the $\mathscr{L}_p$ spaces $(1 \leq p < \infty)$ are of cotype $q = \min(2, p)$. It is also known that these spaces admit Schauder bases (Johnson, Rosenthal and Zippin (1971))

2. Integrability of accompanying Poisson laws.

In this section we prove the following:

2.1 THEOREM. *The following are equivalent for any* $q \geq 2$:

(i) c_0 *is not finitely representable in* B.

(ii) *There exists* $L = L(B, q) < \infty$ *such that for every finite sequence* $\{X_j\}_{j=1}^n$ *of symmetric, independent* B*-valued* rv's *with* $E\|X_j\|^q < \infty, j = 1, \ldots, n$,

$$E \| \textstyle\sum_{j=1}^n \sum_{i=1}^{N_j} X_{ji} \|^q \leq LE \| \sum_{j=1}^n X_j \|^q, \tag{2.1}$$

with the notation as in (1.1).

We need a very simple lemma:

2.2. LEMMA. *If* $\{X_j\}_{j=1}^n$ *are identically distributed* B*-valued* rv's *and* X_0 *is independent of* $\{X_j\}$, *then for any* $q \geq 1$,

$$E \| \textstyle\sum_{j=0}^n X_j \|^q \leq E \| X_0 + nX_1 \|^q.$$

PROOF. By Minkowski's inequality,

$$(E \| X_0 + \textstyle\sum_{j=1}^n X_j \|^q)^{1/q} = (E \| \sum_{j=1}^n \left(\frac{X_0}{n} + X_j\right) \|^q)^{1/q}$$

$$\leq n\left(E \| \frac{X_0}{n} + X_1 \|^q\right)^{1/q} = (E \| X_0 + nX_1 \|^q)^{1/q}$$

as $\{X_0/n + X_j\}_{j=1}^n$ are identically distributed. □

PROOF OF THEOREM 2.1. (ii) ⇒ (i). Let $\{x_j\}_{j=1}^n \subseteq B$, $n \in N$. Note that if $\{\epsilon_j\}_{j=1}^\infty$ is a Rademacher sequence, $\epsilon_0 = 0$, N a Poisson rv with parameter 1 independent of $\{\epsilon_j\}$, $\{\xi_j\}_{j=1}^n$ independent Poisson rv's with parameter ½ and $\{\bar{\xi}_j\}$ independent symmetrizations of $\{\xi_j\}$, then

$$e(\mathscr{L}(x\epsilon_i)) = \mathscr{L}(x \textstyle\sum_{j=0}^N \epsilon_j), \qquad \mathscr{L}(\sum_{j=0}^N \epsilon_j) = e(\tfrac{1}{2}\,\delta_{-1} + \tfrac{1}{2}\,\delta_1) = e(\tfrac{1}{2}\,\delta_{-1})*e(\tfrac{1}{2}\,\delta_1) = \mathscr{L}(\bar{\xi}_1),$$

where δ_1 and δ_{-1} denote respectively the unit masses at 1 and -1.
Inequality (2.1) then gives

$$E\|\textstyle\sum x_j\bar{\xi}_j\|^q \le LE\|\sum x_j\epsilon_j\|^q$$

and (i) follows from Theorem 1.2 (a) because $E|\xi_1|^P < \infty$ for every $p > 0$ and $P\{|\xi_1| > t\} > 0$ for all t (i) ⇒ (ii). Since $\xi_j = N_j$ is as in Theorem 1.2 (a), if $\{X_j\}_{j=1}^n$ are independent, symmetric B-valued rv's, independent of the $\{N_j\}$, and if c_0 is not finitely representable in B, then by Theorem 1.2 (a) and Fubini's theorem we have:

$$E\|\textstyle\sum_j X_j(N_j - 1)_j\|^q \le CE\|\sum_j X_j\epsilon_j\|^q = CE\|\sum_j X_j\|^q \tag{2.2}$$

where $\{\epsilon_j\}$, as usual, is a Rademacher sequence independent of $\{X_j\}$. Let now $\{X_{ji}\}$ be an independent set of B-valued rv's such that $X_{j0} = 0$ and $\mathscr{L}(X_{ji}) = \mathscr{L}(X_i)$ for every $i \ge 1$ and $j = 1, \ldots, n$. Let $\{N_j\}$ be independent $e(\delta_1)$ real random variables independent of $\{X_{ji}\}$; in fact assume the X_{ji} defined on $(\Omega_1, \mathscr{F}_1, P_1)$ and the N_j on $(\Omega_2, \mathscr{F}_2, P_2)$ and let E, E_1, E_2 denote integration with respect to $P_1 \times P_2$, P_1 and P_2 respectively. By Lemma 2.2 and inequality (2.2) we then have:

$$\begin{aligned} E\|\textstyle\sum_j \sum_{i=0}^{N_j} X_{ji}\|^q &= E_2E_1\|\textstyle\sum_j \sum_{i=0}^{N_j} X_{ji}\|^q \le E_2E_1\|\textstyle\sum_j N_jX_{ji}\|^q \\ &\le 2^{q-1}E_2E_1\|\textstyle\sum_j (N_j - 1)X_j\|^q + 2^{q-1}E_2E_1\|\textstyle\sum_j X_j\|^q \\ &\le 2^{q-1}(C + 1)E\|\textstyle\sum_j X_j\|^q, \end{aligned}$$

which proves (2.1). (Note that in the first inequality Lemma 2.2 is applied recursively taking first $X_0 = \sum_{j=2}^n \sum_{i=0}^{N_j} X_{ji}$, then $X_0 = N_1X_{11} + \sum_{j=3}^n \sum_{i=0}^{N_j} X_{ji}$, and so on). □

3. A decomposition theorem for bounded convergent triangular arrays. Let $\{X_{nj}\}$ be a symmetric infinitesimal array of B-valued random variables. Then, if $\mathscr{L}(S_n) \to_w \lambda$, it is known that λ is infinitely divisible and in fact $\lambda = \rho*\nu$ where ρ is symmetric Gaussian, $\nu = w - \lim_n e(\mu B_{\delta_n}^c)$ ([9]) with $\delta_n \downarrow 0$ and μ a symmetric σ-finite Borel measure on B finite outside every neighborhood of zero; μ is the *Lévy measure* associated to ν, and by abuse of notation we will write $\nu = e(\mu)$. $e(\mu)$ has no Gaussian factor (aside from δ_0). It is also known that if $C(\mu) = \{\delta : \mu\{\|x\|\,\delta\} = 0\}$, then

$$\mu_n | B_\delta^c \to {}_w\mu | B_\delta^c \tag{3.1}$$

for every $\delta \in C(\mu)$, where $\mu_n = \sum_j \mathscr{L}(X_{nj})$ as defined above. Proofs of this fact can be found in de Acosta et al, (1978), Section 2 (Theorem 2.10) and in Mandrekar and Zinn (1977), Section 3 (Theorem 3.10)).

3.1 THEOREM. *Let $\{X_{nj}\}$ be a symmetric infinitesimal array of B-valued* rv's *such that $\mathscr{L}(S_n) \to_w \lambda$. Then there exists $\delta_n \downarrow 0$ such that $\mathscr{L}(S_{n,\delta_n}) \to_w \rho$, the Gaussian factor of λ, and $\mathscr{L}(\tilde{S}_{n,\delta_n}) \to_w \nu$, the non-Gaussian factor of λ.*

PROOF. As noted above $\tilde{\mu}_{n,\delta} \to_w \mu|_{B_\delta^c}$ and therefore $e(\tilde{\mu}_{n,\delta}) \to_w e(\mu|_{B_\delta^c})$. (It is well known that for finite measures $\nu_n \to_w \nu$ finite implies $e(\nu_n) \to_w e(\nu)$; the tightness of $e(\nu_n)$ follows by decomposing $e(\nu_n) = e^{-\nu_n(B)} \sum_1^r \nu_n^k/k! + e^{-\nu_n(B)} \sum_{k=r+1}^\infty \nu_n^k/k!$ and observing that the first term is tight for fixed r and the second can be made as small in total variation as one wishes just taking r large enough. Then the convergence follows using characteristic functions). Let $\tilde{Y}_\delta$ be a random variable such that $\mathscr{L}(\tilde{Y}_\delta) = e(\mu \mid B_\delta^c)$. Then $\mathscr{L}(\tilde{S}_{n,\delta}) \to_w \mathscr{L}(\tilde{Y}_\delta)$ by using linear functionals. Clearly $\tilde{Y}_\delta$ is non-Gaussian and for each $\delta \in C(\mu)$, $d(\mathscr{L}(\tilde{S}_{n,\delta}), \mathscr{L}(\tilde{Y}_\delta)) \to 0$ where d is Lévy-Prohorov metric on the probability measures on B. Hence there exist $\delta_n \to 0$ such that $d(\mathscr{L}(\tilde{S}_{n,\delta_n}), \mathscr{L}(\tilde{Y}_{\delta_n})) \to 0$. Since $\mu \mid_{B_{\delta_n}^c}$ increases to μ, we have $\mathscr{L}(\tilde{Y}_{\delta_n}) \to_w e(\mu)$. Also $\{S_{n,\delta_n} = \tilde{S}_n - S_{n,\delta_n}\}$ is relatively compact. Now for $f \in B'$ with $X'_{nj} = X_{nj\delta_n}$ and $X''_{nj} = \tilde{X}_{nj,\delta_n}$ we get

$$\sum_1^{k_n} E(f, X'_{ng} + X''_{nj})^2\, 1\,(|(f, X'_{nj} + X''_{nj})| \leq \delta)$$
$$= \sum_{j=1}^{k_n} E(f, X'_{nj})^2\, 1\,(|(f, X'_{nj}) + (f, X''_{nj})| \leq \delta)$$
$$+ \sum_{j=1}^{k_n} E(f, X''_{nj})^2\, 1\,(|(f, X'_{nj}) + (f, X''_{nj})| \leq \delta) = R_n(\delta) \qquad \text{(say)}.$$

By multiplying each integrand with each of $1(\|X_{nj}\| \leq \delta_n)$ and $1\,(\|X_{nj}\| > \delta_n)$ we see that $\lim_{\delta\to 0} \limsup_n R_n(\delta) = \lim_{\delta\to 0} \limsup_n \sum_1^n E(f, X'_{nj})^2\, 1\,((f, X'_{nj})| \leq \delta) + \sum_1^{k_n} E(f, X''_{nj})^2$ $1(|(f, X''_n)| \leq \delta)$. But $(f, \tilde{S}_{n,\delta_n})$ converges to non-Gaussian limit, hence the limit of the second term in the right-hand side of the above expression is zero, giving,

$$\lim_{\delta\to 0} \limsup_n R_n(\delta) = \lim_{\delta\to 0} \limsup_n \sum_1^{k_n} E(f, X'_{nj})^2\, 1\,(|(f, X'_{nj})| \leq \delta).$$

Similar arguments give

$$\lim_{\delta\to 0} \liminf_n R_n(\delta) = \lim_{\delta\to 0} \liminf_n \sum_1^{k_n} E(f, X'_{nj})^2\, 1\,(|(f, X'_{nj})| \leq \delta).$$

Hence we get the Gaussian component of the limit law of $\{S_n\}$ is the limit law of $\{S_{n,\delta_n}\}$. But $\{S_{n,\delta_n}\}$, being relatively compact we get $\mathscr{L}(S_{n,\delta_n}) \to_w \rho$. □

The following result is contained in the proof of Theorem 2.10 of Mandrekar and Zinn (1977); thus we state it with no proof.

3.2. Lemma. *Let $\{X_{nj}\}$ be a symmetric infinitesimal array of uniformly bounded B-valued* rv's *such that $\mathscr{L}(S_n) \to_w \nu = e(\lambda)$. Then for every $\epsilon > 0$ and $q > 0$ there exists a symmetric simple function $t: B \to B$ such that $\{\mathscr{L}(\sum_j t(X_{nj}))\}$ is tight and* $\sup_n E\|\sum_j X_{nj} - \sum_j t(X_{nj})\|^q < \epsilon$.

In the Gaussian convergence case we have:

3.3. Lemma. *Let $\{X_{nj}\}$ be a symmetric infinitesimal array of uniformly bounded B-valued* rv's *such that $\mathscr{L}(S_n) \to_w \rho$ Gaussian. Then for every $\epsilon > 0$ and $q > 0$ there exists a linear operator π_N with finite-dimensional range such that*

$$\lim_{n\to\infty} E\|\sum_j X_{nj} - \sum_j \pi_N(X_{nj})\|^q < \epsilon.$$

Proof. Let Z be a B-valued rv with $\mathscr{L}(Z) = \rho$. Then it is well known that $Z = \sum_{j=1}^\infty f_j(Z)x_j$ a.s. for suitable $f_j \in B'$ and $x_j \in B$ (see, e.g., Jain (1977), Theorem 6 and Remark (3) thereafter). Let $\pi_N(\cdot) = \sum_{j=1}^N f_j(\cdot)x_j$. Then since $\mathscr{L}(S_n) \to_w \mathscr{L}(Z)$, we get that $\mathscr{L}(S_n - \pi_N(S_n)) \to_w \mathscr{L}(Z - \pi_N(Z))$. Also, $\{\|S_n - \pi_N(S_n)\|^q\}$ is uniformly integrable for every $q > 0$ by Theorem 2.1 in Mandrekar and Zinn (1977) or Theorem 2.3 in de Acosta, et al., (1978), and therefore $E\|S_n - \pi_N(S_n)\|^q \to E\|Z - \pi_N(Z)\|^q$. Now the proof is completed by choosing π_N so that $E\|Z - \pi_N(Z)\|^q < \epsilon$. □

The previous propositions yield the following theorem, which is the main result in this section.

3.4. Theorem. *Let $\{X_{nj}\}$ be a symmetric infinitesimal array of uniformly bounded*

B-valued rv's *such that* $\{\mathscr{L}(S_n)\}$ *converges weakly. For any* $q > 0$ *and* $\epsilon > 0$ *there exists a symmetric infinitesimal array* $\{W_{nj}\}$ *such that*:

(i) W_{nj} *is a measurable function of* X_{nj} *only for every* n *and* j;

(ii) *there exists a finite dimensional subspace* $F \subset B$ *such that* $P\{W_{nj} \in F\} = 1$;

(iii) $\{\mathscr{L}(\sum_j W_{nj})\}$ *is relatively compact; and*

(iv) $\sup_n E \| \sum_j X_{nj} - \sum_j W_{nj} \|^q < \epsilon$.

PROOF. Choose δ_n as in Theorem 3.1 and let $S'_n = S_{n,\delta_n}$, $S''_n = \tilde{S}_{n,\delta_n}$. Then $\mathscr{L}(S'_n) \to_w \rho$ Gaussian, $\mathscr{L}(S''_n) \to_w \nu = e(\mu)$ and $\mathscr{L}(S_n) \to_w \lambda = \rho * \nu$, as in Theorem 3.1. By Lemmas 3.2 and 3.3 for every $\epsilon > 0$, $q > 0$ there exist $t: B \to B$ simple symmetric, π linear with finite dimensional range and $n_0 \in N$ such that $E \| S''_n - \sum_j t(X_{nj}) \|^q < \epsilon/4$ and for $n > n_0$, also $E \| S'_n - \pi (X_{nj}) \|^q < \epsilon/4$. Then, for $n > n_0$, $W_{nj} = t(X_{nj}) + \pi (X_{nj})$ satisfies the stated requirements; but for $n \le n_0$, W_{nj} can be obviously obtained by simple function approximation. □

If F is finite-dimensional subspace of B, then let $q_F(x) = \inf\{\| x - y \|; y \in F\}$.

REMARKS. (a) The above result should be compared with Theorem 2.3 of de Acosta, et al, (1978); this last theorem implies the existence of $\{W_{nj}\}$ satisfying the properties in Theorem 3.4 even in the centered nonsymmetric, noninfinitesimal case provided that B *has a Schauder basis*. [Let $\{f_j\} \subseteq B$ be the coordinate functionals, $B_k = \operatorname{Ker}(f_1, f_2, \cdots f_k)]$ if $\{e_n\}_{n=1}^{\infty}$ is a Schauder basis, F_k the linear span of $\{e_1, \ldots, e_k\}$, $\pi_k (\sum_{n=1}^{\infty} \lambda_n e_n) = \sum_{n=1}^{k} \lambda_n e_n)$ and $T_k: B_k \to B/F_k$ is defined as $T_k(x) = x + F_k$, then $(\sup_k \| T_k^{-1} \|) < \infty$ and $x - \pi_k x \| \le q_{F_k}(x) \le \| x - \pi_k (x) \|$.

(b) The above theorem is not a generalization of Pisier (1975, Theorem 3.1) because the approximating row sums are not related to each other in any specific way as in the i.i.d. $(\sqrt{n})$ case. It is not clear how to obtain a general construction of finite dimensional 'natural' triangular arrays so as to deduce the i.i.d. case.

4. Consequences of the foregoing. In the first, main part of this section we present results on accompanying laws and in the second we give necessary conditions for the CLT in cotype p spaces.

4.1. THEOREM. *The following are equivalent for any separable Banach space* B:

(i) c_0 *is not finitely representable in* B;

(ii) *for any symmetric, infinitesimal array* $\{X_{nj}\}$, *convergence of* $\{\mathscr{L}(S_n)\}$ *implies convergence of* $\{e(\mu_n)\}$.

In which case, $w - \lim_n e(\mu_n) = w - \lim_n \mathscr{L}(S_n)$.

PROOF. (i) $\Rightarrow$ (ii). If $\delta \in C(\mu)$ then $\mu_n|_{B_\delta^c} \to_w u|_{B_\delta^c}$ and therefore $e(\mu_n|_{B_\delta^c}) \to_w e(\mu|_{B_\delta^c})$. Since for every Lévy measure ν, $e(\nu) = e(\nu|_{B_\delta^c}) * e(\nu|_{B_\delta})$, and since for $\delta \in C(\mu)$, convergence of $\{\mathscr{L}(S_n)\}$ implies convergence of $\{\mathscr{L}(S_{n,\delta})\}$ and $\{\mathscr{L}(\tilde{S}_{n,\delta})\}$ as observed before, it follows that we need only to consider *uniformly bounded* triangular arrays. So, we assume $\{X_{nj}\}$ uniformly bounded. Let $\{X_{nji}, N_j: n, i \in N, j = 1, \ldots, k_n\}$ be as in (1.1) and recall that $e(\mu_n) = \mathscr{L}(\sum_j \sum_{i=0}^{N_j} X_{nji})$. Given $\epsilon > 0$, for every $i \ge 1$ associate W_{nji} to X_{nji} as in Theorem 3.4, and let $W_{nj0} = 0$. Then, Theorem 2.1 applied to $X_j = X_{nj} - W_{nj}$ gives by Theorem 3.4 that

$$\sup_n E \| \textstyle\sum_{j=1}^{k_n} \sum_{i=0}^{N_j} (X_{nji} - W_{nji}) \|^q \le L\epsilon. \tag{4.1}$$

Since $\{\mathscr{L}(\sum_{j=1}^{k_n} W_{nj})\}$ is relatively compact and this is a finite dimensional array, Theorem 1.1 implies that $\{\mathscr{L}(\sum_j \sum_{i=0}^{N_j} W_{nji})\}$ is also relatively compact, in particular, it is *flatly concentrated*, i.e., given $\delta > 0$ there exists a finite dimensional subspace $F \subset B$ such that $P\{q_F(\sum_j \sum_{i=0}^{N_j} W_{nji}) > \delta\} \le \delta$. Hence (4.1) gives the flat concentration of $\{\mathscr{L}(\sum_j \sum_{i=0}^{N_j}$

$X_{nji})\}$ by Chebyshev's inequality. On the other hand, also by theorem 1.1, if $f \in B'$ and $\mathscr{L}(S_n) \to_w \lambda$, then $e(\mu_n) \circ f^{-1} \to {}_w\lambda \circ f^{-1}$. Therefore, by Theorem 2.4 in de Acosta (1970), $e(\mu_n) \to_w \lambda$.

(ii) $\Rightarrow$ (i). Assume c_0 is finitely representable in B. Then, by Theorem 1.2 there exists a sequence $\{x_j\} \subset B$ such that $\sum_j \epsilon_j x_j$ converges a.s. but there exist $\{k_n\}$, $\{l_n\}$ integers such that $k_n, l_n \to \infty$ and $\sum_{j=l_n+1}^{k_n+l_n} \xi_j x_j \not\to 0$ in probability (hence in law), with $\{\xi_j\}$ as in the proof of Theorem 2.1. Since $l_n \to \infty$ and $\sum_j \epsilon_j x_j$ converges, we get $x_{j+l_n} \to 0$ as $n \to \infty$. Then $\{X_{nj} = \epsilon_{j+l_n} x_{j+l_n}, 1 \le j \le k_n, n \in N\}$ is a symmetric infinitesimal array such that $\mathscr{L}(S_n) \to {}_w\delta_0$. However, $\{\mathscr{L}(\sum_{j=l_n+1}^{k_n+l_n} \xi_j x_j) = e(\sum_j \mathscr{L}(X_{nj}))\}$ does not converge (in fact it is not even tight: otherwise Theorem 1.1 (c) gives a contradiction). □

REMARK. The example in Le Cam (1970) showing that in c_0, $\{\mathscr{L}(S_n)\}$ may be tight without $\{e(\mu_n)\}$ being tight can also be easily adapted to provide a proof of (ii) $\Rightarrow$ (i) in the previous theorem. However the present example, given Theorem 1.2, is much simpler.

In view of Le Cam's theorem (Le Cam (1970)) and Theorem 4.1, we get:

4.2. COROLLARY. *The following are equivalent for a Banach space B*:

(i) *c_0 is not finitely representable in B*;

(ii) *for every B-valued symmetric infinitesimal array $\{X_{nj}\}$, $\{\mathscr{L}(S_n)\}$ is relatively compact if and only if $\{e(\mu_n)\}$ is relatively compact.*

From the previous corollary and Theorem 2.5 in de Acosta, et al, (1978), we obtain the following:

4.3. THEOREM. *The following are equivalent*:

(i) *c_0 is not finitely representable in B*;

(ii) *for any not necessarily symmetric though infinitesimal array $\{X_{nj}\}$, $\{\mathscr{L}(S_n)\}$ is relatively shift compact if and only if $\{e(\sum_j \mathscr{L}(X_{nj} - EX_{nj\delta}))\}$ is relatively compact for every (some) $\delta > 0$.*

In the noninfinitesimal case, the following partial result can be obtained:

4.4. THEOREM. *Let B be a separable Banach space with a Schauder basis. Then the following are equivalent*:

(i) *c_0 is not finitely representable in B*;

(ii) *for any symmetric triangular array, $\{X_{nj}\}$, $\{\mathscr{L}(S_n)\}$ is relatively compact if and only if $\{e(\sum_j \mathscr{L}(X_{nj}))\}$ is relatively compact.*

PROOF. (ii) $\Rightarrow$ (i) is contained in the proof of Theorem 4.1. To prove (i) $\Rightarrow$ (ii) proceed as in 4.1 but using Remark (a) after Theorem 3.4 instead of Theorem 3.4. □

For examples of spaces where condition (ii) in any of the previous theorems holds, see Remark after Theorem 1.2.

REMARKS. (a) Unlike the i.i.d. ($\sqrt{n}$) case the fact that the law of S_n converges need not imply that any of the X_{nj} is pre-Gaussian even in the case of bounded random variables (a B-valued rv X is pre-Gaussian if there exists a Gaussian p.m. with covariance of X). To see this last remark consider $B = c_0$, ϵ_{ij} independent Rademacher, and

$$X_{nj} = n^{-1/2}\,[\textstyle\sum_{i=3}^{N_n-1} \epsilon_{ij} e_i/(2 \log i \log\log i)^{1/2} + \sum_{i \ge N_n} \epsilon_{ij} e_i/(2 \log i)^{1/2}]$$

where N_n is chosen such that

$$P\{\textstyle\sum_{j=1}^{n} \| n^{-1/2} \sum_{i \le N_n} \epsilon_{ij} e_i/(2 \log i)^{1/2} \|_{c_0} > \delta\} = P\{\sum_{j=i}^{n} (2n \log N_n)^{-1/2} > \delta\} \to 0$$

for any $\delta > 0$ (for example, $N_n = e^{n^2}$). (Here $\{e_i\}$ is the canonical basis of c_0). But X_{nj} is not pre-Gaussian for any fixed n, j since

$$\sum_{i \geq N_n} \gamma_i e_i/(2 \log i)^{1/2} \notin c_0$$

where $\{\gamma_i\}$ are i.i.d. $N(0, 1)$ ($\limsup_i |\gamma_i|/(2 \log i)^{1/2} = 1$).

(b) It is known that if $\{X_{nj}\}$ is i.i.d. for each n, then $\{\mathscr{L}(S_n)\}$ is tight if and only if $\{e(\mu_n)\}$ is tight. (de Acosta and Samur, 1977, Theorem 2.2). Hence, there exist in spaces in which c_0 is finitely representable examples of triangular arrays $\{X_{nj}\}$, $\{Y_{nj}\}$, such that $\{\mathscr{L}(\sum_{j=1}^{k_n} X_{nj})\}$ converges $\{\mathscr{L}(\sum_{j=1}^{k_n} Y_{nj})\}$ does not converge, but the μ_n's are the same. Hence in such spaces one cannot obtain conditions on μ_n alone to ensure convergence of the triangular array.

We now use the approximation theorem to improve parts of Theorem 6.6 in de Acosta, et al, (1978).

4.5. THEOREM. *Let B be of cotype p and let $\{X_{nj}\}$ be a symmetric infinitesimal triangular array of uniformly bounded B-valued* rv's *such that $\{\mathscr{L}(S_n)\}$ is relatively compact. Then there exists an increasing sequence $\{F_k\}$ of finite dimensional subspaces with $\cup_k F_k = B$ such that* $\lim_k \sup_n \sum_{j=1}^{k_n} Eq_{F_k}^p(X_{nj}) = 0$.

PROOF. From Theorem 3.4 for each $\epsilon > 0$ there exists a finite dimensional triangular array $\{W_{nj}\}$ approximating $\{X_{nj}\}$. Let F_ϵ denote the range of $\{W_{nj}\}$. Then since $\|X_{nj} - W_{nj}\| \geq q_{F_\epsilon}(X_{nj})$ and B is of cotype p we have

$$\epsilon > E\|S_n - \textstyle\sum_{j=1}^{k_n} W_{nj}\|^p \geq c \sum_{j=1}^{k_n} E\|X_{nj} - W_{nj}\|^p \geq c \sum_{j=1}^{k_n} Eq_{F_\epsilon}^p(X_{nj}). \qquad \square$$

REMARKS. In Theorem 6.6 of de Acosta, et al, (1978), infinitesimality is not assumed and $\{F_k\}$ can be *any* increasing sequence of finite dimensional subspaces with $\overline{\cup_k F_k} = B$. However, an additional assumption such as B being the dual of a type 2 space or B having a Schauder basis must be assumed (see the remark following Theorem 6.6 there).

(b) In view of Theorem 1.2 (b) it follows that if c_0 is not finitely representable in B, then there exists $r > 2$ such that whenever X_{nj} is as in Theorem 4.5, then the conclusion of Theorem 4.5 holds for that r.

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