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WEAK CONVERGENCE OF THE EMPIRICAL CHARACTERISTIC FUNCTION

BY MICHAEL B. MARCUS

Northwestern University

Let X be a real valued random variable with probability distribution function $F(x)$ and characteristic function $c(t)$. Let $F_n(x)$ be the nth empirical distribution function associated with X and $c_n(t)$ the characteristic function of $F_n(x)$. Necessary and sufficient conditions are obtained for the weak convergence of $\sqrt{n}[c_n(t) - c(t)]$ on the space of continuous complex valued functions on $[-\frac{1}{2}, \frac{1}{2}]$.

Let X be a real valued random variable with probability distribution function $F(x)$ and characteristic function $c(t) = \int_{-\infty}^{\infty} e^{itx}\, dF(x)$. Let $F_n(x)$ be the nth empirical distribution function associated with $F(x)$, i.e., $F_n(x) = \frac{1}{n} \sum_{x=1}^{n} I_{\{[X_k,\infty)\}}(x)$ where $I_{\{A\}}$ is the characteristic function of the set A and $X_1, X_2, \ldots$ are independent identically distributed copies of X. $F_n(x)$ is an increasing, right continuous stochastic process; similarly the empirical characteristic function

$$c_n(t) = \int_{-\infty}^{\infty} e^{itx}\, dF_n(x) = \frac{1}{n} \sum_{k=1}^{n} e^{iX_k t}, \qquad t \in [-\tfrac{1}{2}, \tfrac{1}{2}]$$

is a stochastic process. The proper normalization to use in considering weak limits involving $c_n(t)$ is to consider

$$(1) \qquad C_n(t) = n^{1/2}(c_n(t) - c(t)), \qquad t \in [-\tfrac{1}{2}, \tfrac{1}{2}].$$

We can rewrite (1) as

$$(2) \qquad C_n(t) = \sum_{k=1}^{n} \frac{e^{iX_k t} - c(t)}{n^{1/2}}, \qquad t \in [-\tfrac{1}{2}, \tfrac{1}{2}]$$

to see that the question of the weak convergence of $C_n(t)$ is that of the standard central limit theorem for the random variable $e^{iXt} - c(t)$ which satisfies $E[e^{iXt} - c(t)] = 0$ and $E[(e^{iXt} - c(t))(\overline{e^{-iXs} - c(s)})] = c(t-s) - c(t)c(-s)$. To be more explicit, each process $C_n(t)$ induces a measure on the Banach space $C([-\frac{1}{2}, \frac{1}{2}])$ of continuous complex valued functions on $[-\frac{1}{2}, \frac{1}{2}]$ with the usual sup-norm ($\|\ \|_\infty = \sup_{t\in[-1/2,1/2]} |\ |$). We say that $C_n(t)$, $t \in [-\frac{1}{2}, \frac{1}{2}]$, converges weakly or equivalently that $e^{iXt} - c(t)$, $t \in [-\frac{1}{2}, \frac{1}{2}]$ satisfies the central limit theorem on $C([-\frac{1}{2}, \frac{1}{2}])$ if the measures induced by $C_n(t)$ on $C([-\frac{1}{2}, \frac{1}{2}])$ converge weakly.

A version of this problem was considered in [8] and also in [4]. We first read about it in a paper by S. Csörgo, [2], in which he points out that there is a fundamental error in [4]. Our notation and introduction to the problem is taken from [2]. Our approach is based on recent results in the central limit theorem for random trigonometric series [9], [10], [11], [3].

By the finite dimensional central limit theorem we see that if $C_n(t)$, $t \in [-\frac{1}{2}, \frac{1}{2}]$, converges weakly then the limiting process must be a Gaussian process with covariance $c(t$

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$- s) - c(t)c(-s)$. Thus a necessary condition for the weak convergence of $C_n(t)$, $t \in [-\frac{1}{2}, \frac{1}{2}]$, is that the Gaussian process with covariance $c(t - s) - c(t)c(-s)$ has continuous sample paths on $[-\frac{1}{2}, \frac{1}{2}]$. This process can be represented by the stochastic integral

$$Y(t) = \int_{-\infty}^{\infty} e^{itx}\, dB(F(x))$$

where B is a Brownian bridge. (i.e., let b be standard Brownian motion and define, for $0 \le y \le 1$, $B(y) = b(y) - yb(1)$). Note that $Y(t) = \int_{-\infty}^{\infty} e^{itx}\, db(F(x)) - b(1)c(t)$. Therefore $Y(t)$, $t \in [-\frac{1}{2}, \frac{1}{2}]$, has continuous sample paths if and only if the stationary Gaussian process

$$G(t) = \int_{-\infty}^{\infty} e^{itx}\, db(F(x)), \qquad t \in [-\tfrac{1}{2}, \tfrac{1}{2}] \tag{3}$$

has continuous sample paths. Along with (3) consider the real valued stationary Gaussian process

$$\tilde{G}(t) = \int_{-\infty}^{\infty} \cos tx\, db(F(x)) + \int_{-\infty}^{\infty} \sin tx\, db'(F(x)), \qquad t \in [-\tfrac{1}{2}, \tfrac{1}{2}] \tag{4}$$

where b' is an independent copy of b. We have

$$\begin{aligned} \sigma^2(|t - s|) &= E|G(t) - G(s)|^2 = E(\tilde{G}(t) - \tilde{G}(s))^2 \\ &= 4 \int_{-\infty}^{\infty} \sin^2 \frac{|t - s|x}{2}\, dF(x). \end{aligned} \tag{5}$$

Furthermore either both G and $\tilde{G}$ have continuous sample paths a.s. or else neither does.

The domain of σ defined in (5) is $[-1, 1]$. For this function we define

$$m_\sigma(x) = \lambda\{u \in [-1, 1] \mid \sigma(u) < x\}$$

where λ is Lebesgue measure and

$$\bar{\sigma}(u) = \sup\{y \mid m_\sigma(y) < u\}.$$

The function $\bar{\sigma}$ is called the nondecreasing rearrangement of σ. It is a nondecreasing function on $[0, 2]$ and has the same distribution function with respect to Lebesgue measure on $[0, 2]$ that $\sigma(u)$ has with respect to Lebesgue measure on $[-1, 1]$. Let

$$I(\sigma(s)) = I(\sigma) = \int_0^2 \frac{\bar{\sigma}(s)}{s\left(\log \frac{16}{s}\right)^{1/2}}\, ds. \tag{6}$$

By the Dudley-Fernique necessary and sufficient condition for the continuity of stationary Gaussian processes (see [7] Chapter IV, Theorem 7.6 and Corollary 6.3) we have that $\tilde{G}(t)$ and consequently $G(t)$ has continuous sample paths if and only if $I(\sigma) < \infty$. Therefore $I(\sigma) < \infty$ is a necessary condition for the weak convergence of $C_n(t)$. We will show that it is also sufficient.

THEOREM 1. *Let X be a real valued random variable with distribution function $F(x)$ and characteristic function $c(t)$. Let*

$$\sigma^2(t) = 4 \int_{-\infty}^{\infty} \sin^2 \frac{xt}{2}\, dF(x) = 2(1 - \operatorname{Re} c(t)) \tag{7}$$

and consider $I(\sigma)$ as defined in (6). *If $I(\sigma) < \infty$ the normalized empirical characteristic function $C_n(t)$ given in* (2) *converges weakly on $C([-\frac{1}{2}, \frac{1}{2}])$ to a Gaussian process with covariance $c(t-s) - c(t)c(-s)$. If $I(\sigma) = \infty$ the Gaussian process with covariance $c(t-s) - c(t)c(-s)$ does not have continuous sample paths and consequently $C_n(t)$ does not converge weakly on $C([-\frac{1}{2}, \frac{1}{2}])$.*

The method of proof presents us with some interesting examples of stochastic integrals of the type considered in [3]. These will be considered briefly following the proof of Theorem 1. We will also apply our method to a problem considered by Kent [8].

We pause for some preliminaries. A Rademacher sequence $\{\epsilon_k\}$ is a sequence of independent symmetric random variables each one taking on the values ± 1. The following lemma is well known in certain circles and deserves wider exposure.

LEMMA 2. *Let X be a random variable with values in a linear space L and let $\|\ \|$ be a norm or seminorm on L. Assume that $EX = 0$. Let X' be an independent copy of X. Then*

$$E\|X\| \le E\|X - X'\| \le 2E\|X\|. \tag{8}$$

Let $\{X_k, k = 1, 2, \cdots\}$ be a sequence of independent random variables with values in L, $EX_k = 0$, and let $\{\epsilon_k\}$ be a Rademacher sequence independent of $\{X_k\}$. Then

$$\tfrac{1}{2}E\|\textstyle\sum_{k=1}^n \epsilon_k X_k\| \le E\|\sum_{k=1}^n X_k\| \le 2E\|\sum_{k=1}^n \epsilon_k X_k\|. \tag{9}$$

Also, let $\{Y_k, k = 1, 2, \cdots\}$ be a sequence of independent random variables with values in L and let $\{\epsilon_k\}$ be a Rademacher sequence independent of $\{Y_k\}$. Then

$$E\|\textstyle\sum_{k=1}^n (Y_k - EY_k)\| \le 2E\|\sum_{k=1}^n \epsilon_k Y_k\|. \tag{10}$$

PROOF. The right side of (8) is trivial. For the left side we have

$$E\|X - X'\| = E_X E_{X'}\|X - X'\| \ge E_X\|S - EX'\|$$

where we write $E = E_X E_{X'}$ to indicate integration with respect to the components of the product space induced by X and X'. To obtain (9) we have

$$\begin{aligned} E\|\textstyle\sum_{k=1}^n X_k\| &\le E\|\textstyle\sum_{k=1}^n (X_k - X_k')\| = E\|\textstyle\sum_{k=1}^n \epsilon_k(X_k - X_k')\| \\ &\le 2E\|\textstyle\sum_{k=1}^n \epsilon_k X_k\|, \end{aligned}$$

where for the first inequality we use (8) and the equality follows because $\{(X_k - X_k')\}$ and $\{\epsilon_k(X_k - X_k')\}$ are equal in distribution. The left side of (9) follows similarly:

$$\begin{aligned} 2E\|\textstyle\sum_{k=1}^n X_k\| &\ge E\|\textstyle\sum_{k=1}^n (X_k - X_k')\| = E_\epsilon E_X E_{X'}\|\textstyle\sum_{k=1}^n \epsilon_k(X_k - X_k')\| \\ &\ge E_\epsilon E_X\|\textstyle\sum_{k=1}^n \epsilon_k(X_k - EX_k')\| = E\|\textstyle\sum_{k=1}^n \epsilon_k X_k\|. \end{aligned}$$

Here we denote by E_ϵ, E_X and $E_{X'}$ expectation with respect to the components of the product space induced by $\{\epsilon_k\}$, $\{X_k\}$ and $\{X_k'\}$. Finally for (10) we have

$$\begin{aligned} E\|\textstyle\sum_{k=1}^n (Y_k - EY_k)\| &\le E\|\textstyle\sum_{k=1}^n (Y_k - Y_k')\| \\ = E\|\textstyle\sum_{k=1}^n \epsilon_k(Y_k - Y_k')\| &\le 2E\|\textstyle\sum_{k=1}^n \epsilon_k Y_k\| \end{aligned}$$

by arguments similar to those given above.

PROOF OF THEOREM 1. We will give two proofs of this theorem in order to explore different ideas and techniques in the study of the central limit theorem on the Banach space of continuous functions on a compact set with the sup-norm. Let $I(\sigma) < \infty$; the case $I(\sigma) = \infty$ was considered in the remarks preceeding the statement of the theorem. We first obtain the weak convergence of a symmetrized version of $C_n(t)$. Let

$$S_n(t) = \frac{1}{\sqrt{n}} \sum_{k=1}^{n} (e^{iX_k t} - e^{iX'_k t}), \qquad t \in [-\tfrac{1}{2}, \tfrac{1}{2}],$$

where $\{X'_k\}$ is an independent copy of $\{X_k\}$. We have $ES_n(t)S_n(s) = 2(c(t-s) - c(t) c(-s))$. Furthermore, the finite dimensional distributions of $S_n(t)$ converge weakly to the corresponding finite dimensional distributions of the Guassian process with this covariance. For $f \in C([-\tfrac{1}{2}, \tfrac{1}{2}])$ we define a seminorm on $C([-\tfrac{1}{2}, \tfrac{1}{2}])$ by

$$\|f\|_d = \sup_{|s-t| \le d;\, s,t \in [-1/2,1/2]} |f(t) - f(s)|.$$

In order to prove the weak convergence of $S_n(t)$ it remains for us to show that given $\epsilon > 0$ there exists a d such that for all n

$$P(\|S_n\|_d > \epsilon) < \epsilon. \tag{11}$$

Let $(\Omega_1, \mathscr{F}_1, P_1)$ be the probability space of $\{X_k\}$ and $\{X'_k\}$ and $(\Omega_2, \mathscr{F}_2, P_2)$ the probability space of $\{\epsilon_k\}$ and denote the corresponding expectation operators by E_1, E_2. $S_n(t)$ is defined on $(\Omega_1 \times \Omega_2, \mathscr{F}_1 \times \mathscr{F}_2, P_1 \times P_2)$. We shall denote this space by $(\Omega, \mathscr{F}, P)$ and the corresponding expectation operator by E. Note that

$$\tilde{S}_n(t) = n^{-1/2} \sum_{k=1}^{n} \epsilon_k (e^{iX_k t} - e^{iX'_k t}), \qquad t \in [-\tfrac{1}{2}, \tfrac{1}{2}],$$

is stochastically equivalent to $S_n(t)$. Consider, for $w_1 \in \Omega_1$,

$$\tilde{S}_n(t, w_1) = n^{-1/2} \sum_{k=1}^{n} \epsilon_k (e^{iX_k(w_1)t} - e^{iX'_k(w_1)t}), \qquad t \in [-\tfrac{1}{2}, \tfrac{1}{2}],$$

as a stochastic process on $(\Omega_2, \mathscr{F}_2, P_2)$. We have

$$\begin{aligned} &(E_2 |\tilde{S}_n(t+u, w_1) - \tilde{S}_n(t, w_1)|^2)^{1/2} \\ &\le 2(2)^{1/2} \left[\left(\frac{1}{n} \sum_{k=1}^{n} \sin^2 \frac{X_k(w_1)u}{2} \right)^{1/2} + \left(\frac{1}{n} \sum_{k=1}^{n} \sin^2 \frac{X'_k(w_1)u}{2} \right)^{1/2} \right] \\ &\equiv \tau_n(u, w_1) \le 8. \end{aligned} \tag{12}$$

Let Re $\tilde{S}_n(t, w_1)$ denote the real part of $\tilde{S}_n(t, w_1)$. We have that Re $\tilde{S}_n(t, w_1)$ is a stochastic process with subgaussian increments ([7] Chapter II, Definition 5.4) and of course $(E_2 |\mathrm{Re}\, \tilde{S}_n(t+u, w_1) - \mathrm{Re}\, \tilde{S}_n(t, w_1)|^2)^{1/2} \le \tau_n(u, w_1)$. The same holds for the imaginary part of $\tilde{S}_n(t, w_1)$. Let $\hat{\tau}_n(\delta, w_1) = \sup_{|u| \le \delta} \tau_n(u, w_1)$ and set $\gamma_n(u, w_1) = \tau_n(u, w_1)/16$ and $\hat{\gamma}_n(\delta, w_1) = \hat{\tau}_n(\delta, w_1)/16$. We also observe that $|u| \le \delta$ implies $\gamma_n(u, w_1) \le \hat{\gamma}_n(\delta, w_1) \le \tfrac{1}{2}$. Taking these observations into consideration we apply Theorem 4.1 [9] to the real and imaginary parts of $\tilde{S}_n(t, w_1)/16$ and obtain

$$\begin{aligned} &\frac{1}{16} E_2[\sup_{|s-t| \le \delta;\, s,t \in [-1/2,1/2]} |\tilde{S}_n(t, w_1) - \tilde{S}_n(s, w_1)|] \\ &\le \frac{1}{16} E_2[\sup_{\gamma_n(u,w_1) \le \hat{\gamma}_n(\delta,w_1);\, u=|s-t|,\, s,t \in [-1/2,1/2]} |\tilde{S}_n(t, w_1) - \tilde{S}_n(s, w_1)|] \\ &\le K \left[\int_0^{\hat{\gamma}_n(\delta,w_1)} [\log N_{\gamma_n(\cdot,w_1)}([-\tfrac{1}{2}, \tfrac{1}{2}], s)]^{1/2}\, ds \right. \\ &\qquad \left. + g(\hat{\gamma}_n(\delta, w_1)) \right] \end{aligned} \tag{13}$$

where $g(x) = x(\log \log 1/x)^{1/2}$, K is a constant and $N_{\gamma_n(\cdot,w_1)}([-\tfrac{1}{2}, \tfrac{1}{2}], u)$ is the minimum number of open balls in the $\gamma_n(\cdot, w_1)$ metric or pseudometric with centers in $[-\tfrac{1}{2}, \tfrac{1}{2}]$ that covers $[-\tfrac{1}{2}, \tfrac{1}{2}]$.

We need to compute $E_1(\hat{\gamma}_n(\delta, w_1))$. To that end consider

$$(14)\qquad E_1 \sup_{|u|\le\delta}\left(\frac{1}{n}\sum_{k=1}^{n}\sin^2\frac{X_k(w_1)u}{2}\right)^{1/2} \le \frac{1}{2}E_1\left(\frac{1}{n}\sum_{k=1}^{n}(X_k^2(w_1)\delta^2 \wedge 1)\right)^{1/2} \le \frac{1}{2}(E|X^2\delta^2 \wedge 1|)^{1/2}.$$

Furthermore

$$E|X^2\delta^2 \wedge 1| = P[X^2 > 1/\delta^2] + 16\int_0^{1/\delta}\frac{\delta^2 X^2}{16}\,dF(x)$$

and since $u \le 2\sin u$ for $0 \le u \le 1$ we get

$$(15)\qquad E[X^2\delta^2 \wedge 1] \le P[X^2 > 1/\delta^2] + 4\sigma^2(\delta) \equiv h_\sigma^2(\delta)$$

and using (14) and (15) in (12) we get

$$(16)\qquad E_1(\hat{\gamma}_n(\delta, w_1)) \le 2\sqrt{2}\,h_\sigma^2\,(\delta)$$

By Lemma 6.1, Chapter IV [7] we have for all $j \ge 1$

$$(17)\qquad \frac{1}{2m_{\gamma_n(\cdot,w_1)}(2^{-j})} \le N_{\gamma_n(\cdot,w_1)}\left(\left[-\frac{1}{2},\frac{1}{2}\right], 2^{-j}\right) \le \frac{8}{m_{\gamma_n(\cdot,w_1)}(2^{-j-2})}$$

where $m_{\gamma_n(\cdot,w_1)}$ is defined in the same way as m_σ after (5). It follows that

$$(18)\qquad \int_0^{\hat{\gamma}_n(\delta,w_1)}[\log N_{\gamma_n(\cdot,w_1)}([-½, ½], s)]^{1/2}\,ds \le 4\int_0^{\hat{\gamma}_n(\delta,w_1)/4}\left[\log\left(1+\frac{8}{m_{\gamma_n(\cdot,w_1)}(s)}\right)\right]^{1/2} ds$$

and from Proposition 1.4.2 [3] and (16) that

$$(19)\qquad E_1\left[\int_0^{\hat{\gamma}_n(\delta,w_1)}[\log N_{\gamma_n(\cdot,w_1)}([-½, ½], s)]^{1/2}\,ds\right] \le 4\int_0^{2\sqrt{2}h_\sigma(\delta)}\left[\log\left(1+\frac{8}{m_{E\gamma_n(\cdot,w_1)}(s)}\right)\right]^{1/2} ds \le 4\int_0^{2\sqrt{2}h_\sigma(\delta)}\left[\log\left(1+\frac{8}{m_\sigma(2s)}\right)\right]^{1/2} ds \equiv H_\sigma(\delta)$$

where, at the last step, we use the facts that $E\gamma_n(u, w_1) \le \sigma(u)/2$ and $m_{\sigma/2}(u) = m_\sigma(2u)$. We apply E_1 to each side of (13) and since g is concave we obtain

$$(20)\qquad E[\sup_{|s-t|\le\delta; s,t\in[-1/2,1/2]}|\tilde{S}_n(t, w_1) - \tilde{S}_n(s, w_1)|] \le K'[H_\sigma(\delta) + g(2\sqrt{2}h_\sigma(\delta))].$$

It follows from (17) and integration by parts that $I(\sigma) < \infty$ implies that the last integral in (19) is finite (see also Lemma 6.2, Chapter IV [7]). Thus, whenever $I(\sigma) < \infty$ the last term in (19) goes to zero as δ goes to zero. Because of Lemma 2 this enables us to conclude that (11) holds and thus that $X_n(t)$ converges weakly on $C([-½, ½])$. It follows from Lemma 2 [6] that $C_n(t)$ converges weakly on $C([-½, ½])$ since $(E|C_n(t) - C_n(s)|^2)^{1/2}$, $t, s \in [-½, ½]$, is uniformly continuous.

The second proof depends even more heavily on Fernique's work. We use Lemma 2 to convert our problem into one of considering the stochastic integrals studied in [3]. We begin by quoting Theorem 1.3 [3]: "Let E be a Banach space and let Z be a random variable with values in E. In order for Z to satisfy the (standard) central limit theorem in E it is sufficient that for all $\epsilon > 0$ there exists a random variable Y with values in E that satisfies the central limit theorem in E and such that for all $n > 0$

$$(21)\qquad E\left\|\frac{1}{\sqrt{n}}\sum_{k=1}^{n}(Z_k - Y_k)\right\| < \epsilon$$

where Z_k, Y_k; $k = 1, \cdots$ are independent copies of Z and Y respectively." This theorem is a minor variant of a result of Pisier [12] (see also [5]).

Consider

$$Y_N(t) = \int_{-N}^{N} e^{ixt}\, d[I_{\{[X,\infty)\}}(x) - F(x)], \qquad t \in [-\tfrac{1}{2}, \tfrac{1}{2}].$$

Following Theorem 3.1 [4] (which is correct for $E|X|^{1+\delta} < \infty$)

$$\begin{aligned} E|Y_N(t) - Y_N(s)|^2 &\le 2\int_{-N}^{N} (1 - \cos x(t-s))\, dF(x) \\ &\le 2\int_{-N}^{N} |x(t-s)|^{1+\delta}\, dF(x) \\ &\le 8|t-s|^{1+\delta} N^{1+\delta} \end{aligned}$$

for $0 \le \delta \le 1$. Therefore by Theorem 12.3 [1] $Y_N(t)$ satisfies the central limit theorem. Our object is to show that

$$Z(t) = \int_{-\infty}^{\infty} e^{ixt}\, d[I_{[X,\infty)}(x) - F(x)], \qquad t \in [-\tfrac{1}{2}, \tfrac{1}{2}]$$

satisfies the central limit theorem (see (2)). Let

$$W_N(t) = \left(\int_{-\infty}^{N} + \int_{N}^{\infty}\right) e^{ixt}\, d[I_{\{[X,\infty)\}}(x) - F(x)], \qquad t \in [-\tfrac{1}{2}, \tfrac{1}{2}]$$

By the theorem quoted above we must show that

$$E\| n^{-1/2} \textstyle\sum_{k=1}^{n} W_{N,k}(t) \|_\infty < \epsilon_N \tag{22}$$

where $\epsilon_N \downarrow 0$ as $N \to \infty$. Here $W_{N,k}$; $k = 1, \cdots$ are independent copies of W_N. By (10) the left side of (22)

$$\le 2E\left\| \frac{1}{\sqrt{n}} \textstyle\sum_{k=1}^{n} \epsilon_k \left(\int_{-\infty}^{-N} + \int_{N}^{\infty}\right) e^{ixt}\, dI_{\{[X_k,\infty)\}}(x) \right\|_\infty \tag{23}$$

where $\{\epsilon_k\}$ is a Rademacher sequence independent of $\{W_{N,k}\}$. Let us write

$$\begin{aligned} U_N(t) &= \frac{1}{\sqrt{n}} \textstyle\sum_{k=1}^{n} \left(\int_{-\infty}^{-N} + \int_{N}^{\infty}\right) e^{ixt}\, dI_{\{[X_k,\infty)\}}(x) \\ &= \int_{-\infty}^{\infty} e^{ixt}\, dm_N(w, x) \end{aligned}$$

where $m_N(w, x)$ is a random measure given by

$$m_N(w, x) = n^{-1/2} \textstyle\sum_{k=1}^{n} \epsilon_k I_{\{[X'_k,\infty)\}}(x)$$

for $\{X'_k\}$ a sequence of independent identically distributed real valued random variables defined by

$$\begin{aligned} X'_1 &= 0 \qquad |X| < N \\ X'_1 &= X \qquad |X| \ge N. \end{aligned}$$

(To avoid problems we choose those N for which $-N$ and N are not atoms of X.) Clearly

$$\begin{aligned} m_N(x) = E|m_N(w, x)|^2 &= F(x) & x &\le -N \\ &= F(-N) & |x| &< N \\ &= F(-N) + (F(x) - F(N)) & x &\ge N. \end{aligned}$$

Also $m_N(x)$ is a measure on R and $m_N(R) = 1 - F(N) + F(-N)$. By Proposition 2.3 [3]

$$E\| U_N\|_\infty \le 4[1 - F(N) + F(-N)]^{1/2} + \sqrt{2}K \int_0^{2[1-F(N)-F(-N)]^{1/2}} \left[\log\left(1 + \frac{1}{m_o(u/2)}\right)\right]^{1/2} du \tag{24}$$

where we use the fact that

$$E\,|\,U_N(t+u) - U_N(t)\,|^2 = 4\left(\int_{-\infty}^{-N} + \int_N^\infty\right) \sin^2 \frac{xu}{2}\, dF(x) \le 4\sigma^2(u).$$

From (23), (24) and the argument following (20), we obtain (22). This completes the second proof of the theorem.

The second proof of Theorem 1 calls our attention to a class of stochastic processes which generalize the random trigonometric series considered in [10], [11] and give examples of the stochastic integrals considered in [3]. For complex $\{a_n\} \in l^2$ we consider

$$X(t) = \textstyle\sum_{k=1}^\infty a_k \epsilon_k \xi_k e^{iX_k t}, \qquad t \in [-\tfrac{1}{2}, \tfrac{1}{2}] \tag{25}$$

where $\{\epsilon_k\}$, $\{\xi_k\}$ and $\{X_k\}$ are independent of each other, $\{\epsilon_k\}$ is a Rademacher sequence and $\{\xi_k\}$ and $\{X_k\}$ are complex and real valued random variables respectively. Note that the $\{\xi_k\}$ and $\{X_k\}$ are not necessarily independent. Along with $(\Omega_1, \mathscr{F}_1, P_1)$ and $(\Omega_2, \mathscr{F}_2, P_2)$ defined above we define $(\Omega_3, \mathscr{F}_3, P_3)$ to be the probability space of $\{\xi_k\}$ with corresponding expectation operator E_3 and set $(\Omega, \mathscr{F}, P) = (\Omega_1 \times \Omega_2 \times \Omega_3, \mathscr{F}_1 \times \mathscr{F}_2 \times \mathscr{F}_3, P_1 \times P_2 \times P_3)$. For fixed $w_1 \in \Omega_1$, $w_3 \in \Omega_3$

$$X(t; w_1, w_3) = \textstyle\sum_{k=1}^\infty a_k \epsilon_k \xi_k(w_3) e^{iX_k(w_1)t}, \qquad t \in [-\tfrac{1}{2}, \tfrac{1}{2}].$$

is a random Fourier series of the type considered in [10], [11] and converges uniformly a.s. if and only if

$$I\left(\left(\textstyle\sum_{k=1}^\infty |\,a_k|^2 |\,\xi_k(w_3)|^2 \sin^2 \dfrac{X_k(w_1)u}{2}\right)^{1/2}\right) < \infty$$

a.s. $P_1 \times P_3$.

Let $\sup_k E_3 |\,\xi_k|^2 \le M^2$ and $\{X_k\}$ be independent and identically distributed with distribution function F and characteristic function $c(t)$ and let $\sigma^2(u) = 2(1 - \mathrm{Re}\ c(u))$ as above. Then following [10]

$$E_3 E_1 I\left(\left(\textstyle\sum_{k=1}^\infty |\,a_k|^2 |\,\xi_k(w_3)|^2 \sin^2 \dfrac{X_k(w_1)u}{2}\right)^{1/2}\right) \le CM(\textstyle\sum |\,a_k|^2)^{1/2} + MI(\sigma(u)). \tag{26}$$

Thus the series (25) converges uniformly a.s. if $I(\sigma) < \infty$. Now let X_k have distribution function F_k and characteristic function $c_k(t)$ and let $\sigma_k^2(u) = 2(1 - \mathrm{Re}\ c_k(u))$. Then, as in (26),

$$I((\textstyle\sum_{k=1}^\infty |\,a_k|^2 \sigma_k^2(u))^{1/2}) < \infty \tag{27}$$

implies the uniformly convergence a.s. of the series (25). The integral (27) depends on $\sigma_k(u)$. For example, if $P(X_k = \lambda_k) = 1$ where $\{\lambda_k\}$ is a sequence of real numbers then (25) is a random Fourier series and, under the additional hypothesis $\liminf_{k\to\infty} E\,|\,\xi_k| > 0$, (27) is also a necessary condition for the uniform convergence a.s. of (25). But (27) is not a necessary condition for the uniform convergence of (25) in general. The counterexample is very simple. Let $a_1 = 1$, $a_k = 0$, $k \ne 1$ and $\xi_1 \equiv 1$ in (25) so that (25) is simply

$$\epsilon_1 e^{iX_1 t}, \qquad t \in [-\tfrac{1}{2}, \tfrac{1}{2}]. \tag{28}$$

Let F be the distribution function of X_1 and let $c(t)$ be the corresponding characteristic function with $\sigma(u) = 2(1 - \text{Re } c(u))$. The stochastic process in (28) always has continuous sample paths, even for $I(\sigma) = \infty$.

Now let $\{a_k\}$ and $\{\xi_k\}$ be real in (25). We write

$$X(t) = \int_{-\infty}^{\infty} e^{ixt} \, d\{\textstyle\sum_{k=1}^{\infty} a_k \epsilon_k \xi_k I_{\{[X_k, \infty)\}}(x) \qquad t \in [-\tfrac{1}{2}, \tfrac{1}{2}] \tag{29}$$

This is a stochastic integral of the kind considered in [3]. The main theorem in [3] also shows that (27) is a sufficient condition for the uniform convergence of (25) (see Corollary 6.3, Chapter IV, [7]). Note that the example in (28) shows that Fernique's condition is not a necessary condition. Perhaps it is necessary if the random measures have independent and not just sign-invariant increments.

Kent [8] considers the weak convergence of

$$\varphi_n(t) = n^{-1/2} \textstyle\sum_{k=1}^{n} e^{iX_k(T_n + t)}, \qquad t \in [-\tfrac{1}{2}, \tfrac{1}{2}]$$

for some sequence of real numbers T_n, where $\{X_k\}$ is as in Theorem 1. As in the proof of Theorem 1 consider

$$\tilde{\varphi}_n(t, w_1) = n^{-1/2} \textstyle\sum_{k=1}^{n} \epsilon_k (e^{iX_k(w_1)(T_n+t)} - e^{iX_k'(w_1)(T_n+t)}), \qquad t \in [-\tfrac{1}{2}, \tfrac{1}{2}]$$

where $\{X_k'\}$ is an independent copy of $\{X_k\}$. We have

$$(E_2 \,|\, \tilde{\varphi}_n(t + u, w_1) - \tilde{\varphi}_n(t, w_1) \,|^2)^{1/2} \le \tau_n(u, w_1)$$

as in (12). Therefore, following the proof of Theorem 1 we see that the measures induced by

$$n^{-1/2} \textstyle\sum_{k=1}^{n} (e^{iX_k(T_n+t)} - c(T_n + t)), \qquad t \in [-\tfrac{1}{2}, \tfrac{1}{2}] \tag{30}$$

are relatively compact as long as $I(\sigma) < \infty$. Thus the only conditions needed in [8], besides $I(\sigma) < \infty$ which is necessary, are those that insure the weak convergence of the finite dimensional distributions of (30).

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Department of Mathematics
Northwestern University
Evanston, Illinois 60201