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THE MEAN NUMBER OF REAL ROOTS FOR ONE CLASS OF RANDOM POLYNOMIALS

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Let $\xi_0, \xi_1, \cdots, \xi_n, \cdots$ be a Gaussian stationary sequence of random variables. We study the asymptotic behavior of the mean number of real roots of the polynomial $P_n(x) = \xi_0 + \xi_1 x + \cdots + \xi_n x^n$ as $n \to \infty$.

Let $\xi_0, \xi_1, \cdots, \xi_n, \ldots$ be a Gaussian stationary sequence satisfying the conditions
(*) $E\xi_n = 0$, $\text{Var}\, \xi_n = 1$
(**) for any n, the distribution of $(\xi_0, \xi_1, \cdots, \xi_n)$ is nonsingular. We put $\rho_k = E\xi_0 \xi_k$ and denote by $\nu_n(a, b)$ the number of real roots of the polynomial

$$P_n(z) = \xi_0 + \xi_1 z + \cdots + \xi_n z^n$$

in the interval (a, b).

Let us note that:

(1) in view of the assumption (**), $P_n(z)$ has no multiple roots with probability 1;

(2) $E\nu_n(0, 1) = E\nu_n(1, \infty)$, $E\nu_n(-\infty, -1) = E\nu_n(-1, 0)$. $E\nu_n(-1, 0)$ and $E\nu_n(0, 1)$ may be different.

The purpose of this paper is study of the behavior of $E\nu_n(-\infty, \infty)$ as $n \to \infty$.

The problem reduces to studying $E\nu_n(0, 1)$, for $\nu_n(-1, 0)$ is the number of real roots in the interval (0, 1) of the polynomial $Q_n(z) = P_n(-z)$ whose coefficients $\xi_0, -\xi_1, \cdots, (-1)^n\xi_n$ form a Gaussian stationary sequence with the correlation function $\tilde{\rho}_k = (-1)^k\rho_k$. Kac proved (see [1]) that if $\xi_0, \xi_1, \cdots$ are independent ($\rho_k = 0$ for $k > 0$) then, as $n \to \infty$,

$$(1) \qquad E\nu_n(0, 1) \sim \frac{1}{2\pi} \ln n \quad \text{and} \quad E\nu_n(-\infty, \infty) \sim \frac{2}{\pi} \ln n$$

for in this case $\rho_k = \hat{\rho}_k$.

It turns out that these asymptotics remain valid if all the correlations are small. One of the main results of this paper is

THEOREM 1. *If*

$$\textstyle\sum_{k=1}^{\infty} |\rho_k| < ½$$

then

$$E\nu_n(0, 1) \sim \frac{1}{2\pi} \ln n$$

and

$$E\nu_n(-\infty, \infty) \sim \frac{2}{\pi} \ln n$$

as $n \to \infty$.

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In the condition of this theorem, the strict inequality cannot be replaced by a nonstrict one.

EXAMPLE 1. Let $\eta_0, \eta_1, \cdots, \eta_n, \cdots$ be a Gaussian stationary sequence of independent random variables, $E\eta_0 = 0$, Var $\eta_0 = 1$. We define $\xi_k = (-1)^k(\eta_k + \eta_{k+1})/\sqrt{2}$. Then $\xi_0, \xi_1, \cdots, \xi_n, \cdots$ is a Gaussian stationary sequence satisfying (*), (**); $\rho_1 = \frac{1}{2}$, $\rho_k = 0$ for $k > 1$, and in this case $E\nu_n(0, 1) = o(\ln n)$, $E\nu_n(-\infty, \infty) \sim (1/\pi) \ln n$.

It turns out that if $\rho_k \geq 0$ then the asymptotics (1) on (0, 1) remain valid under weaker restrictions on ρ_k.

THEOREM 2. *If $\rho_k \geq 0$ for all k and $\sum_{k=1}^n k\rho_k = o(n)$ as $n \to \infty$ then $E\nu_n(0, 1) \sim (1/2\pi) \ln n$ as $n \to \infty$.*

EXAMPLE 2. Let $\eta, \eta_0, \eta_1, \cdots, \eta_n, \cdots$ be a Gaussian stationary sequence of independent random variables with $E\eta = 0$, Var $\eta = 1$. We define $\xi_k = \sqrt{\rho}\eta + \sqrt{1 - \rho}\eta_k$ where $0 < \rho < 1$. Then $\xi_0, \xi_1, \cdots, \xi_n, \cdots$ is a Gaussian stationary sequence satisfying (*), (**), $\rho_k = \rho$ when $k > 0$.

In this case $E\nu_n(0, 1) = o(\ln n)$ as $n \to \infty$.

Theorem 2 asserts nothing about $E\nu_n(-1, 0)$, for the signs of the coefficients $\tilde{\rho}_k = (-1)^k\rho_k$ alternate. Note that in Example 1 the function $\tilde{\rho}_k$ satisfies the conditions of Theorem 2, yet the asymptotics (1) in $(-\infty, \infty)$ are not valid. In the case of alternating sign coefficients, strong assumptions concerning their regularity are necessary.

THEOREM 3. *If*

(1) $\rho_{2k} \geq 0$;

(2) *the function* $|\rho_k|$ *is strictly concave*;

(3) $\sum_{k=1}^n k|\rho_k| = o(n)$ *as* $n \to \infty$,

then

$$E\nu_n(0, 1) \sim \frac{1}{2\pi} \ln n \quad \textit{and} \quad E\nu_n(-\infty, \infty) \sim \frac{2}{\pi} \ln n \textit{ as } n \to \infty.$$

This theorem is an easy corollary from the following

LEMMA. *If*

(1) $\rho_{2k} \geq 0$;

(2) $\rho_{2k} \geq -\rho_{2k+1}$;

(3) $\rho_{2k} + 2\rho_{2k+1} + \rho_{2k+2} \geq 0$ *and there exists j such that* $\rho_{2j} + 2\rho_{2j+1} + \rho_{2j+2} > 0$;

(4) $\sum_{k=1}^n k|\rho_k| = o(n)$ *as* $n \to \infty$

then $E\nu_n(0, 1) \sim (1/2\pi) \ln n$ *as* $n \to \infty$.

Example 1 shows that without any regularity assumptions, no restrictions on the rate of convergence of ρ_k to 0 are sufficient for the asymptotics (1) to hold, and $E\nu_n(-1, 0)$ and $E\nu_n(0, 1)$ may differ even by order.

Let us sketch the proof of these results.

Under the assumptions (*), (**), the Kac-Rice formula holds (see [2]):

$$E\nu_n(0, 1) = \frac{1}{\pi}\int_0^1 (AC\text{-}B^2)^{1/2}A^{-1}dx \quad \text{where} \quad A = A_n(x) = EP_n^2(x),$$

$$B = B_n(x) = E[P_n(x)dP_n(x)/dx], \quad C = C_n(x) = E[dP_n(x)/dx]^2.$$

It is easy to see that

$$A = \textstyle\sum_{i=0}^n x^{2i} + 2\sum_{k=1}^n (\rho_k \sum_{i=0}^{n-k} x^{2i})$$

$$= \frac{1 - x^{2n}}{1 - x^2} + 2 \textstyle\sum_{k=1}^{n} \rho_k x^k (1 - x^{2n-2k+2})/(1 - x^2).$$

Similar expressions can be obtained for B and C. One finds that if ε is sufficiently large then in the interval $[0, 1 - \varepsilon n^{-1} \ln n]$ the terms containing x^n are negligibly small because $x^n \le (1 - \varepsilon n^{-1} \ln n)^n = O(n^{-\varepsilon})$. Taking it into account we obtain that

$$\int_0^{1-\varepsilon n^{-1} \ln n} (AC\text{-}B^2)^{1/2} A^{-1}\, dx \sim \tfrac{1}{2} \ln n;$$

and in the interval $[1 - \varepsilon n^{-1} \ln n, 1]$ there are not many roots for the interval itself is small.

The following example gives an idea of the significance of the condition (**).

EXAMPLE 3. Let m be a natural number, $\xi_k = \xi_j$ if $k = j \pmod m$, ξ_k and ξ_j independent otherwise. Evidently,

$$\rho_k = \begin{cases} 1 & \text{if} \quad k = 0 (\text{mod } m) \\ 0 & \text{if} \quad k \neq 0 (\text{mod } m) \end{cases}$$

and the condition (**) does not hold. Let n be such that $(n + 1/m) = j$ is an integer. Then

$$\begin{aligned} P_n(z) &= \xi_0 + \xi_1 z + \cdots + \xi_{m-1} z^{m-1} \\ &\quad + \xi_0 z^m + \cdots + \xi_{m-1} z^{2m-1} + \cdots \\ &\quad + \xi_0 z^{m(j-1)} + \cdots + \xi_{m-1} z^{mj-1} \\ &= (\xi_0 + \xi_1 z + \cdots + \xi_{m-1} z^{m-1})(1 + z^m + \cdots + z^{m(j-1)}) \end{aligned}$$

has at most m real roots counted according to multiplicity.

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